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THE ORDER-DUAL OF A TRL GROUP, I

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Abstract

Conditions are found for several intrinsically defined partial orders on $\mathfrak{Lb}$, the vector space of order-bounded additive functionals on a commutative pogroup, to have Riesz interpolation properties, and to make $\mathfrak{Lb}$ a TRL group.

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1. Introduction

We begin a study of the vector lattice $\mathfrak{Lb}$ of all order-bounded additive functionals on a commutative partially ordered group G, with particular attention to tight Riesz properties of $\mathfrak{Lb}$. G is assumed to be an l-group with respect to a partial order $\preccurlyeq$, and to carry a compatible tight Riesz order and its open-interval topology. Thus besides the usual notion of positivity for a functional $f \in \mathfrak{Lb}$ there are others, some of which (here written $\leqslant$, $\leqslant_o$, $\preccurlyeq_a$) we describe.

A fundamental theorem due to F. Riesz describes the vector-lattice structure of $\mathfrak{Lb}$ under its principal partial order $\preccurlyeq$. We show that two orders $\leqslant$ and $\leqslant_o$ are determining orders for this $\preccurlyeq$. The main aim of the paper is to formulate conditions under which $\leqslant$ on $\mathfrak{Lb}$ is a compatible tight Riesz order for $\preccurlyeq$. The interest in this question stems from the fact that, by Riesz's formula, the lattice operations on $\mathfrak{Lb}$ with respect to $\preccurlyeq$ are not pointwise on G^+; this is unlike the situation in most previously studied examples of compatible tight Riesz orders on l-groups. Two types of conditions are found; one based on compactness properties in G (9° and § 5), the other on properties of basic elements of G (10°). The latter are the more delicate.

We also find sufficient conditions for $\leqslant$ to be non-secular (9°, 11°, 12°).

By examples it is shown that not all continuous additive functionals need be order-bounded (4°, 5°).

Thanks are due to Robert Redfield who supplied the present form of Theorem 10° and the example in 13°, thus substantially improving an earlier version of this paper.

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Functionals f which are positive with respect to $\preccurlyeq$, but not with respect to some compatible tight Riesz order for $\preccurlyeq$, lie on the surface of the positive cone of $\mathfrak{Lb}$, so a study of such orders gives information about surface structure. A subsequent paper will deal with these questions.

2. Preliminaries

2.1 We summarize some definitions and results which are needed later. All order symbols $<, <_0, \prec, \ldots$ in this paper should be read as excluding equality, with $\leqslant$ meaning "$<$ or $=$", and so on. The (m,n) *tight Riesz property* for a poset $(X, \leqslant)$, abbreviated $\mathrm{TR}(m,n)$, asserts the following: For any set of elements a_i, b_j $(i = 1, 2, \ldots, m;\, j = 1, 2, \ldots, n)$ in X such that $a_i < b_j$ for all i, j, there exists $x \in X$ such that $a_i < x < b_j$ for all i, j. We have

$$\mathrm{TR}(2,2) \Rightarrow \mathrm{TR}(1,2);\ \mathrm{TR}(1,2) \Rightarrow \mathrm{TR}(1,1);\ \mathrm{TR}(1,2) \Leftrightarrow \mathrm{TR}(2,1)$$

when X is a pogroup; $\mathrm{TR}(1,2)$ does not imply $\mathrm{TR}(2,2)$. The loose Riesz property $\mathrm{LR}(m,n)$ is defined by replacing $<$ by $\leqslant$ at each occurrence. For any order $\leqslant$ on X, its *associated preorder* $\preccurlyeq$ is defined thus:

$$x \preccurlyeq y \text{ if and only if } (\forall\, u \in X)\,[u < x \Rightarrow u < y]\ \&\ (\forall\, t \in X)\,[t > y \Rightarrow t > x]. \tag{1}$$

When $(X, \leqslant)$ is a pogroup, this is equivalent to saying:

$$z \succcurlyeq 0 \quad \text{if and only if} \quad a > 0 \Rightarrow a + z > 0;$$

that is, the positive wedge of $\preccurlyeq$ is got by adjoining to the positive cone of $\leqslant$ all the pseudopositives of $\leqslant$. We consider only cases where $\preccurlyeq$ is a partial order. We call $\leqslant$ a *determining order* for $\preccurlyeq$. A partial order may have many determining orders.

A *tight Riesz group* (abbreviated TR group) is here defined to be a directed commutative† pogroup $(G, \leqslant)$ with the $\mathrm{TR}(1,2)$ property, and without pseudozeros, so that $(G, \preccurlyeq)$ is likewise a directed pogroup. We call $\leqslant$ a compatible tight Riesz order (CTRO) for $\preccurlyeq$. It is generally assumed that $G \neq (0)$ and neither $\leqslant$ nor $\preccurlyeq$ is trivial. The open-interval topology $\mathscr{U}$ defined from $\leqslant$ makes $(G, \mathscr{U})$ a non-discrete Hausdorff topological group, non-compact though quite possibly locally compact. Thus a TR group has a structure $(G, \leqslant, \preccurlyeq, \mathscr{U})$. By a *TR*$(2,2)$ *group* we mean a TR group for which $\leqslant$ is $\mathrm{TR}(2,2)$. For elementary consequences of these various definitions see Loy and Miller (1972) or Cameron and Miller (1975). We write $P = \{x \in G : x \geqslant 0\}$, $P^* = P \setminus \{0\}$, $G^+ = \{x : x \succcurlyeq 0\}$ for the positive cones. Order-intervals are written $(a,b) = \{x : a < x < b\}$, $[a,b] = \{x : a \leqslant x \leqslant b\}$, and similarly $((a,b))$, $[\![a,b]\!]$ for $\preccurlyeq$. The intervals (a,b), $a < b$, form a base for $\mathscr{U}$.

By a *TRL group* we mean a structure $(G, \leqslant, \preccurlyeq, \mathscr{U})$ in which $(G, \leqslant)$ is a TR group and $(G, \preccurlyeq)$ is an l-group, $\preccurlyeq$ being of course the associated order of $\leqslant$ and $\mathscr{U}$

† We assume that all groups in this paper are commutative: "group" means "abelian group".

being the open-interval topology of $\leqslant$. (By an "l-group", we mean a "lattice-ordered group" in the usual sense, as in Birkhoff (1967). The lattice operations of $(G, \preccurlyeq)$ are written $\wedge, \vee$.) For a TRL group G, $\leqslant$ is isolated and $(G, \mathscr{U})$ has no compact subgroups other than (0); see Loy and Miller (1972). If G is a TR group for which $\preccurlyeq$ is LR(2, 2) (in particular, if G is a TRL group), then $\leqslant$ is necessarily TR(2, 2). See Cameron and Miller (1975).

A TRL group G is called *secular* (or *androgynous*) if any of the following pairwise equivalent properties hold:

(i) G contains a pair of elements x, y satisfying

$$x > 0, \quad y \succ 0, \quad x \wedge y = 0.$$

(ii) The set $\Upsilon = \{x \in G: x^+ > 0, x^- \succ 0\}$ is non-empty. (Here $x^+ = x \vee 0$, $x^- = -(x \wedge 0)$.)

(iii) $P^* \nsubseteq \mathfrak{w}$. (Here $\mathfrak{w} = \{w \succ 0: w \wedge x = 0 \Rightarrow x = 0\}$ is the set of weak units of $(G, \preccurlyeq)$; we may have $\mathfrak{w} = \varnothing$.)

There are other characterizations; see Miller (1976). The property expresses a certain relationship of $\leqslant$ to its associated order, resulting in P^* occupying a greater portion of G^+ than is sometimes desirable; it can lead to computational difficulties. Since $\leqslant$ determines $\preccurlyeq$, it is allowable to call $\leqslant$ secular, rather than G. Secular groups are discussed in some detail in Miller (1976).

2.2 For any pogroup $(G, \leqslant)$, its *order-dual* is the real vector space $\mathfrak{Ld}(G)$ (briefly, $\mathfrak{Ld}$) of all order-bounded additive functionals in G, that is, additive functions mapping order-intervals of G to bounded subsets of $\mathbf{R}$. When G is a TR group, it does not matter which of its two orders is used here: they produce the same set $\mathfrak{Ld}$. However, when it comes to ordering $\mathfrak{Ld}$, as usual by ordering functionals pointwise on the positive cone of G, several possibilities arise. For $f \in \mathfrak{Ld}$ we shall write

$$f > 0 \quad \text{if and only if} \quad (\forall x \in G)[x \succ 0 \Rightarrow f(x) > 0], \tag{2}$$

$$f >_0 0 \quad \text{if and only if} \quad (\forall x \in G)[x > 0 \Rightarrow f(x) > 0], \tag{3}$$

$$f \succcurlyeq 0 \quad \text{if and only if} \quad (\forall x \in G)[x \succ 0 \Rightarrow f(x) \geqslant 0], \tag{4}$$

$$f \succcurlyeq_a 0 \quad \text{if and only if} \quad (\forall x \in G)[x > 0 \Rightarrow f(x) \geqslant 0]; \tag{5}$$

and $f \leqslant g$ will mean $g - f > 0$ or $g = f$, etc. These definitions make $\mathfrak{Ld}$ a partially ordered vector space with respect to each of $\leqslant, \leqslant_0, \preccurlyeq, \preccurlyeq_a$.

When G is a TRL group it is natural to think of the l-group structure of $(G, \preccurlyeq)$ as the dominating one, since much is known about l-groups. If we accept this view then $\preccurlyeq$ in (4) is the natural order to place on $\mathfrak{Ld}$. Notice that the orders $\leqslant$ and $\preccurlyeq$ in (2) and (4) are wholly determined by $\preccurlyeq$ on G, that is, are defined for any pogroup $(G, \preccurlyeq)$ whether or not $\preccurlyeq$ is an associated order. Our principal concern is with (2) and (4); nevertheless, the $\leqslant$-structure on G is relevant.

We write $\mathfrak{L}^+$ for the positive cone of $\preccurlyeq$ in $\mathfrak{Lb}$. It is clear that

$$f > 0 \Rightarrow f >_{\mathrm{o}} 0 \Rightarrow f \succ_{\mathrm{a}} 0 \quad \text{and} \quad f > 0 \Rightarrow f \succ 0 \Rightarrow f \succ_{\mathrm{a}} 0.$$

From results due to Hayes (1962) we know that $\mathrm{Hom}(G, \mathbf{R})$ contains non-zero elements, for any non-trivial group G. Bonsall (1954) has pointed out that if $(G, \preccurlyeq)$ is an everywhere non-archimedean pogroup, that is, if

$$G = \{x\colon \text{there exists } a \succcurlyeq 0 \text{ such that } -a \preccurlyeq nx \preccurlyeq a \text{ for all } n \in \mathbf{N}\},$$

then $\mathfrak{L}^+ = (0)$. On the other hand, another result of Hayes (1962) shows that $\mathfrak{L}^+$ contains non-zero elements if G has a strong unit. When G is a locally compact TR group with $\leqslant$ isolated, Mackey's theorem shows that the continuous additive functionals on G are sufficiently numerous to separate points. (Compare Hewitt and Ross (1963), Theorems (24.34) and (24.35).) It is easy to construct examples in which $\leqslant$ on $\mathfrak{Lb}$ is non-trivial; see 6° below.

We note some preliminary results for the structure $(G, \leqslant, \preccurlyeq, \mathscr{U})$.

1°. *When G is a TR group, the orders $\preccurlyeq$ and $\preccurlyeq_{\mathrm{a}}$ on $\mathfrak{Lb}$ coincide, and $f \succcurlyeq 0$ implies that f is continuous. We have, for all $f \in \mathfrak{Lb}$,*

$$f > 0 \Rightarrow f >_{\mathrm{o}} 0 \Rightarrow f \succ 0 \Leftrightarrow f \succ_{\mathrm{a}} 0. \tag{6}$$

Proof. Certainly $f \succcurlyeq 0$ implies $f \succcurlyeq_{\mathrm{a}} 0$. We prove that $f \succcurlyeq_{\mathrm{a}} 0$ implies that f is continuous. Let $f \succcurlyeq_{\mathrm{a}} 0$. Suppose $\ker(f)$ meets P^*, say $f(a) = 0$ with $a > 0$. Then for any $x \in G$, $x + (-a, a)$ is a neighbourhood of x on which f is constant. Hence f is continuous. Suppose, on the other hand, that $\ker(f)$ does not meet P^*, that is $f >_{\mathrm{o}} 0$. Let $(x_i)_{i \in I}$ be a net converging to x in G. Given any $\varepsilon > 0$ in $\mathbf{R}$ choose any $a \in P^*$, then $n \in \mathbf{N}$ so that $0 < f(a)/n < \varepsilon$, then $b \in G$ such that $0 < b < nb < a$, and take $V = (x - b, x + b)$. (The existence of such an element b is easily shown.) Eventually the net is in V and $|f(x_i) - f(x)| < \varepsilon$; so again f is continuous.

Finally, $f \succcurlyeq_{\mathrm{a}} 0$ implies $f \succcurlyeq 0$. For if $f \succcurlyeq_{\mathrm{a}} 0$ then when $x \succcurlyeq 0$ we can find a net $(x_i)_{i \in I}$ in P^* converging to x and continuity of f gives $f(x) \geqslant 0$, so $f \succcurlyeq 0$. The implication (6) is clear. //

2°. *When $(G, \leqslant)$ is a TR group:*

(i) *If $f \succ 0$ and $f(c) \neq 0$ for some $c > 0$, then f does not vanish identically on* $(0, c)$.

(ii) *If $f \succ 0$ then* $\ker(f)$ *is a closed convex subgroup of* $(G, \preccurlyeq, \mathscr{U})$, *and* $f(b) > 0$ *for some* $b > 0$.

(iii) *For $f \in \mathfrak{Lb}$, $f >_{\mathrm{o}} 0$ if and only if $f \succ 0$ and* $\ker(f)$ *is not open. If $f >_{\mathrm{o}} 0$ then* $\mathrm{ran}(f)$ *is dense in* $\mathbf{R}$.

Proof. (i) and (ii) are straightforward. (iii) Clearly $f >_{\mathrm{o}} 0$ if and only if $f \succ 0$ and $\ker(f)$ does not meet P^*. On the other hand, $\ker(f)$ is open if and only if $\ker(f)$ meets P^*. For if $\ker(f)$ is open then it contains some interval (a, b), and taking

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$a<x<y<b$ we get $y-x\in\ker(f)\cap P^*$; conversely, if $\ker(f)$ meets P^*, say $f(a)=0$, $a>0$, then by (ii) $\ker(f)$ contains the open interval $(a,2a)$, hence it has an interior point, hence it is open.

Suppose $f>_0 0$; let $0<\varepsilon\in\mathbf{R}$. As in the proof of 1° there exists $b>0$ with $0<f(b)<\varepsilon$. (For this, $f\succ 0$ does not suffice; we need $f>_0 0$.) So if $0<\alpha<\beta$ in $\mathbf{R}$ write $\varepsilon=\frac{1}{2}(\beta-\alpha)$ and find a corresponding b. Then $(m-1)f(b)\leqslant\alpha<mf(b)$ for some $m\in\mathbf{N}$, and this implies $\alpha<f(mb)<\beta$. Thus $f(P^*)$ is dense in $\mathbf{R}^+$ and hence $f(G)$ is dense in $\mathbf{R}$. //

3°. *When $(G,\leqslant)$ is a TR group, each of the orders $\leqslant$ and $\leqslant_0$, if non-trivial, is a determining order on $\mathfrak{Lb}$ for $\preccurlyeq$, and their cones have bases. The cone $\mathfrak{L}^+$ of $\preccurlyeq$ has a base if G has a strong unit.*

The same conclusions follow for $\leqslant$ and $\preccurlyeq$ on $\mathfrak{Lb}$ if $(G,\preccurlyeq)$ is any l-group.

Proof. Consider $\leqslant_0$. By 1°, $f\leqslant_0 g$ implies $f\preccurlyeq g$. Now if $h\succ 0$ then

$$f>_0 0 \quad\text{implies}\quad f+h>_0 0, \tag{7}$$

for when $x\in P^*$ we have $(f+h)(x)=f(x)+h(x)>0$; (7) shows that h is positive in the associated order of $\leqslant_0$. Conversely, suppose $0\not\preccurlyeq h$, so that $h(x)<0$ for some $x\succ 0$. If $f>_0 0$ then $f(x)\geqslant 0$ and by multiplying f by a small positive real if necessary we can arrange that $0\leqslant f(x)<-h(x)$, so $0\not\preccurlyeq f+h$ and hence $f+h\not>_0 0$. Therefore (7) does not hold. This proves that $\leqslant_0$ determines $\preccurlyeq$. The proof for $\leqslant$ is the same.

Concerning bases for the cones, by Peressini (1976), p. 26, it suffices to produce a strictly positive linear functional in each case, that is, a linear map $\alpha\colon\mathfrak{Lb}\to\mathbf{R}$ such that $f>0\Rightarrow\alpha(f)>0$, or $f>_0 0\Rightarrow\alpha(f)>0$, or $f\succ 0\Rightarrow\alpha(f)>0$, respectively. For the first two cases take any $x\in P^*$ and define $\alpha(f)=f(x)$. In the third define $\alpha(f)=f(s)$ where s is a strong unit of G.†

When $(G,\preccurlyeq)$ is any l-group (that is, no determining order for $\preccurlyeq$ is given) the statements about $\leqslant$ and $\preccurlyeq$ on $\mathfrak{Lb}$ still make sense and are proved in the same way. //

For any pogroup $(G,\preccurlyeq)$ (whether or not $\preccurlyeq$ is an associated order) there is F. Riesz's theorem (see, for example, Peressini (1967), §2.3):

If $(G,\preccurlyeq)$ is an $LR(2,2)$ directed pogroup then $(\mathfrak{Lb}(G),\preccurlyeq)$ is a complete vector lattice, the lattice operations being given (for $a\in G^+$) by the formulae

$$(f\vee g)(a)=\sup\{f(x)+g(y)\colon x,y\in G^+,\ x+y=a\}, \tag{8}$$

$$(f\wedge g)(a)=\inf\{f(x)+g(y)\colon x,y\in G^+,\ x+y=a\}; \tag{9}$$

and

$$\mathfrak{Lb}=\mathfrak{L}^+-\mathfrak{L}^+. \tag{10}$$

† The element s is a strong unit for $(G,\leqslant)$ if and only if $(\forall\, x\in G^+)\,(\exists\, n\in N)\,(x\leqslant ns)$. The order $\leqslant$ and its associated order have the same set of strong units.

The conditions here are met when $(G, \preccurlyeq)$ is an l-group. The conclusions of the theorem are also deducible from the following modified hypothesis: $(G, \leqslant)$ is a TR(2, 2) group. The proof is like that for the theorem itself, and uses also 1° above. Since $\leqslant$ is TR(2, 2) it is also LR(2, 2), and the formulae (8) and (9) also hold in their modified forms

$$(f \vee g)(a) = \sup\{f(x)+g(y) \colon x, y \in P^*, x+y = a\} \tag{8'}$$

and its dual, (9').

3. Examples

For a TR group G we can form the real vector space $\mathfrak{L} \equiv \mathfrak{L}(G)$ of all continuous additive functionals on G. By 1° and (10) we have

$$\mathfrak{L}\mathfrak{b} \subseteq \mathfrak{L}.$$

The following two examples illustrate cases where $\mathfrak{L}\mathfrak{b} \neq \mathfrak{L}$. The first is due to R. H. Redfield.

4°. Let $G = \mathbf{R} \circ \mathbf{R}$, the lexicographic product of $\mathbf{R}$ with itself, in which $\langle x, y\rangle > 0$ if and only if $x > 0$ or $x = 0$, $y > 0$. Here $\leqslant$ is full and so coincides with its associated order $\preccurlyeq$; G is a TRL group. Let f be the map $f\langle x, y\rangle = y$. Then $f \in \mathfrak{L}$. However, if $A = \{\langle 0, y\rangle \colon y \in \mathbf{R}\}$ then A is bounded since $\langle -1, 0\rangle < A < \langle 1, 0\rangle$, but $f(A) = \mathbf{R}$. Thus $f \notin \mathfrak{L}\mathfrak{b}$.

5°. Let G be the subgroup of $C[0, 1]$ consisting of all continuous functions x for which the derivative $x'(0)$ exists. Take $\preccurlyeq$ to be the weak pointwise order ($x \succcurlyeq 0$ if and only if $x(t) \geqslant 0$ for all $0 \leqslant t \leqslant 1$), and define $x > 0$ to mean $x(t) > 0$ for all $0 < t \leqslant 1$. Then $(G, \preccurlyeq)$ is an l-subgroup of $(C[0, 1], \preccurlyeq)$, though not convex, and

$$(x \vee y)'(0) = \begin{cases} y'(0) & (\text{if } x(0) < y(0)), \\ \max\{x'(0), y'(0)\} & (\text{if } x(0) = y(0)). \end{cases}$$

Moreover, $\leqslant$ is a TR(2, 2) determining order for $\preccurlyeq$. With respect to its open-interval topology $\mathscr{U}$, convergence of a net $(x_i)_{i \in I}$ to x implies that x_i converges uniformly to x on $[0, 1]$ and $x_i(0) = x(0)$ for $i \geqslant$ some i_0. G is a nonsecular TRL group; $\preccurlyeq$ is archimedean, $\leqslant$ is not eudoxian.

Let f be the map defined by $f(x) = x'(0)$; we have $f \in \mathfrak{L} \setminus \mathfrak{L}\mathfrak{b}$.

Instead, let G be as above, except that $\leqslant$ is now defined thus: $x > 0$ means

$$x(t) > 0 \text{ if } 0 < t < \tfrac{1}{2}, \quad x(t) \geqslant 0 \text{ if } \tfrac{1}{2} \leqslant t \leqslant 1 \text{ or } t = 0.$$

We still have $f \in \mathfrak{L} \setminus \mathfrak{L}\mathfrak{b}$, but $\leqslant$ is now a secular order for the TRL group G.

6°. Let c denote the sequence space of all real sequences $\alpha = (\alpha_n)_{n\in\mathbf{N}}$ for which the limit

$$\lambda(\alpha) = \lim_{n\to\infty} \alpha_n$$

exists; c is a vector lattice, *a fortiori* a commutative l-group, with respect to the weak pointwise order $\preccurlyeq$ on sequences. For any real sequence $\zeta = (\zeta_n)_{n\in\mathbf{N}}$ write

$$f(\alpha) = \sum_{n=1}^{\infty} \zeta_n \alpha_n. \tag{11}$$

This equation defines an element $f\in\mathfrak{L}^+(c)$ if and only if $\zeta\in l^1$ and $\zeta_n \geqslant 0$ for all n. The general element of $\mathfrak{L}^+$ has the form $f+\rho\lambda$, where $\rho\in\mathbf{R}^+$, so that $\mathfrak{L}\mathfrak{b}$ can be identified with $l^1\oplus(\lambda)$.

For f in (11) we have $f>0$ if and only if $\zeta_n>0$ for all n; on the other hand, $\lambda\succ 0$, $\lambda \not> 0$. The only lattice homomorphisms in $\mathfrak{L}^+$ are λ and those f for which $\operatorname{supp}(\zeta)$ is a singleton.

Let a filter $\mathscr{F}$ of subsets of $\mathbf{N}$ be given; define $\leqslant$ on c by

$$\alpha>0 \quad \text{if and only if} \quad \alpha\succcurlyeq 0 \text{ and } \operatorname{supp}(\alpha)=\{n\in\mathbf{N}\colon \alpha_n>0\}\in\mathscr{F}. \tag{12}$$

Then $\leqslant$ is a compatible TR(2, 2) order for $(c, \preccurlyeq)$. For f in (11) we have

$$f>_0 0 \quad \text{if and only if} \quad \operatorname{supp}(\zeta) \text{ meets every set in } \mathscr{F};$$

we have $\lambda \not>_0 0$.

Every sequence α in c^+ for which $\inf\alpha_n>0$ is a strong unit of c; $\mathfrak{L}\mathfrak{b}$ has no strong units.

4. Tight interpolation for $\leqslant$ and $\leqslant_0$

4.1 The orders $\leqslant$ and $\leqslant_0$ on $\mathfrak{L}\mathfrak{b}$ are TR(1, 1), that is, order-dense, since for example if $f<g$ then $f<\frac{1}{2}(f+g)<g$. If $\leqslant$ is non-trivial and TR(1, 2) then since its associated order is LR(2, 2), $\leqslant$ is a compatible TR(2, 2) order for $\preccurlyeq$. The same remark applies to $\leqslant_0$. Let $\mathscr{T}$ denote the open-interval topology of $\leqslant$ on $\mathfrak{L}\mathfrak{b}$. We have, in view of previous remarks:

7°. *If $(G, \leqslant, \preccurlyeq, \mathscr{U})$ is a TR(2, 2) group (or if $(G, \preccurlyeq)$ is any l-group) and if $\leqslant$ on $\mathfrak{L}\mathfrak{b}$ is non-trivial and TR(1, 2), then*

$$(\mathfrak{L}\mathfrak{b}(G), \leqslant, \preccurlyeq, \mathscr{T})$$

is a TRL group.

We ask if either order is TR(1, 2). First, we note that $\leqslant_0$ need not be TR(1, 2). This failure is simply illustrated by the following example.

8°. Take $G=\mathbf{R}^2$ with the strong and weak pointwise orders $\leqslant$, $\preccurlyeq$, and functionals

$$f\langle x_1, x_2\rangle = x_1, \quad g\langle x_1, x_2\rangle = x_2.$$

We have $f, g \in \mathfrak{L}^+$, in fact $f, g >_0 0$, and for any $a = \langle a_1, a_2 \rangle > 0$ in $\mathbf{R}^2$,

$$(f \wedge g)(a) = \inf\{f(x) + g(y) \colon 0 \preccurlyeq x, y;\ x + y = a\} = 0,$$

the infimum being attained by taking $x = \langle 0, a_2 \rangle$, $y = \langle a_1, 0 \rangle$. Thus $f \wedge g = 0$, and so $0 <_0 h <_0 f, g$ is possible for no $h \in \mathfrak{Lb}$.

This example also shows that $\wedge$ for $\mathfrak{Lb}$ need not be pointwise on G^+, since $f(a) \wedge g(a) = \min\{a_1, a_2\} > 0$. If for any TRL group G it is the case that $\wedge$ (and so $\vee$) is pointwise on G^+ then $\leqslant$ and $\leqslant_0$ are TR(2,2) on $\mathfrak{Lb}$. For with $0 < f, g$ in $\mathfrak{Lb}$ and $a \succ 0$ we should have $0 < f(a) \wedge g(a) = (f \wedge g)(a)$, so $f \wedge g > 0$, whence

$$0 < \tfrac{1}{2}(f \wedge g) < f, g$$

so $\leqslant$ is TR(1,2), hence TR(2,2). However, the proviso is rather special, as we have seen. The counterexample shows that $\leqslant_0$ is not really the appropriate order to expect to be TR(2,2). For $\leqslant$ the property is more apt, but is still a delicate matter. The remainder of this section deals with the question for $\leqslant$ on $\mathfrak{Lb}$. We describe two cases where $\leqslant$, if non-trivial, can be shown to be TR(2,2): when $(G, \mathscr{U})$ is interval-compact, and when $(G, \preccurlyeq)$ has a basis.

4.2 A TR(2,2) group is called *interval-compact* if $[\![a, b]\!]$ is compact for every $a \preccurlyeq b$. We have

$$(a, b)^- = ((a, b))^- = [\![a, b]\!] \text{ whenever } a < b$$

(where $^-$ denotes closure); and equivalent formulations of the property are: $(a, b)^-$ is compact for every $a < b$; $(0, a)^-$ is compact for every $a > 0$; $[\![0, a]\!]$ is compact for every $a \succcurlyeq 0$.

If $(G, \leqslant, \preccurlyeq, \mathscr{U})$ is an interval-compact TR(2,2) group then $(G, \preccurlyeq)$ is a lattice-complete l-group, and $(G, \mathscr{U})$ is locally compact. For these and related results see Loy and Miller (1972).

9° THEOREM. *If G is an interval-compact TR(2,2) group then $\leqslant$ on $\mathfrak{Lb}(G)$ if non-trivial is TR(2,2), and $(\mathfrak{Lb}, \leqslant, \preccurlyeq, \mathscr{T})$ is a non-secular TRL group.*

PROOF. Let $f, g \in \mathfrak{L}^+$, and $a \succ 0$ in G. From (9) we have

$$(f \wedge g)(a) = g(a) - \sup\{g(x) - f(x) \colon 0 \preccurlyeq x \preccurlyeq a\} \geqslant 0. \tag{13}$$

There exists a net $(x_i)_{i \in I}$ in $[\![0, a]\!]$ such that

$$\lim (g(x_i) - f(x_i)) = \sup\{g(x) - f(x) \colon 0 \preccurlyeq x \preccurlyeq a\}. \tag{14}$$

Suppose that $(f \wedge g)(a) = 0$. Then $\lim(g(x_i) - f(x_i)) = g(a)$. Since

$$g(x) - f(x) \leqslant g(x) \leqslant g(a)$$

for $x \in [\![0, a]\!]$, it follows that

$$\lim g(x_i) = g(a), \quad \lim f(x_i) = 0. \tag{15}$$

By assumption, $[\![0,a]\!]$ is compact, so replacing $(x_i)_{i\in I}$ by a subset if necessary we can assume that $\lim x_i$ exists, $= x_0 \in [\![0,a]\!]$. Since f and g are continuous by 1°, $g(a) = \lim g(x_i) = g(x_0)$ and $0 = \lim f(x_i) = f(x_0)$.

Now suppose that $f > 0$ and $g > 0$. Since $x_0 \prec a$ would imply $g(x_0) < g(a)$ we have $x_0 = a$ and hence $0 < f(a) = f(x_0) = 0$, contradiction. We have thus shown that $f, g > 0$ implies $(f \wedge g)(a) > 0$ for all $a \succ 0$, that is, $f \wedge g > 0$. Therefore $\leqslant$ is TR$(1,2)$, and so TR$(2,2)$, and 7° shows that $\mathfrak{Lb}$ is a TRL group.

Suppose instead that $f \succ 0$ and $g > 0$. The above considerations in this case show that $f(a) = 0$ whenever $(f \wedge g)(a) = 0$ and $a \succ 0$. Thus $f \succ 0$ and $g > 0$ imply $f \wedge g \succ 0$, which means (by 2.1) that $\mathfrak{Lb}$ is non-secular. //

Remark: The trivial order is always TR$(2,2)$; on the other hand, for $\leqslant$ to determine $\preccurlyeq$ on $\mathfrak{Lb}$, $\leqslant$ must be non-trivial.

A less direct proof of 9° is possible using 10° and 11° below and a result due to Wirth (1975) characterizing interval-compact tight Riesz groups.

4.3 For any l-group $(G, \preccurlyeq)$, a *basic element* is by definition an element $a \in G^+ \setminus \{0\}$ such that $[\![0,a]\!]$ is a fully-ordered subset of G^+. Alternative characterizations are: (i) The carrier $\tilde{a}$ determined by a is an atom of the carrier lattice $\mathfrak{C}$ of G; (ii) If $0 \preccurlyeq s, t \preccurlyeq a$ and $s \wedge t = 0$ then $s = 0$ or $t = 0$; (iii) $a^{\perp\perp}$ is fully-ordered; (iv) $a^{\perp\perp}$ is an atom of the lattice Pol(G) of all polars of G. (For any subset $A \subseteq G$, the *polar* of A is $A^{\perp} = \{x \in G: |x| \wedge |a| = 0 \text{ for all } a \in A\}$, and $c^{\perp} = \{c\}^{\perp}$. The polars form a complete Boolean algebra Pol(G) with respect to inclusion and $^{\perp}$ as complementation. For $c \in G^+$ the *carrier* determined by c is $\tilde{c} = \{x \in G^+: x^{\perp} = c^{\perp}\}$. The carriers form a distributive disjunctive lattice $(\mathfrak{C}, \preccurlyeq)$ when ordered by writing $\tilde{a} \preccurlyeq \tilde{b}$ if and only if $a^{\perp\perp} \subseteq b^{\perp\perp}$.)

A *basis* of the l-group $(G, \preccurlyeq)$ is any subset of $G^+ \setminus \{0\}$ which is maximal with respect to the property: each element of the subset is basic, and the elements are pairwise disjoint. G has a basis if and only if $\mathfrak{C}$ is atomic, that is, every element $\tilde{x} \in \mathfrak{C}$ dominates some atom (it is then the join of the atoms it dominates); equivalently, every $x \succ 0$ dominates some basic element.

The following result is due to R. H. Redfield; it subsumes a number of special cases proved earlier by the author using more complicated arguments.

10°. Theorem. *Let $(G, \preccurlyeq)$ be any l-group with a basis. Then $\leqslant$ on $\mathfrak{Lb}$, if non-trivial, is TR$(2,2)$, and $\mathfrak{Lb}$ is a TRL group.*

Proof. Let $f, g > 0$ in $\mathfrak{Lb}$ and suppose $u \succ 0$ in G. Then $f \wedge g \geqslant 0$ and we wish to show that $(f \wedge g)(u) > 0$. By assumption there exists some basic element $a \preccurlyeq u$. Since $(2a)^{\sim} = \tilde{a}$, $2a$ is also basic. Take any $x, y \in G^+$ with $x + y = 2a$; since $[\![0, 2a]\!]$ is fully-ordered, either $a \preccurlyeq x$ or $a \preccurlyeq y$ so either $f(x) + g(y) \geqslant f(x) \geqslant f(a)$ or

$f(x)+g(y) \geqslant g(y) \geqslant g(a)$ and therefore by (9),

$$(f \wedge g)(2a) \geqslant \min\{f(a), g(a)\} > 0$$

since $f > 0$ and $g > 0$. Thus in either case

$$2(f \wedge g)(u) \geqslant (f \wedge g)(2a) > 0.$$

This proves that $f \wedge g > 0$ and hence that $\leqslant$ is TR(2, 2). Again 7° shows that $\mathfrak{L}\mathfrak{b}$ is a TRL group. //

The question of whether $\leqslant$ is non-secular is not so immediately settled in this case as it is in 9°. We have the following sufficient condition.

11°. *Let* $(G, \preccurlyeq)$ *be any l-group with a basis. Suppose that* $\leqslant$ *on* $\mathfrak{L}\mathfrak{b}$ *is non-trivial, and that for every* $f \succ 0$ *in* $\mathfrak{L}\mathfrak{b}$ *there exists a basic element a such that* $f(a) > 0$. *Then* $\mathfrak{L}\mathfrak{b}$ *is non-secular.*

PROOF. We have to show that in $\mathfrak{L}\mathfrak{b}$, $f \succ 0$ and $g > 0$ imply $f \wedge g \succ 0$. Suppose a is basic and $f(a) > 0$. Necessarily $g(a) > 0$, and consequently the same argument as in the proof of 10° leads to $(f \wedge g)(a) > 0$. This proves $f \wedge g \succ 0$. //

Call an l-group $(G, \preccurlyeq)$ *Jaffard projectable* if G has a basis, and

$$G = a^{\perp\perp} \oplus a^{\perp}$$

for every basic element. Call $(G, \preccurlyeq)$ *finitely based* if G has a basis, and for every non-zero $x \in G^+$ there exists no infinite subset of $\{y : 0 \prec y \preccurlyeq x\}$ the elements of which are pairwise disjoint (equivalently: the carrier lattice $\mathfrak{C}$ of G is atomic and each non-zero carrier $\tilde{x}$ dominates only finitely many atoms). P. Jaffard (1953) showed that an l-group is expressible as a direct sum

$$\sum_{i \in I} \oplus H_i$$

of fully-ordered convex subgroups if and only if it is Jaffard projectable and finitely based. In this case the H_i's are precisely the principal bipolars $a_i^{\perp\perp}$, where $\tilde{a}_i$ runs through the atoms of $\mathfrak{C}$. Here I need not be finite. From 11° we deduce:

12°. *Let* $(G, \preccurlyeq)$ *be a Jaffard projectable and finitely based l-group. Then* $\leqslant$, *if non-trivial, makes* $\mathfrak{L}\mathfrak{b}$ *a non-secular TRL group.*

PROOF. If f vanishes on every basic element then by the representation $G = \sum_{i \in I} \oplus a_i^{\perp\perp}$, f vanishes on G. Thus $f \succ 0$ implies $f(a) > 0$ for some basic element a, and 11° gives the result. //

It is reasonable to conjecture that the condition that $\mathfrak{C}$ is finitely based can be dropped from 15°. When $(G, \preccurlyeq)$ is Jaffard projectable its basic subgroup B (the subgroup generated by the set of all the basic elements) has G for its lattice-closure (this result is due to R. H. Redfield), so if g vanishes on B it might be supposed that it should vanish on G. However, if a continuous functional vanishes on a subset X of a TRL group it need not vanish on $\sup(X)$. In fact the conjecture is false, as the following counterexample (also due to Redfield) shows.

13°. EXAMPLE of a TRL group G for which $(G, \preccurlyeq)$ is order-dense, Jaffard projectable and archimedean, G is non-secular, but $\leqslant$ on $\mathfrak{Lb}$ is secular. Let G be the subset of $\prod_{k=1}^{\infty} \mathbf{R}$ consisting of functions x of the form

$$x = s + p,$$

where $s \in \sum_{k=1}^{\infty} \mathbf{R}$ and $p = (p_k)_{k \in \mathbf{N}}$, $p_k = 2^{-k} \alpha_x$ for some $\alpha_x \in \mathbf{Q}$. Clearly x determines s and p uniquely, and G is an l-subgroup of the cardinal product $(\prod \mathbf{R}, \preccurlyeq)$. Let $\leqslant$ be the strong pointwise order on G. Then $(G, \leqslant, \preccurlyeq)$ is a TRL group with the asserted properties. We produce two functionals $f, g \in \mathfrak{Lb}$ such that $f \succ 0$, $g > 0$ and $f \wedge g = 0$, namely

$$f(x) = \alpha_x, \quad g(x) = \sum_{k=1}^{\infty} x_k.$$

To prove $f \wedge g = 0$, let $a \in G^+$. If $a \in \sum \mathbf{R}$ then $f(a) = 0$ so $f \wedge g(a) = 0$. Suppose $a \notin \sum \mathbf{R}$, and let $\varepsilon > 0$ in $\mathbf{R}$; pick n such that $\sum_{k>n} 2^{-k} \alpha_a < \frac{1}{2}\varepsilon$, define $b \in \prod \mathbf{R}$ by

$$b_k = \varepsilon / 2^n \quad (1 \leqslant k \leqslant n), \qquad \alpha_a / 2^k \quad (k > n),$$

and let $c = a \vee b - b$. Then $0 \preccurlyeq c \in \sum \mathbf{R}$, so $f(c) = 0$, $g(b) < \varepsilon$ and therefore

$$0 \leqslant (f \wedge g)(a) \leqslant (f \wedge g)(a \vee b) \leqslant f(c) + g(b) < \varepsilon.$$

Thus $f \wedge g = 0$.

4.4 The question of whether $\leqslant$ on $\mathfrak{Lb}$ is TR(1, 2), for l-groups G not covered in 9° and 10°, can be formulated in terms of certain sets of the form

$$A_{s,t}(a) = \{x \colon 0 \preccurlyeq x \preccurlyeq a, \text{ but neither } s \preccurlyeq x \text{ nor } x \preccurlyeq t\}.$$

Here a is some element in $G^+ \setminus \{0\}$, and $s, t \in ((0, a))$. It is found most useful to choose s, t so that $0 \prec s \prec t \prec a$. Let $f > 0$ and $g > 0$ in $\mathfrak{Lb}$, and assume that $(f \wedge g)(a) = 0$, and consider the sequence $(x_i)_{i \in I}$ in the proof of 9°, with the properties (14) and (15). If for some s we have $0 \prec s \preccurlyeq x_i$ for all i in some cofinal subset I_0 of I, then $0 < f(s) \leqslant f(x_i)$ for all $i \in I_0$, contradicting (15). (Note that we need $f > 0$ here, not merely $f \succ 0$.) Similarly, if $x_i \preccurlyeq t \prec a$ for some t and all i in some cofinal subset we get $g(x_i) \leqslant g(t) < g(a)$, contradicting (15). Therefore, for all $s, t \in ((0, a))$, x_i is eventually in $A_{s,t}(a)$.

If $A_{s,t}(a) \neq \varnothing$ for all such s, t, this means roughly speaking that the net $(x_i)_{i \in I}$ migrates towards the boundary of $[\![0, a]\!]$ and away from 0 and a: this can be

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illustrated by considering the group $G = \mathbf{R}^2$, taking $a > 0$. If, on the contrary, it can be shown that $A_{s,t}(a) = \emptyset$ for some pair $s, t \in ((0, a))$ then we have a contradiction implying $(f \wedge g)(a) > 0$. Thus

14°. *Let* $(G, \preccurlyeq)$ *be an l-group. For* $\leqslant$ *on* $\mathfrak{Lb}$ *to be* $TR(1,2)$ *it is sufficient that* G *satisfy the following condition*:

[*] *For every* $a \succ 0$ *in* G *there exists a pair of elements* $s, t \in ((0, a))$ *such that* $A_{s,t}(a) = \emptyset$.

The use of basic elements in 10° reduces the discussion to the case where $[\![0, a]\!]$ is fully-ordered; here [*] is satisfied trivially by any s, t such that $0 \prec s \prec t \prec a$. Since the sets $A_{s,t}(a)$ for fixed a do not form a filterbase, [*] does not seem to be a necessary condition.

5. Another CTRO for $(\mathfrak{Lb}, \preccurlyeq)$

There is another result establishing a CTRO for $(\mathfrak{Lb}, \preccurlyeq)$, suggested by the compactness argument in 9°. It concerns not $\leqslant$ but yet another partial order on $\mathfrak{Lb}$, which we write $\leqslant_1$. This time we assume that $(G, \leqslant, \preccurlyeq, \mathscr{U})$ is a locally compact TR group. In this case the set

$$D_1 = \{x \succcurlyeq 0 \colon [\![0, x]\!] \text{ is compact}\}$$

is non-empty, and generates a subgroup $G_1 = D_1 - D_1$, for which $G_1 \cap G^+ = D_1$. For $f \in \mathfrak{Lb}(G)$ write

$$f >_1 0 \quad \text{if and only if} \quad f \succcurlyeq 0, \text{ and } f(x) > 0 \text{ for every } x \in D_1 \setminus \{0\}. \tag{16}$$

This makes $(\mathfrak{Lb}(G), \leqslant_1)$ a partially ordered vector space, and by almost the same arguments as were used in proving 3° and 9° we find that $\leqslant_1$, if non-trivial, is a TR(2, 2) determining order for $\preccurlyeq$ on $\mathfrak{Lb}(G)$, and $(\mathfrak{Lb}, \leqslant_1, \preccurlyeq)$ is a TRL group.

When $(G, \mathscr{U})$ is locally compact, 10° is a particular case of this result.

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