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Global smooth solutions and singularity formation for the relativistic Euler equations with radial symmetry*

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Abstract. In this paper, we consider the global-in-time existence and singularity formation of smooth solutions for the radially symmetric relativistic Euler equations of polytropic gases. We introduce the rarefaction/compression character variables for the supersonic expanding wave with relativity and derive their Riccati type equations to establish a series of priori estimates of solutions by the characteristic method and the invariant domain idea. It is verified that, for rarefactive initial data with vacuum at the origin, smooth solutions will exist globally. On the other hand, the smooth solution develops a singularity in finite time when the initial data are compressed and include strong compression somewhere.

1 Introduction

We consider the global-in-time existence and singularity formation of radially symmetric relativistic smooth flows. The system of equations for the relativistic fluid dynamics, as an important representative model of nonlinear hyperbolic conservation laws, has wide applications in several fields such as astrophysics, plasma physics and nuclear physics, and has been extensively studied by many mathematicians and physicists. The motion of perfect inviscid fluids in the $(d+1)$-dimensional Minkowski space-time $(t,x)\in\mathbb{R}_+\times\mathbb{R}^d$ $(d\geq 1)$ in special relativity can be described by the isentropic relativistic Euler equations (e.g. [48, 53, 15, 50])

$$\begin{cases} \left(\dfrac{\rho+\varepsilon p}{1-\varepsilon\mathbf{u}^2}-\varepsilon p\right)_t+\nabla\cdot\left(\dfrac{\rho+\varepsilon p}{1-\varepsilon\mathbf{u}^2}\mathbf{u}\right)=0,\\ \left(\dfrac{\rho+\varepsilon p}{1-\varepsilon\mathbf{u}^2}\mathbf{u}\right)_t+\nabla\cdot\left(\dfrac{\rho+\varepsilon p}{1-\varepsilon\mathbf{u}^2}\mathbf{u}\otimes\mathbf{u}+p\right)=0, \end{cases} \tag{1.1}$$

where ρ is the mass-energy density of fluid, $p=p(\rho)$ is the pressure, $\mathbf{u}=(u_1,\cdots,u_d)$ is the particle speed, $\varepsilon=1/c^2$ and $c>0$ is the light speed. The

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first and second equations in (1.1) are the conservation laws in energy and momentum for the relativistic fluids. We are interested in system (1.1) with the classical equation of state for the polytropic ideal gas

$$p(\rho) = A\rho^{\gamma}, \tag{1.2}$$

where A is a positive constant and $\gamma > 1$ is the adiabatic constant. The relativistic constraint leads to the corresponding physical area

$$\Pi = \Big\{(\rho, \mathbf{u})|\ 0 \le \rho < \rho^*; |\mathbf{u}| = \sqrt{u_1^2 + \cdots + u_d^2} < \frac{1}{\sqrt{\varepsilon}}\Big\},$$

where the number ρ^* is determined by $p'(\rho^*) = 1/\varepsilon$.

For a radially symmetric relativistic flow, that is the solution has the following geometric structure

$$\begin{aligned} &\rho(\mathbf{x}, t) = \rho(r, t), \quad u_i(\mathbf{x}, t) = \frac{x_i}{r} u(r, t), \\ &r = |\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_d^2} \quad \text{with} \quad d > 1, \end{aligned} \tag{1.3}$$

where $u(r, t)$ is a scalar function, system (1.1) can be reduced to

$$\begin{cases} \left(\dfrac{\rho + \varepsilon p}{1 - \varepsilon u^2} - \varepsilon p\right)_t + \left(\dfrac{\rho + \varepsilon p}{1 - \varepsilon u^2} u\right)_r = -\dfrac{(d-1)(\rho + \varepsilon p)u}{r(1 - \varepsilon u^2)}, \\ \left(\dfrac{\rho + \varepsilon p}{1 - \varepsilon u^2} u\right)_t + \left(\dfrac{\rho + \varepsilon p}{1 - \varepsilon u^2} u^2 + p\right)_r = -\dfrac{(d-1)(\rho + \varepsilon p)u^2}{r(1 - \varepsilon u^2)}. \end{cases} \tag{1.4}$$

In this paper, we tackle the problems of global existence and singularity formation of smooth solutions to system (1.4) in all dimensions $d > 1$. Particularity, system (1.4) with $d = 2$ and $d = 3$ correspond to the cylindrically and spherically symmetric relativistic Euler equations, respectively. In the Newtonian limit ($c \to \infty$), system (1.4) formally reduces to the classical radially symmetric isentropic Euler equations

$$\begin{cases} \rho_t + (\rho u)_r = -\dfrac{(d-1)\rho u}{r}, \\ (\rho u)_t + (\rho u^2 + p)_r = -\dfrac{(d-1)\rho u^2}{r}. \end{cases} \tag{1.5}$$

Many excellent pieces of work have been contributed on the study of one-dimensional isentropic relativistic Euler equations (1.1) with the equation of state (1.2). We refer the reader to, e.g., works on the Riemann solutions [15, 13], on the local existence of smooth solutions [21, 47], on the formation of singularities of smooth solutions [1, 2, 24, 48] and on the studies of weak solutions [14, 20, 30, 50, 53]. In [51], Ruan and Zhu established an existence theorem of global smooth solutions to its Cauchy problem by applying the characteristic method and maximum principle. The global existence of smooth solutions for some general one-dimensional quasilinear hyperbolic systems was presented in [40, 42].

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As a special but physically interesting multi-dimensional flow, the radially symmetric relativistic Euler equations (1.4) has also been broadly investigated. By using Sideris's framework [52], singularity formation results of smooth solutions to the multi-dimensional relativistic Euler equations with polytropic gases were presented in [48, 24, 54, 2]. The framework of Sideris is based on the establishment of differential inequalities for some averaged quantities, and therefore can not provide a local analysis of the singularity. For more general solutions in multiple space dimensions, a geometric framework was introduced by Christodoulou [16] to study the singularity formation of smooth solutions for the relativistic Euler equations. Also see the related works in [3, 4, 5, 17, 19, 44, 45] etc for the classical Euler equations and in [26, 34, 35] etc for the more general quasilinear wave equations. The singularity formation problem for the radially symmetric relativistic Euler equations with the so-called Chaplygin gas equation of state $p = -\rho^{-1}$ was analyzed by the characteristic method in [27, 28].

In another related area, the global existence of bounded weak entropy solutions for (1.4), (1.2) with $\gamma = 1$ was established outside a unit ball by Mizohata [46] employing Glimm's method. This method was also applied by Hao, Li and Wang [25] to achieve the non-relativistic global limits of entropy solutions for (1.4). In [31], Hsu, Lin and Makino utilized the Lax-Friedrichs scheme to establish the global existence of spherically symmetric weak solutions of (1.4) with initial data containing the vacuum state. The spherical piston problem for the relativistic Euler equations was discussed in [18, 37]. In [55], Wei analyzed the stabilizing effect of the power law inflation on relativistic Eulerian fluids. Moreover, Lai [36] constructed a family of self-similar bounded weak entropy solutions for (1.4). Recently, the third author and Zhang [29] constructed a piecewise smooth solution containing a single shock wave for the radially symmetric relativistic Euler equations (1.4) with polytropic gas equation of state (1.2).

An important feature of quasilinear hyperbolic systems is that, even for sufficiently smooth and small initial data, their smooth solutions may form singularities in finite time. Therefore, it is rather meaningful and interesting to investigate what conditions are necessary to guarantee the global existence of smooth solutions for the relativistic system (1.4). Grassin [23] studied the multi-dimensional classical Euler equations with polytropic gases and acquired global smooth solutions for sufficiently small initial density and sufficiently smooth initial velocity that forces particles to spread out, also see Lecureux-Mercier [39] for the extension of van der Waals gases. Godin [22] derived the lifespan of smooth solutions for the spherically symmetric Euler equations with initial data that are a small perturbation of a constant state.

To study the global existence and singularity formation of smooth solutions for the one-dimensional quasilinear hyperbolic systems, the characteristic method is undoubtedly a significant idea. It is proved by using the characteristic method that smooth solutions of one-dimensional classical isentropic Euler equations exist globally if and only if the initial data are rarefactive everywhere, see among others [6, 8, 10, 11, 40, 59]. Also see the works by Ruan and

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Zhu [51] and Athanasiou, Bayles-Rea and Zhu [1, 2] for the one-dimensional relativistic Euler equations. Based on the fact that these one-dimensional hyperbolic systems are homogeneous, the rarefaction/compression (R/C) character of solutions can be directly defined by the sign of the gradient on a pair of Riemann invariants. Due to the presence of non-constant entropy, the situation for the one-dimensional full, non-isentropic Euler equations is relatively more complex. Some singularity formation results of its smooth solutions were established in [7, 12] by utilizing the strong compressibility of initial data to overcome the influence of entropy. In [10], Chen, Pan and Zhu shown that initial weak compressions do not necessarily form singularity in finite time, unless the compression is strong enough for general data. Pan and Zhu [49] found a class of solutions of the non-isentropic Euler equations, developing singularity in finite time even though their initial data do not contain any compression. In [1], the authors demonstrated the development of singularities for the one-dimensional non-isentropic relativistic Euler equations with strong compressive initial data by discussing some non-homogeneous ordinary differential equations along characteristics.

For the radially symmetric Euler and relativistic Euler systems, a natural idea is to use the gradient of Riemann variables to characterize the R/C character of solutions by mimicking the one-dimensional cases. However, due to the influence of source terms, directly selecting gradient variables in this way often fails to obtain the desired a priori estimates of solutions. In [58, 57], the authors studied the radially symmetric Euler equations (1.5) satisfying a polytropic gas equation of state (1.2) with damping. Under the assumption that the damping is strong enough, they applied the characteristic method to establish the global existence of smooth solutions outside a ball under some sufficient conditions. Some global existence results for the p-system with damping were presented in [56, 32, 33]. In [38], Lai and Zhu manipulated a class of initial data that are constant state near the origin and meet tedious conditions elsewhere to acquire a global existence result for (1.5) with $d = 2$. The constant initial data allow them to construct a smooth self-similar solution and then gain an expanding vacuum region centered at the origin.

In [9], Chen et al introduced the concepts of rarefaction and compression characters for expanding wave for the radially symmetric Euler equations (1.5) with (1.2). The corresponding R/C character variables are not only related to the gradients of Riemann variables but also to the non-homogeneous terms of the governing system. The idea in [9] originates from the observation that the stationary solution of the governing system is neither rarefactive nor compressive, that is the R/C character variables should be vanished in the stationary solution. Then they derived the Riccati type equations for the R/C character variables and utilized them to prove that smooth solutions with rarefactive initial data will exist for any time, while strong initial compression will make solutions to form finite time singularities.

In the current paper, we are inspired by the previous work [9] to discover a pair of appropriate gradient variables from the stationary solution of the relativistic Euler equations (1.4) and apply them to discuss the global existence and singularity formation of its smooth solutions. For the classical Euler system (1.5), the authors [9] introduced its R/C characters

$$\alpha_e = -\frac{\partial_{e1}(r^{d-1}\rho u)}{r^{d-1}\rho\cdot\lambda_{e2}}, \qquad \beta_e = -\frac{\partial_{e2}(r^{d-1}\rho u)}{r^{d-1}\rho\cdot\lambda_{e1}}, \tag{1.6}$$

where $\lambda_{ei} = u+(-1)^i\sqrt{p'(\rho)}$ are two eigenvalues of (1.5), and $\partial_{ei} = \partial_t+\lambda_{ei}\partial_r$. It is noted that $r^{d-1}\rho u$ is constant in the stationary solution of (1.5). For the relativistic Euler equations (1.4), we can easily see that the quantity

$$r^{d-1}\frac{\rho+\varepsilon p(\rho)}{1-\varepsilon u^2}u$$

is constant in its stationary solution. Then we define the variables α and β by

$$\begin{aligned} \alpha &= -\frac{\partial_1\Big(r^{d-1}\frac{\rho+\varepsilon p(\rho)}{1-\varepsilon u^2}u\Big)}{r^{d-1}(\rho+\varepsilon p(\rho))\cdot\frac{u+\sqrt{p'(\rho)}}{1-\varepsilon\sqrt{p'(\rho)}u}}, \\ \beta &= -\frac{\partial_2\Big(r^{d-1}\frac{\rho+\varepsilon p(\rho)}{1-\varepsilon u^2}u\Big)}{r^{d-1}(\rho+\varepsilon p(\rho))\cdot\frac{u-\sqrt{p'(\rho)}}{1+\varepsilon\sqrt{p'(\rho)}u}}, \end{aligned} \tag{1.7}$$

where

$$\partial_1 = \partial_t+\lambda_1\partial_r, \quad \partial_2 = \partial_t+\lambda_2\partial_r, \tag{1.8}$$

are the directional derivatives along the characteristics, and

$$\lambda_1 = \frac{u-\sqrt{p'(\rho)}}{1-\varepsilon\sqrt{p'(\rho)}u}, \quad \lambda_2 = \frac{u+\sqrt{p'(\rho)}}{1+\varepsilon\sqrt{p'(\rho)}u}, \tag{1.9}$$

are two eigenvalues of the relativistic Euler equations (1.4). It is worth emphasizing that the last two terms in the denominators of (1.7) are not the eigenvalues λ_1 and λ_2, which are different from the constructions in (1.6). These special resultants are used to determine the signs of some coefficients in the Riccati equations of (α,β), which are important for our analysis in the paper.

We present the main process of this paper below. For any number $b > 0$, we first consider the Cauchy problem to (1.4) with (1.2) on $[b,\infty)$. By using the equations of the Riemann variables, we can gain the a priori C^0 estimates of the solutions. Then we derive the Riccati type equations for the rarefaction characters (α,β) and apply these equations and initial data $0 \le \alpha(r,0), \beta(r,0) \le M$ to establish their invariant domain, which leads to the a priori C^1 estimates of the solutions. Moreover, the invariant domain of (α,β) allows us to deduce a lower bound estimate of density independent of the spatial variable r. For any fixed time $T>0$, the lower bound of density and the C^1-estimates of the solutions depend only on the numbers b and T. Hence we can achieve a unique global solution of (1.4) on the region Ω_b,

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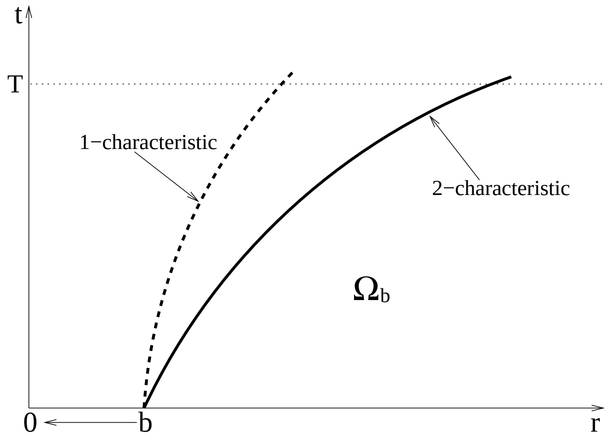

Figure 1: The region Ω_b..

where $\Omega_b = \{(r,t):\ 0 \le t \le T, r \ge r_2(t;b,0)\}$ and $r = r_2(t;b,0)$ is the 2-characteristic curve starting from point $(b,0)$. See Figure 1 for illustration. Based on the conclusion that the characteristic starting from $(r,t) = (0,0)$ must be $r = 0$, we let $b \to 0$ to establish the global existence result on the entire half line $[0,\infty)$. On the other hand, for the compressive initial data $\alpha(r,0) < 0, \beta(r,0) < 0$, we employ their Riccati type equations to prove that $\alpha(r,t) < 0, \beta(r,t) < 0$ in the region of existence of smooth solutions. If $-\alpha(r,0)$ or $-\beta(r,0)$ is further assumed to be sufficiently large somewhere at a certain point, then the smooth solution may form singularities in finite time.

The rest of the paper is organized as follows. In Section 2, we introduce the Riemann variables for system (1.4) and provide the main result of the paper. Section 3 is devoted to deriving the Riccati type equations of (α,β). In Section 4, we establish invariant domains for the Riemann variables and the rarefaction characters (α,β), from which we obtain the a priori C^1 estimates of the solutions. A positive lower bound estimate for density that is only time-dependent while spatially independent is also deduced in this section. In Section 5, we demonstrate the global existence of C^1 solution on the entire domain $t \ge 0$, $r \ge 0$ for rarefactive initial data. Finally, the singularity formation of smooth solutions is discussed in Section 6.

2 The main results

In this section, we introduce a pair of Riemann variables to transform system (1.4) into a diagonal form and then present the main results of the paper.

We consider smooth solutions of system (1.4) with (1.2), in which case the system can be rewritten as

$$\mathbf{A}\mathbf{U}_t + \mathbf{B}\mathbf{U}_r = \mathbf{F}, \tag{2.1}$$

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where the variables and the coefficient matrices are

$$\mathbf{U} = \begin{pmatrix} \rho \\ u \end{pmatrix}, \qquad \mathbf{F} = -\frac{(d-1)(\rho + \varepsilon A\rho^\gamma)u}{(1-\varepsilon u^2)r} \begin{pmatrix} 1 \\ u \end{pmatrix},$$

and

$$\mathbf{A} = \begin{pmatrix} \dfrac{1+\varepsilon^2 A\gamma\rho^{\gamma-1}u^2}{1-\varepsilon u^2} & \dfrac{2\varepsilon(\rho+\varepsilon A\rho^\gamma)u}{(1-\varepsilon u^2)^2} \\ \dfrac{(1+\varepsilon A\gamma\rho^{\gamma-1})u}{1-\varepsilon u^2} & \dfrac{(\rho+\varepsilon A\rho^\gamma)(1+\varepsilon u^2)}{(1-\varepsilon u^2)^2} \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} \dfrac{(1+\varepsilon A\gamma\rho^{\gamma-1})u}{1-\varepsilon u^2} & \dfrac{(\rho+\varepsilon A\rho^\gamma)(1+\varepsilon u^2)}{(1-\varepsilon u^2)^2} \\ \dfrac{A\gamma\rho^{\gamma-1}+u^2}{1-\varepsilon u^2} & \dfrac{2(\rho+\varepsilon A\rho^\gamma)u}{(1-\varepsilon u^2)^2} \end{pmatrix}.$$

The eigenvalues of (2.1) are defined by finding the roots of $|\lambda\mathbf{A} - \mathbf{B}| = 0$, which are given in (1.9). The corresponding right eigenvectors of λ_1 and λ_2 are, respectively,

$$\mathbf{R}_1 = \left(-\frac{\sqrt{A\gamma\rho^{\gamma-1}}}{\rho+\varepsilon A\rho^\gamma}, \frac{1}{1-\varepsilon u^2} \right)^\mathsf{T}, \quad \mathbf{R}_2 = \left(\frac{\sqrt{A\gamma\rho^{\gamma-1}}}{\rho+\varepsilon A\rho^\gamma}, \frac{1}{1-\varepsilon u^2} \right)^\mathsf{T}.$$

One can check that there hold $\nabla\lambda_i \cdot \mathbf{R}_i \neq 0$ for $0 < \rho < \rho^*$, which means that system (2.1) is strictly hyperbolic and genuinely nonlinear.

Let the symbol a represent the sound speed, that is, $a = \sqrt{p_\rho} = \sqrt{A\gamma}\rho^{\frac{\gamma-1}{2}}$ with $\gamma > 1$ satisfying the relativistic constraint $a < 1/\sqrt{\varepsilon}$. We rewrite system (2.1) as follows

$$\begin{cases} \dfrac{2\gamma}{\gamma-1}\dfrac{\varepsilon a u(1-\varepsilon u^2)}{\gamma+\varepsilon a^2}\partial_t a + \partial_t u + \dfrac{2\gamma}{\gamma-1}\dfrac{(1-\varepsilon u^2)a}{\gamma+\varepsilon a^2}\partial_r a + u\partial_r u = 0, \\ \dfrac{2\gamma}{\gamma-1}\dfrac{1-\varepsilon u^2}{\gamma+\varepsilon a^2}\partial_t a + \varepsilon a u \partial_t u + \dfrac{2\gamma}{\gamma-1}\dfrac{1-\varepsilon u^2}{\gamma+\varepsilon a^2}u\partial_r a + a\partial_r u \\ = -\frac{d-1}{r}(1-\varepsilon u^2)ua. \end{cases} \tag{2.2}$$

The eigenvalues λ_1 and λ_2 in (1.9) become

$$\lambda_1 = \frac{u-a}{1-\varepsilon a u}, \quad \lambda_2 = \frac{u+a}{1+\varepsilon a u}, \tag{2.3}$$

Introduce the Riemann variables $w_\pm = w_\pm(u, a)$

$$\begin{aligned} w_+ &= \frac{1}{2\sqrt{\varepsilon}} \ln\left(\frac{1+\sqrt{\varepsilon}u}{1-\sqrt{\varepsilon}u}\right) + \frac{2\sqrt{\gamma}}{\sqrt{\varepsilon}(\gamma-1)} \arctan\left(\sqrt{\frac{\varepsilon}{\gamma}}a\right). \\ w_- &= \frac{1}{2\sqrt{\varepsilon}} \ln\left(\frac{1+\sqrt{\varepsilon}u}{1-\sqrt{\varepsilon}u}\right) - \frac{2\sqrt{\gamma}}{\sqrt{\varepsilon}(\gamma-1)} \arctan\left(\sqrt{\frac{\varepsilon}{\gamma}}a\right). \end{aligned} \tag{2.4}$$

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from which one has

$$
\begin{aligned}
u(w_+,w_-) =&\frac{e^{\sqrt{\varepsilon}(w_++w_-)}-1}{\sqrt{\varepsilon}(e^{\sqrt{\varepsilon}(w_++w_-)}+1)},\\
a(w_+,w_-) =&\sqrt{\frac{\gamma}{\varepsilon}}\tan\Big(\frac{\sqrt{\varepsilon}(\gamma-1)}{4\sqrt{\gamma}}(w_+-w_-)\Big),
\end{aligned}
\tag{2.5}
$$

Then we can obtain the governing equations for (w_+,w_-)

$$
\frac{\partial w_+}{\partial t}+\lambda_2\frac{\partial w_+}{\partial r}=-\frac{d-1}{r}\frac{au}{1+\varepsilon au},\tag{2.6}
$$

$$
\frac{\partial w_-}{\partial t}+\lambda_1\frac{\partial w_-}{\partial r}=\frac{d-1}{r}\frac{au}{1-\varepsilon au}.\tag{2.7}
$$

We next supply the assumptions of the paper.

Assumption 1 *Let the adiabatic constant γ be in the interval $1<\gamma<3$. For any given $b>0$, the initial data $(\rho_0(r),u_0(r))=(\rho(r,0),u(r,0))$ of (2.1) make the following inequalities hold*

$$
0\le w_-(r,0)<w_+(r,0)\le C:=\frac{1}{2\sqrt{\varepsilon}}\ln\Big(\frac{7-\gamma}{1+\gamma}\Big),\tag{2.8}
$$

for any $r\in(b,\infty)$, where

$$
w_\pm(r,0)=\frac{1}{2\sqrt{\varepsilon}}\ln\Big(\frac{1+\sqrt{\varepsilon}u_0(r)}{1-\sqrt{\varepsilon}u_0(r)}\Big)\pm\frac{2\sqrt{\gamma}}{\sqrt{\varepsilon}(\gamma-1)}\arctan\Big(\sqrt{\frac{\varepsilon}{\gamma}}a_0(r)\Big),\tag{2.9}
$$

and $a_0(r)=\sqrt{A\gamma}\rho_0^{\frac{\gamma-1}{2}}$.

Remark 1 The inequalities in (2.8) form an invariant domain of (w_-,w_+) holding for any time before singularity formation.

We now set

$$
\begin{aligned}
\alpha_0(r) =&\frac{u_0'(r)}{1-\varepsilon u_0^2(r)}+\frac{2\gamma}{\gamma-1}\frac{a_0'(r)}{\gamma+\varepsilon a_0^2(r)}+\frac{d-1}{r}\frac{u_0(r)a_0(r)}{u_0(r)+a_0(r)},\\
\beta_0(r) =&\frac{u_0'(r)}{1-\varepsilon u_0^2(r)}-\frac{2\gamma}{\gamma-1}\frac{a_0'(r)}{\gamma+\varepsilon a_0^2(r)}-\frac{d-1}{r}\frac{u_0(r)a_0(r)}{u_0(r)-a_0(r)}.
\end{aligned}
\tag{2.10}
$$

Assumption 2 *Let the adiabatic constant γ be in the interval $1<\gamma<3$. For the initial data $(\rho_0(r),u_0(r))=(\rho(r,0),u(r,0))$ of (2.1), there exists a uniform constant $M>0$ such that*

$$
\min_{r\in[0,\infty)}\{\alpha_0(r),\beta_0(r)\}\ge 0,\quad \max_{r\in[0,\infty)}\{\alpha_0(r),\beta_0(r)\}<M,\tag{2.11}
$$

where $\alpha_0(0)=\lim_{r\to 0^+}\alpha_0(r)$ and $\beta_0(0)=\lim_{r\to 0^+}\beta_0(r)$ are assumed to exist and nonnegative.

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Remark 2 The inequalities in (2.10) form an invariant domain of (α, β) holding for any time before singularity formation.

Then the main results of the paper can be stated as follows.

Theorem 2.1 *Let the initial data $(\rho_0(r), u_0(r)) \in C^1([0, \infty))$ satisfy Assumptions 1 and 2. We further assume $\rho_0(0) = u_0(0) = 0$, $\rho_0(r) > 0$ for $r > 0$. Then the radially symmetric relativistic Euler equations (1.4) with (1.2) admit a global C^1 solution $(\rho, u)(r, t)$ on the entire domain $r \geq 0$, $t \geq 0$. Moreover, the solution satisfies $\rho(0, t) = 0$ and $u(0, t) = 0$ and*

$$\sqrt{\frac{2A\gamma}{\gamma - 1}}\rho^{\frac{\gamma-1}{2}}(r, t) \leq u(r, t) \leq \frac{3-\gamma}{4\sqrt{\varepsilon}}, \quad \rho(r, t) > 0, \quad \forall\ r \geq 0, t \geq 0, \tag{2.12}$$

and

$$\min_{r\geq 0, t\geq 0}(\alpha, \beta)(r, t) \geq 0, \qquad \max_{r\geq 0, t\geq 0}(\alpha, \beta)(r, t) < M. \tag{2.13}$$

Theorem 2.2 *Let the initial data $(\rho_0(r), u_0(r)) \in C^1((b, \infty))$ with $b > 0$ satisfy Assumption 1 and the following compression conditions*

$$\alpha_0(r) < 0, \quad \beta_0(r) < 0, \quad \forall\ r \in [b, \infty). \tag{2.14}$$

Then there exists a constant $N(b, T)$, depending on b and T, such that, if

$$\alpha_0(r^*) \leq -N(b, T), \text{ or } \beta_0(r^*) \leq -N(b, T), \tag{2.15}$$

for some number $r^* > b$, then singularity forms before time T.

Remark 3 The precise form of the constant $N(b, T)$ will be given along with the analysis of the problem in Section 6.

3 The Riccati type equations

In this section, we derive the Riccati type equations of (α, β), which play a very important role in our analysis below.

We first calculate the rarefaction characters α and β introduced in (1.7). According to the definition of ∂_1 in (1.8), we directly compute by (2.2)

$$\begin{aligned}
&\partial_1\Big(r^{d-1}\frac{\rho+\varepsilon p}{1-\varepsilon u^2}u\Big)\\
=&\partial_t\Big(r^{d-1}\frac{\rho+\varepsilon p}{1-\varepsilon u^2}u\Big)+\frac{u-a}{1-\varepsilon au}\partial_r\Big(r^{d-1}\frac{\rho+\varepsilon p}{1-\varepsilon u^2}u\Big)\\
=&-\partial_r\Big(r^{d-1}\frac{\rho+\varepsilon p}{1-\varepsilon u^2}u^2\Big)-r^{d-1}\partial_r p+\frac{u-a}{1-\varepsilon au}\partial_r\Big(r^{d-1}\frac{\rho+\varepsilon p}{1-\varepsilon u^2}u\Big)\\
=&-r^{d-1}\frac{\rho+\varepsilon p}{1-\varepsilon u^2}\Big\{u\partial_r u+\frac{a(1+\varepsilon u^2)}{1-\varepsilon au}\partial_r u+\frac{d-1}{r}\frac{au(1-\varepsilon u^2)}{1-\varepsilon au}\Big\}\\
&-r^{d-1}\frac{a(a+u)}{1-\varepsilon au}\partial_r\rho\\
=&-r^{d-1}(\rho+\varepsilon p)\frac{a+u}{1-\varepsilon au}\Big(\frac{\partial_r u}{1-\varepsilon u^2}+\frac{d-1}{r}\frac{ua}{u+a}\Big)\\
&-r^{d-1}\frac{2}{\gamma-1}\frac{\rho(a+u)}{1-\varepsilon au}\partial_r a.
\end{aligned}\tag{3.1}$$

The last step in (3.1) is used the relation $\rho\partial_r a=\frac{\gamma-1}{2}a\partial_r\rho$. It follows by (3.1) that

$$\begin{aligned}
\alpha=&-\frac{\partial_1(r^{d-1}\frac{\rho+\varepsilon p}{1-\varepsilon u^2}u)}{r^{d-1}(\rho+\varepsilon p)\frac{u+a}{1-\varepsilon ua}}\\
=&\frac{\partial_r u}{1-\varepsilon u^2}+\frac{2}{\gamma-1}\frac{\rho\partial_r a}{\rho+\varepsilon p}+\frac{d-1}{r}\frac{ua}{u+a}\\
=&\frac{\partial_r u}{1-\varepsilon u^2}+\frac{2\gamma}{\gamma-1}\frac{\partial_r a}{\gamma+\varepsilon a^2}+\frac{d-1}{r}\frac{ua}{u+a}\\
=&\partial_r w_+ +\frac{d-1}{r}\frac{ua}{u+a}.
\end{aligned}\tag{3.2}$$

Analogously, one has

$$\begin{aligned}
&\partial_2\Big(r^{d-1}\frac{\rho+\varepsilon p}{1-\varepsilon u^2}u\Big)\\
=&\partial_t\Big(r^{d-1}\frac{\rho+\varepsilon p}{1-\varepsilon u^2}u\Big)+\frac{u+a}{1+\varepsilon au}\partial_r\Big(r^{d-1}\frac{\rho+\varepsilon p}{1-\varepsilon u^2}u\Big)\\
=&-r^{d-1}(\rho+\varepsilon p)\frac{u-a}{1+\varepsilon au}\Big(\frac{\partial_r u}{1-\varepsilon u^2}-\frac{d-1}{r}\frac{ua}{u-a}\Big)\\
&+r^{d-1}\frac{2}{\gamma-1}\frac{\rho(u-a)}{1+\varepsilon au}\partial_r a,
\end{aligned}\tag{3.3}$$

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which gives

$$
\begin{aligned}
\beta &= -\frac{\partial_2\left(r^{d-1}\frac{\rho+\varepsilon p}{1-\varepsilon u^2}u\right)}{r^{d-1}(\rho+\varepsilon p)\frac{u-a}{1+\varepsilon ua}} \\
&= \frac{\partial_r u}{1-\varepsilon u^2} - \frac{2\gamma}{\gamma-1}\frac{\partial_r a}{\gamma+\varepsilon a^2} - \frac{d-1}{r}\frac{ua}{u-a} \\
&= \partial_r w_- - \frac{d-1}{r}\frac{ua}{u-a}.
\end{aligned} \tag{3.4}
$$

We next derive the Riccati equations for the radially symmetric isentropic relativistic Euler equations.

Lemma 3.1 *For smooth solutions of* (2.1) *with* $\lambda_1\lambda_2 \neq 0$, *we have the following Riccati type equations on* α *and* β

$$
\begin{aligned}
\partial_1\beta = &-\frac{1-\varepsilon u^2}{(1-\varepsilon au)^2}\frac{(\gamma+1)(\gamma-\varepsilon a^2)}{4\gamma}\beta^2 \\
&-\frac{1-\varepsilon u^2}{(1-\varepsilon au)^2}\frac{\gamma(3-\gamma)+\varepsilon a^2(1-3\gamma)}{4\gamma}\alpha\beta + A_1\alpha - B_1\beta,
\end{aligned} \tag{3.5}
$$

where

$$
\begin{aligned}
A_1 &= \frac{(d-1)(u+a)}{2r(1+\varepsilon au)(u-a)^2}\left(\frac{\gamma-1}{2}u^2 - a^2 + \frac{3\gamma-1}{2\gamma}\varepsilon u^2 a^2\right) \\
B_1 &= \frac{(d-1)(1-\varepsilon u^2)}{r(u-a)(1-\varepsilon au)^2}\left\{\frac{ua}{u+a}\left(a(1-\varepsilon a^2) + \frac{\gamma-1}{2\gamma}(\gamma+\varepsilon a^2)u\right)\right. \\
&\quad \left. + \frac{1-\varepsilon au}{2(1-\varepsilon u^2)}\left(a^2(1-\varepsilon u^2) + \frac{\gamma-1}{2\gamma}u^2(\gamma+\varepsilon a^2)\right)\right\},
\end{aligned} \tag{3.6}
$$

and

$$
\begin{aligned}
\partial_2\alpha = &-\frac{1-\varepsilon u^2}{(1+\varepsilon au)^2}\frac{(\gamma+1)(\gamma-\varepsilon a^2)}{4\gamma}\alpha^2 \\
&-\frac{1-\varepsilon u^2}{(1+\varepsilon au)^2}\frac{\gamma(3-\gamma)+\varepsilon a^2(1-3\gamma)}{4\gamma}\alpha\beta + A_2\beta - B_2\alpha,
\end{aligned} \tag{3.7}
$$

where

$$
\begin{aligned}
A_2 &= \frac{(d-1)(u-a)}{2r(1-\varepsilon au)(u+a)^2}\left(\frac{\gamma-1}{2}u^2 - a^2 + \frac{3\gamma-1}{2\gamma}\varepsilon u^2 a^2\right) \\
B_2 &= \frac{(d-1)(1-\varepsilon u^2)}{r(u+a)(1+\varepsilon au)^2}\left\{\frac{ua}{u-a}\left(a(1-\varepsilon a^2) - \frac{\gamma-1}{2\gamma}(\gamma+\varepsilon a^2)u\right)\right. \\
&\quad \left. + \frac{1+\varepsilon au}{2(1-\varepsilon u^2)}\left(a^2(1-\varepsilon u^2) + \frac{\gamma-1}{2\gamma}u^2(\gamma+\varepsilon a^2)\right)\right\}.
\end{aligned} \tag{3.8}
$$

Remark 4 It is obvious that equations (3.5) and (3.7) are homogeneous, i.e. their right hand sides vanish when $\alpha = \beta = 0$. This important property

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originated from the special selection of α and β is crucial for us to establish the a priori C^1 estimates of solutions by constructing the invariant domains.

Proof We only derive (3.5) since the derivation of (3.7) is parallel. In view of (3.4), we first have

$$
\begin{aligned}
\partial_1\beta &= \partial_1\Big(\partial_r w_- - \frac{d-1}{r}\frac{ua}{u-a}\Big)\\
&= \partial_1\partial_r w_- - \partial_t\Big(\frac{d-1}{r}\frac{ua}{u-a}\Big) - \frac{u-a}{1-\varepsilon au}\partial_r\Big(\frac{d-1}{r}\frac{ua}{u-a}\Big).
\end{aligned}
\tag{3.9}
$$

Moreover, thanks to (2.7) and (3.4), one obtains

$$
\begin{aligned}
\partial_1\partial_r w_- =&\frac{u-a}{1-\varepsilon au}\partial_r\Big(\frac{d-1}{r}\frac{ua}{u-a}\Big)\\
&+\frac{d-1}{r}\frac{ua}{u-a}\partial_r\Big(\frac{u-a}{1-\varepsilon au}\Big) - \partial_r\Big(\frac{u-a}{1-\varepsilon au}\Big)\partial_r w_-\\
=&\frac{u-a}{1-\varepsilon au}\partial_r\Big(\frac{d-1}{r}\frac{ua}{u-a}\Big) - \beta\partial_r\Big(\frac{u-a}{1-\varepsilon au}\Big),
\end{aligned}
\tag{3.10}
$$

which together with (3.9) yields

$$
\partial_1\beta = -\beta\partial_r\Big(\frac{u-a}{1-\varepsilon au}\Big) - \partial_t\Big(\frac{d-1}{r}\frac{ua}{u-a}\Big) = \mathcal{I}_1 + \mathcal{I}_2, \tag{3.11}
$$

where

$$
\begin{aligned}
\mathcal{I}_1 =& -\frac{\beta}{(1-\varepsilon au)^2}\Big\{(1-\varepsilon a^2)\partial_r u - (1-\varepsilon u^2)\partial_r a\Big\},\\
\mathcal{I}_2 =& \frac{d-1}{r(u-a)^2}(a^2\partial_t u - u^2\partial_t a).
\end{aligned}
\tag{3.12}
$$

To proceed, we next give some relationships among α, β, $\partial_t u$, $\partial_t a$, $\partial_r u$ and $\partial_r a$. Owing to (2.4), one has by directly calculates

$$
\partial_r w_+ + \partial_r w_- = \frac{2\partial_r u}{1-\varepsilon u^2}, \qquad \partial_r w_+ - \partial_r w_- = \frac{4\gamma}{\gamma-1}\frac{\partial_r a}{\gamma+\varepsilon a^2},
$$

which together with (3.2), (3.4) and (2.2), we achieve the following formulas

$$
\begin{aligned}
\frac{2\partial_r u}{1-\varepsilon u^2} &= \alpha + \beta + \frac{d-1}{r}\frac{2ua^2}{(u-a)(u+a)},\\
\frac{4\gamma}{\gamma-1}\frac{\partial_r a}{\gamma+\varepsilon a^2} &= \alpha - \beta - \frac{d-1}{r}\frac{2u^2 a}{(u-a)(u+a)},
\end{aligned}
\tag{3.13}
$$

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and

$$\begin{aligned}
&\frac{2\gamma}{\gamma-1}\frac{(1-\varepsilon au)(1+\varepsilon au)}{\gamma+\varepsilon a^2}\partial_t a\\
&=-a\partial_r u-\frac{2\gamma}{\gamma-1}\frac{u(1-\varepsilon a^2)}{\gamma+\varepsilon a^2}\partial_r a-\frac{d-1}{r}ua,\\
&\frac{(1-\varepsilon au)(1+\varepsilon au)}{1-\varepsilon u^2}\partial_t u\\
&=-\frac{u(1-\varepsilon a^2)}{1-\varepsilon u^2}\partial_r u-\frac{2\gamma}{\gamma-1}\frac{a(1-\varepsilon u^2)}{\gamma+\varepsilon a^2}\partial_r a+\frac{d-1}{r}\varepsilon a^2u^2.
\end{aligned}\tag{3.14}$$

Then putting (3.13) and (3.14) into (3.11) and doing simple rearrangements, we acquire

$$\begin{aligned}
\mathcal{I}_1=&-\frac{\beta}{(1-\varepsilon au)^2}\Big\{(1-\varepsilon a^2)\partial_r u-(1-\varepsilon u^2)\partial_r a\Big\}\\
=&-\frac{\beta}{(1-\varepsilon au)^2}\Big\{\frac{(1-\varepsilon a^2)(1-\varepsilon u^2)}{2}\Big(\alpha+\beta+\frac{d-1}{r}\frac{2ua^2}{(u-a)(u+a)}\Big)\\
&-\frac{\gamma-1}{4\gamma}(1-\varepsilon u^2)(\gamma+\varepsilon a^2)\Big(\alpha-\beta-\frac{d-1}{r}\frac{2u^2a}{(u-a)(u+a)}\Big)\Big\}\\
=&-\frac{d-1}{r}\frac{ua(1-\varepsilon u^2)}{(u-a)(u+a)(1-\varepsilon au)^2}\Big\{a(1-\varepsilon a^2)+\frac{\gamma-1}{2\gamma}(\gamma+\varepsilon a^2)u\Big\}\beta\\
&-\frac{1-\varepsilon u^2}{(1-\varepsilon au)^2}\frac{(\gamma+1)(\gamma-\varepsilon a^2)}{4\gamma}\beta^2\\
&-\frac{1-\varepsilon u^2}{(1-\varepsilon au)^2}\frac{\gamma(3-\gamma)+\varepsilon a^2(1-3\gamma)}{4\gamma}\alpha\beta.
\end{aligned}\tag{3.15}$$

Furthermore, one applies (2.2), (3.13) and (3.14) again to achieve

$$\begin{aligned}
&a^2\partial_t u-u^2\partial_t a\\
=&-\frac{1}{2(1-\varepsilon^2a^2u^2)}\Big(a^3(1-\varepsilon u^2)^2-\frac{\gamma-1}{2\gamma}(\gamma+\varepsilon a^2)(1-\varepsilon a^2)u^3\Big)(\alpha-\beta)\\
&-\frac{ua(1-\varepsilon u^2)}{2(1-\varepsilon^2a^2u^2)}\Big(a(1-\varepsilon a^2)-\frac{\gamma-1}{2\gamma}(\gamma+\varepsilon a^2)u\Big)(\alpha+\beta)\\
&+\frac{d-1}{r(1-\varepsilon^2a^2u^2)}\Big\{\frac{u^4a-u^2a^3}{u^2-a^2}\Big(\varepsilon a^3(1-\varepsilon u^2)+\frac{\gamma-1}{2\gamma}(\gamma+\varepsilon a^2)u\Big)\\
&+\frac{u^2a}{u^2-a^2}\Big(a^3(1-\varepsilon u^2)^2-\frac{\gamma-1}{2\gamma}(\gamma+\varepsilon a^2)(1-\varepsilon a^2)u^3\Big)\\
&-\frac{u^2a^3(1-\varepsilon u^2)}{u^2-a^2}\Big(a(1-\varepsilon a^2)-\frac{\gamma-1}{2\gamma}(\gamma+\varepsilon a^2)u\Big)\Big\}.
\end{aligned}\tag{3.16}$$

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Note that

$$\begin{aligned}
&\frac{u^4a-u^2a^3}{(u-a)(u+a)}\Big(\varepsilon a^3(1-\varepsilon u^2)+\frac{\gamma-1}{2\gamma}(\gamma+\varepsilon a^2)u\Big)\\
&+\frac{u^2a}{(u-a)(u+a)}\Big(a^3(1-\varepsilon u^2)^2-\frac{\gamma-1}{2\gamma}(\gamma+\varepsilon a^2)(1-\varepsilon a^2)u^3\Big)\\
&-\frac{u^2a^3(1-\varepsilon u^2)}{(u-a)(u+a)}\Big(a(1-\varepsilon a^2)-\frac{\gamma-1}{2\gamma}(\gamma+\varepsilon a^2)u\Big)\\
=&\frac{u^2a(1-\varepsilon u^2)}{(u-a)(u+a)}\Big\{\varepsilon a^3(u^2-a^2)+a^3(1-\varepsilon u^2)-a^3(1-\varepsilon a^2)\Big\}\\
&+\frac{\gamma-1}{2\gamma}\frac{u^2a(\gamma+\varepsilon a^2)}{(u-a)(u+a)}\Big\{u(u^2-a^2)-(1-\varepsilon a^2)u^3+a^2u(1-\varepsilon u^2)\Big\}=0.
\end{aligned}$$

Utilizing the above fact into (3.16) leads to

$$\begin{aligned}
a^2\partial_t u-u^2\partial_t a=&-\frac{u+a}{2(1+\varepsilon ua)}\Big(a^2(1-\varepsilon u^2)-\frac{\gamma-1}{2\gamma}(\gamma+\varepsilon a^2)u^2\Big)\alpha\\
&-\frac{u-a}{2(1-\varepsilon au)}\Big(\frac{\gamma-1}{2\gamma}(\gamma+\varepsilon a^2)u^2+a^2(1-\varepsilon u^2)\Big)\beta. \qquad (3.17)
\end{aligned}$$

We insert (3.17) into (3.12) to obtain

$$\begin{aligned}
\mathcal{I}_2=&\frac{d-1}{r(u-a)^2}(a^2\partial_t u-u^2\partial_t a)\\
=&-\frac{d-1}{r(u-a)^2}\Big[\frac{u+a}{2(1+\varepsilon ua)}\Big(a^2(1-\varepsilon u^2)-\frac{\gamma-1}{2\gamma}(\gamma+\varepsilon a^2)u^2\Big)\alpha \qquad (3.18)\\
&+\frac{u-a}{2(1-\varepsilon au)}\Big(\frac{\gamma-1}{2\gamma}(\gamma+\varepsilon a^2)u^2+a^2(1-\varepsilon u^2)\Big)\beta\Big].
\end{aligned}$$

One combines (3.11), (3.15) and (3.18) to easily gain (3.5). The proof of the lemma is complete. ■

4 Invariant domains

In this section, we establish invariant domains for (w_-, w_+) and (α, β) and derive a positive lower bound estimate for the density.

Before studying the solution in the whole domain $(r,t)\in\mathbb{R}^+\times\mathbb{R}^+$, we first consider the Cauchy problem on the domain Ω_b in the (r,t)-plane, where Ω_b is defined as in Figure 1.

Problem 1 *For system (2.1) with the initial data $(\rho_0(r), u_0(r)) = (\rho(r,0), u(r,0))$, we seek the smooth solution on domain of dependence Ω_b with base $t=0$ and $r\in(b,\infty)$ for any $b>0$, i.e. domain to the right of the 2-characteristic starting at $(b,0)$.*

Then we have

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Lemma 4.1 *Let Assumption 1 hold. Then any smooth solution of Problem 1 in the domain of dependence Ω_b based on (b, ∞), with $b > 0$, satisfies*

$$0 \leq w_-(r,t) \leq w_+(r,t) \leq C. \tag{4.1}$$

Remark 5 We shall show the fact that $w_- < w_+$(i.e. $\rho > 0$) in Ω_b is true at the end of this section.

Proof We first prove $w_- \geq 0$ in Ω_b. Assume that there exists a point $(\bar{r}, \bar{t}) \in \Omega$ with $\bar{t} > 0$ such that $w_-(\bar{r}, \bar{t}) < 0$. We use l_1 to denote the 1-characteristic $r = r_1(t)$ through the point $(\bar{r}, \bar{t})$:

$$\frac{\mathrm{d}r_1}{\mathrm{d}t} = \lambda_1(r_1(t), t), \quad r_1(\bar{t}) = \bar{r}.$$

Due to $w_-(r_1(0), 0) \geq 0$ by (2.8), we know that there exist some times $0 \leq \tilde{t} < \hat{t} \leq \bar{t}$ such that along l_1, there hold $w_-(r_1(t), t) \geq 0$ for $t \in [0, \tilde{t})$, $w_-(r_1(\tilde{t}), \tilde{t}) = 0$, and $w_-(r_1(t), t) < 0$ for $t \in (\tilde{t}, \hat{t}]$. Thus $\partial_1 w_-(r_1(\tilde{t}), \tilde{t}) < 0$. On the other hand, one recalls (2.7) to see that

$$\partial_1 w_-(r_1(\tilde{t}), \tilde{t}) = \frac{d-1}{r} \frac{au}{1 - \varepsilon au}(r_1(\tilde{t}), \tilde{t}) \geq 0,$$

which leads to a contradiction. Hence we have $w_-(r,t) \geq w_-(r,0) \geq 0$ in Ω_b.

According to the conclusion $w_-(r,t) \geq 0$ in Ω_b, it suggests by the expression of w_- in (2.4) that

$$\frac{1}{2\sqrt{\varepsilon}} \ln\left(\frac{1 + \sqrt{\varepsilon}u}{1 - \sqrt{\varepsilon}u}\right) \geq \frac{2\sqrt{\gamma}}{\sqrt{\varepsilon}(\gamma - 1)} \arctan\left(\sqrt{\frac{\varepsilon}{\gamma}} a\right) \geq 0,$$

which implies that $u(r,t) \geq 0$ in Ω_b. Thus one has by (2.6)

$$\frac{\partial w_+}{\partial t} + \lambda_2 \frac{\partial w_+}{\partial r} = -\frac{d-1}{r} \frac{au}{1 + \varepsilon au} \leq 0,$$

which means that $w_+(r_2(t), t) \leq w_+(r_2(0), 0)$ along any 2-characteristic curve $r_2(t)$. Therefore, we attain

$$w_+(r,t) \leq w_+(r,0) \leq C. \tag{4.2}$$

The proof of the lemma is finished. ∎

Based on Lemma 4.1, we have following estimates

Lemma 4.2 *Let Assumption 1 hold. Suppose that the smooth solution of Problem 1 in Ω_b satisfies $0 < w_- < w_+$. Then there hold in the domain Ω_b*

$$\sqrt{\frac{2}{\gamma - 1}} a \leq u \leq \frac{3 - \gamma}{4\sqrt{\varepsilon}}, \tag{4.3}$$

and

$$0 \leq A_i \leq B_i, \tag{4.4}$$

where $i=1,2$, A_i and B_i are defined in (3.6) and (3.8).

Proof We first combine (2.5), (4.1) and (4.2) to obtain

$$\sqrt{\varepsilon}u \leq \frac{e^{2\sqrt{\varepsilon}w_+(t,r)}-1}{e^{2\sqrt{\varepsilon}w_+(t,r)}+1} \leq \frac{e^{2\sqrt{\varepsilon}w_+(0,r)}-1}{e^{2\sqrt{\varepsilon}w_+(0,r)}+1}, \tag{4.5}$$

which together with (2.8) arrives at

$$\sqrt{\varepsilon}u \leq \frac{e^{2\sqrt{\varepsilon}w_+(0,r)}-1}{e^{2\sqrt{\varepsilon}w_+(0,r)}+1} \leq \frac{3-\gamma}{4}. \tag{4.6}$$

It is easy to find by the assumption $0 < w_- < w_+$ that

$$\begin{aligned}\frac{u}{1-\sqrt{\varepsilon}u} \geq & \frac{1}{2\sqrt{\varepsilon}} \ln\Big(\frac{1+\sqrt{\varepsilon}u}{1-\sqrt{\varepsilon}u}\Big) \\ \geq & \frac{2\sqrt{\gamma}}{\sqrt{\varepsilon}(\gamma-1)} \arctan\Big(\sqrt{\frac{\varepsilon}{\gamma}}a\Big) \geq \frac{2}{\gamma-1}a\Big(1-\frac{\varepsilon a^2}{3\gamma}\Big).\end{aligned} \tag{4.7}$$

In view of (4.7), we calculate

$$\begin{aligned}\frac{\gamma-1}{2}u^2-a^2 \geq & \frac{2}{\gamma-1}a^2\Big(1-\frac{\varepsilon a^2}{3\gamma}\Big)(1-\sqrt{\varepsilon}u)^2-a^2 \\ \geq & \frac{2}{\gamma-1}a^2\Big[\Big(1-\frac{\varepsilon u^2}{3\gamma}\Big)(1-\sqrt{\varepsilon}u)^2-\frac{\gamma-1}{2}\Big] \\ = & \frac{2}{\gamma-1}a^2\mathcal{I}_3,\end{aligned} \tag{4.8}$$

where

$$\begin{aligned}\mathcal{I}_3 = & 1-2\sqrt{\varepsilon}u-\frac{\gamma-1}{2}+\frac{3\gamma-2}{3\gamma}\varepsilon u^2 \\ & +\frac{2(\sqrt{\varepsilon}u)^3}{3\gamma}(2-\sqrt{\varepsilon}u)+\Big(\frac{\varepsilon u^2}{3\gamma}(1-\sqrt{\varepsilon}u)\Big)^2 \\ = & 2\Big(\frac{3-\gamma}{4}-\sqrt{\varepsilon}u\Big)+\frac{3\gamma-2}{3\gamma}\varepsilon u^2 \\ & +\frac{2(\sqrt{\varepsilon}u)^3}{3\gamma}(2-\sqrt{\varepsilon}u)+\Big(\frac{\varepsilon u^2}{3\gamma}(1-\sqrt{\varepsilon}u)\Big)^2 \geq 0,\end{aligned} \tag{4.9}$$

by (4.6). Combining (4.8) and (4.9) yield for $1 < \gamma < 3$

$$a \leq \sqrt{\frac{2}{\gamma-1}}a \leq u, \quad \text{and then } A_i \geq 0. \tag{4.10}$$

Then (4.3) follows from (4.6) and (4.10).

To show $B_i \geq A_i$, we recall the expressions of A_i, B_i in (3.6), (3.8) to obtain

$$\begin{aligned}
&B_1 - A_1 \\
=&\frac{(d-1)(1-\varepsilon u^2)ua}{r(u-a)(u+a)(1-\varepsilon au)^2}\Big(a(1-\varepsilon a^2)+\frac{\gamma-1}{2\gamma}(\gamma+\varepsilon a^2)u\Big) \\
&+\frac{d-1}{2r(u-a)(1-\varepsilon au)}\Big(a^2(1-\varepsilon u^2)+\frac{\gamma-1}{2\gamma}u^2(\gamma+\varepsilon a^2)\Big) \\
&-\frac{(d-1)(u+a)}{2r(1+\varepsilon au)(u-a)^2}\Big(\frac{\gamma-1}{2}u^2-a^2+\frac{3\gamma-1}{2\gamma}\varepsilon u^2a^2\Big),
\end{aligned} \tag{4.11}$$

which gives through basic simplification

$$\begin{aligned}
B_1 - A_1 =&\frac{(d-1)(1-\varepsilon u^2)(1-\varepsilon a^2)ua^2}{r(u-a)(1-\varepsilon au)} \\
&\times\Big(\frac{1}{(u-a)(1+\varepsilon au)}+\frac{1}{(u+a)(1-\varepsilon au)}\Big) \\
&+\frac{\gamma-1}{2}\frac{(d-1)(1-\varepsilon u^2)(\gamma+\varepsilon a^2)u^2a}{r(u-a)(1-\varepsilon au)} \\
&\times\Big(\frac{1}{(u+a)(1-\varepsilon au)}-\frac{1}{(u-a)(1+\varepsilon au)}\Big) \\
=&\frac{2(d-1)(1-\varepsilon u^2)u^2a^2}{r(u-a)^2(u+a)(1+\varepsilon au)(1-\varepsilon au)^2} \\
&\times\Big((1-\varepsilon a^2)^2-\frac{\gamma-1}{2\gamma}(1-\varepsilon u^2)(\gamma+\varepsilon a^2)\Big),
\end{aligned} \tag{4.12}$$

Similarly, one has

$$\begin{aligned}
B_2 - A_2 =&\frac{2(d-1)(1-\varepsilon u^2)u^2a^2}{r(u+a)^2(u-a)(1-\varepsilon au)(1+\varepsilon au)^2} \\
&\times\Big((1-\varepsilon a^2)^2-\frac{\gamma-1}{2\gamma}(1-\varepsilon u^2)(\gamma+\varepsilon a^2)\Big).
\end{aligned} \tag{4.13}$$

Furthermore, we find by (4.3) that

$$\varepsilon a^2 \leq \frac{(\gamma-1)(3-\gamma)^2}{32},$$

from which one gains

$$\begin{aligned}
&(1-\varepsilon a^2)^2-\frac{\gamma-1}{2\gamma}(1-\varepsilon u^2)(\gamma+\varepsilon a^2) \\
\geq&(1-\varepsilon a^2)\Big((1-\varepsilon a^2)-\frac{\gamma-1}{2}(\gamma+\varepsilon a^2)\Big) \\
=&\frac{(1-\varepsilon a^2)[\gamma(3-\gamma)-\varepsilon a^2(3\gamma-1)]}{2\gamma} \\
\geq&(1-\varepsilon a^2)\frac{3-\gamma}{2}\Big(1-\frac{(\gamma-1)(3-\gamma)(3\gamma-1)}{32\gamma}\Big)>0.
\end{aligned} \tag{4.14}$$

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Substituting (4.14) into (4.12) and (4.13) yield $B_i \geq A_i$. The proof of the lemma is ended. ■

Remark 6 The second to last inequality in (4.14) implies that the coefficients of term $\alpha\beta$ in (3.5) and (3.7) are negative.

We now employ (4.4) to verify that the set $\{\min\{\alpha,\beta\} \geq 0\}$ is an invariant region for Problem 1.

Lemma 4.3 *Consider a smooth solution on $t \in [0, T_0]$ for Problem 1 on Ω_b, satisfying Assumption 1 on (b, ∞) with $b > 0$. We further assume that the functions $\alpha_0(r)$ and $\beta_0(r)$ defined in (2.10) fulfill*

$$\min_{r\in[b,\infty)} \{\alpha_0(r), \beta_0(r)\} \geq 0. \tag{4.15}$$

Then the smooth solution satisfies

$$\min_{\Omega_b \cap \{t \leq T_0\}} \{\alpha(r,t), \beta(r,t)\} \geq 0, \tag{4.16}$$

i.e. $\{(\alpha,\beta)|\min(\alpha,\beta) \geq 0\}$ is an invariant domain on time.

Remark 7 The proof of the invariant domain $\{(\alpha,\beta)|\min(\alpha,\beta) \geq 0\}$ does not require upper bounds on $\alpha_0(r)$ and $\beta_0(r)$.

Proof We first utilize Lemmas 4.1 and 4.2 to estimate the upper bounds of B_i. Indeed, the following facts hold

$$\begin{aligned} \frac{a}{u-a} =& \frac{a}{u - \sqrt{\frac{2}{\gamma-1}}a + \sqrt{\frac{2}{\gamma-1}}a - a} \\ \leq& \frac{a}{\left(\sqrt{\frac{2}{\gamma-1}} - 1\right)a} = \frac{\sqrt{\gamma-1}(\sqrt{2}+\sqrt{\gamma+1})}{3-\gamma}, \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} \frac{u}{u-a} =& 1 + \frac{a}{u-a} \\ \leq& 1 + \frac{\sqrt{\gamma-1}(\sqrt{2}+\sqrt{\gamma+1})}{3-\gamma} = \frac{2+\sqrt{2(\gamma-1)}}{3-\gamma}, \end{aligned} \tag{4.18}$$

which have positive upper bounds for $1 < \gamma < 3$. Moreover, due to (4.3), one gains

$$1 - \varepsilon a^2 \geq 1 - \varepsilon a u \geq 1 - \varepsilon u^2 \geq \frac{16 - (3-\gamma)^2}{16} > 0, \tag{4.19}$$

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for $1 < \gamma < 3$. Thus it concludes by the expressions of B_1, B_2 in (3.6), (3.8) that

$$\begin{aligned}
B_1, B_2 \leq & \frac{d-1}{b} \cdot \frac{1}{(1-\varepsilon a u)^2} \Big\{ \frac{ua}{(u-a)(u+a)} \Big(a + \frac{\gamma^2-1}{2\gamma} u \Big) \\
& + \frac{1}{2(1-\varepsilon u^2)(u-a)} \Big(a^2 + \frac{\gamma^2-1}{2\gamma} u^2 \Big) \Big\} \\
\leq & \frac{(d-1)}{b(1-\varepsilon a u)^2 (1-\varepsilon u^2)} \Big(a + \frac{\gamma^2-1}{2\gamma} u \Big) \\
& \times \Big\{ \frac{\sqrt{\gamma-1}(\sqrt{2}+\sqrt{\gamma+1})}{3-\gamma} + \frac{2+\sqrt{2(\gamma-1)}}{3-\gamma} \Big\} \\
\leq & \frac{16^3 (d-1)}{b[16-(3-\gamma)^2]^3} \frac{2+2\sqrt{2(\gamma-1)}+\sqrt{\gamma^2-1}}{3-\gamma} (1 + \frac{\gamma^2-1}{2\gamma}) \frac{3-\gamma}{4\sqrt{\varepsilon}} \\
\leq & \frac{d-1}{b\sqrt{\varepsilon}} f(\gamma),
\end{aligned} \tag{4.20}$$

where

$$f(\gamma) = \frac{16^3 [(2+2\sqrt{2(\gamma-1)}+\sqrt{\gamma^2-1})(\gamma^2+2\gamma-1)]}{4\gamma[16-(3-\gamma)^2]^3}.$$

Therefore, we have for all $(r,t) \in \Omega_b$

$$0 \leq A_i(r,t) \leq B_i(r,t) \leq \frac{(d-1) f(\gamma)}{b\sqrt{\varepsilon}} =: K_b, \quad i = 1, 2. \tag{4.21}$$

Set $\widehat{K}_b = 2K_b + 2$ and $\eta > 0$ is an arbitrary small number such that

$$\frac{16}{16-(3-\gamma)^2} \cdot \eta e^{\widehat{K}_b t} \leq 1.$$

We introduce two new variables for $t \leq T_0$

$$X = \alpha + \eta e^{\widehat{K}_b t}, \quad Y = \beta + \eta e^{\widehat{K}_b t}. \tag{4.22}$$

According to Lemma 3.1, we can obtain the governing system of (X, Y) as follows

$$\begin{aligned}
\partial_2 X = & \Big\{ - \frac{1-\varepsilon u^2}{(1+\varepsilon a u)^2} \frac{(\gamma+1)(\gamma-\varepsilon a^2)}{4\gamma} (X - 2\eta e^{\widehat{K}_b t}) \\
& - \frac{1-\varepsilon u^2}{(1+\varepsilon a u)^2} \frac{\gamma(3-\gamma)+\varepsilon a^2(1-3\gamma)}{4\gamma} (Y - \eta e^{\widehat{K}_b t}) - B_2 \Big\} X \\
& + \Big\{ \frac{1-\varepsilon u^2}{(1+\varepsilon a u)^2} \frac{\gamma(3-\gamma)+\varepsilon a^2(1-3\gamma)}{4\gamma} \eta e^{\widehat{K}_b t} + A_2 \Big\} Y \\
& + \eta e^{\widehat{K}_b t} \Big\{ \widehat{K}_b - A_2 + B_2 - \frac{(1-\varepsilon u^2)(1-\varepsilon a^2)}{(1+\varepsilon a u)^2} \eta e^{\widehat{K}_b t} \Big\},
\end{aligned} \tag{4.23}$$

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and

$$
\begin{aligned}
\partial_1 Y =& \Big\{ -\frac{1-\varepsilon u^2}{(1-\varepsilon au)^2}\frac{(\gamma+1)(\gamma-\varepsilon a^2)}{4\gamma}(Y-2\eta e^{\widehat{K}_b t}) \\
& -\frac{1-\varepsilon u^2}{(1-\varepsilon au)^2}\frac{\gamma(3-\gamma)+\varepsilon a^2(1-3\gamma)}{4\gamma}(X-\eta e^{\widehat{K}_b t})-B_1\Big\}Y \\
& +\Big\{\frac{1-\varepsilon u^2}{(1-\varepsilon au)^2}\frac{\gamma(3-\gamma)+\varepsilon a^2(1-3\gamma)}{4\gamma}\eta e^{\widehat{K}_b t}+A_1\Big\}X \\
& +\eta e^{\widehat{K}_b t}\Big\{\widehat{K}_b - A_1 + B_1 - \frac{(1-\varepsilon u^2)(1-\varepsilon a^2)}{(1-\varepsilon au)^2}\eta e^{\widehat{K}_b t}\Big\}.
\end{aligned} \tag{4.24}
$$

From (4.19), one has

$$
\begin{aligned}
\frac{(1-\varepsilon u^2)(1-\varepsilon a^2)}{(1+\varepsilon au)^2} \le& \frac{(1-\varepsilon u^2)(1-\varepsilon a^2)}{(1-\varepsilon au)^2} \\
\le& \frac{1-\varepsilon a^2}{1-\varepsilon u^2} \le \frac{16(1-\varepsilon a^2)}{16-(3-\gamma)^2} \le \frac{16}{16-(3-\gamma)^2}.
\end{aligned}
$$

Thanks to the choice of $\widehat{K}_b$ and η, it suggests that

$$
\begin{aligned}
&\widehat{K}_b - A_i - B_i - \frac{(1-\varepsilon u^2)(1-\varepsilon a^2)}{(1-\varepsilon au)^2}\eta e^{\widehat{K}_b t} \\
\ge& \widehat{K}_b - 2K_b - \frac{16\eta e^{\widehat{K}_b t}}{16-(3-\gamma)^2} \ge 1 > 0,
\end{aligned} \tag{4.25}
$$

which mean that the last terms in (4.23) and (4.24) are strictly positive.

Next we apply the contradiction argument to confirm $\{(X,Y)|\min(X,Y)>0\}$ is an invariant domain for $t \le T_0$. It is easy to see by the initial conditions that $X(r,0)>0, Y(r,0)>0$ for any $r\in[b,\infty)$. Suppose that there exists some point (r^*,t^*) with $0<t^*\le T_0$ in Ω_b such that $X(r^*,t^*)=0$ or $Y(r^*,t^*)=0$ holds. Owing to the boundedness of the wave speed on $[0,t^*]$, we can draw the 1- and 2-characteristics starting from (r^*,t^*) downwards up to the line $t=0$ and obtain a characteristic triangle $\triangle^*\subset\Omega_b$. on the basis of the above analysis, we can find the first time $T\le t^*$ such that $X(T)=0$ or $Y(T)=0$ in $\triangle^*$. The proof is now divided into two cases.

Case I: At time T, $Y=0$ and $X\ge 0$. In this case, we apply (4.24) and (4.25) to acquire

$$
\begin{aligned}
\frac{\partial_1 Y}{Y} >& -\frac{1-\varepsilon u^2}{(1-\varepsilon au)^2}\frac{(\gamma+1)(\gamma-\varepsilon a^2)}{4\gamma}(Y-2\eta e^{\widehat{K}_b t}) \\
& -\frac{1-\varepsilon u^2}{(1-\varepsilon au)^2}\frac{\gamma(3-\gamma)+\varepsilon a^2(1-3\gamma)}{4\gamma}(X-\eta e^{\widehat{K}_b t})-B_1,
\end{aligned} \tag{4.26}
$$

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in the interval $[0,T)$. Integrating (4.26) along the 1-characteristic $r=r_1(t)$ from 0 to $s<T$ yields

$$
\begin{aligned}
Y(s)=&Y(r_1(s),s)>Y(r_1(0),0)\\
&\times\exp\Big\{\int_0^s\Big(-\frac{1-\varepsilon u^2}{(1-\varepsilon au)^2}\frac{(\gamma+1)(\gamma-\varepsilon a^2)}{4\gamma}(Y-2\eta e^{\widehat{K}_b t})\\
&-\frac{1-\varepsilon u^2}{(1-\varepsilon au)^2}\frac{\gamma(3-\gamma)+\varepsilon a^2(1-3\gamma)}{4\gamma}(X-\eta e^{\widehat{K}_b t})-B_1\Big)(r_1(t),t)\ \mathrm{d}t\Big\},
\end{aligned}
$$

which contradicts the fact that s is finite.

Case II: At time T, $X=0$, and $Y\geq 0$. In this case, we employ (4.23) and (4.25) to obtain

$$
\begin{aligned}
\frac{\partial_2 X}{X}>&-\frac{1-\varepsilon u^2}{(1+\varepsilon au)^2}\frac{(\gamma+1)(\gamma-\varepsilon a^2)}{4\gamma}(X-2\eta e^{\widehat{K}_b t})\\
&-\frac{1-\varepsilon u^2}{(1+\varepsilon au)^2}\frac{\gamma(3-\gamma)+\varepsilon a^2(1-3\gamma)}{4\gamma}(Y-\eta e^{\widehat{K}_b t})-B_2,
\end{aligned}
\tag{4.27}
$$

subsequently

$$
\begin{aligned}
X(s)=&X(r_2(s),s)>X(r_2(s_0),s_0)\\
&\times\exp\Big\{\int_{s_0}^s\Big(-\frac{1-\varepsilon u^2}{(1+\varepsilon au)^2}\frac{(\gamma+1)(\gamma-\varepsilon a^2)}{4\gamma}(X-2\eta e^{\widehat{K}_b t})\\
&-\frac{1-\varepsilon u^2}{(1+\varepsilon au)^2}\frac{\gamma(3-\gamma)+\varepsilon a^2(1-3\gamma)}{4\gamma}(Y-\eta e^{\widehat{K}_b t})-B_2\Big)(r_2(t),t)\ \mathrm{d}t\Big\},
\end{aligned}
$$

which also means by the initial conditions that s cannot be finite. This leads a contradiction.

Therefore, we have

$$
X=\alpha(r,t)+\eta e^{\widehat{K}_b t}>0,\quad Y=\beta(r,t)+\eta e^{\widehat{K}_b t}>0,
$$

for $t\leq T_0$ and any η satisfying $\eta e^{\widehat{K}_b T_0}<1$. It follows by the arbitrariness of $\eta>0$ that

$$
\alpha(r,t)\geq 0,\quad \beta(r,t)\geq 0,\quad \forall\ (r,t)\in\Omega_b\cap\{t\leq T_0\},
$$

which is the desired result (4.16). The proof of the lemma is complete. ■

We next establish another invariant domain on the upper bounds of α and β.

Lemma 4.4 *Consider a smooth solution on $t\in[0,T_0]$ for Problem 1 on Ω_b, satisfying Assumption 1 on (b,∞) with $b>0$. We further assume that the functions $\alpha_0(r)$ and $\beta_0(r)$ defined in (2.10) fulfill*

$$
\min_{r\in[b,\infty)}\{\alpha_0(r),\beta_0(r)\}\geq 0,\qquad \min_{r\in[b,\infty)}(\alpha_0(r),\beta_0(r))<M,
\tag{4.28}
$$

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for some positive constant M. Then the smooth solution satisfies

$$\max_{\Omega_b\cap\{t\le T_0\}}\{\alpha(r,t),\beta(r,t)\}<M. \tag{4.29}$$

Remark 8 Lemmas 4.3 and 4.4 show that $\{(\alpha,\beta)|\min(\alpha,\beta)\ge 0,\max(\alpha,\beta)<M\}$ is an invariant domain on time.

Proof We first rewrite the equations of α and β in Lemma 3.1 as follows

$$\begin{aligned}\partial_1\beta=&-\frac{1-\varepsilon u^2}{(1-\varepsilon au)^2}\frac{(\gamma+1)(\gamma-\varepsilon a^2)}{4\gamma}\beta^2\\&-\frac{1-\varepsilon u^2}{(1-\varepsilon au)^2}\frac{\gamma(3-\gamma)+\varepsilon a^2(1-3\gamma)}{4\gamma}\alpha\beta\\&+A_1(\alpha-\beta)-(B_1-A_1)\beta,\end{aligned}\tag{4.30}$$

and

$$\begin{aligned}\partial_2\alpha=&-\frac{1-\varepsilon u^2}{(1+\varepsilon au)^2}\frac{(\gamma+1)(\gamma-\varepsilon a^2)}{4\gamma}\alpha^2\\&-\frac{1-\varepsilon u^2}{(1+\varepsilon au)^2}\frac{\gamma(3-\gamma)+\varepsilon a^2(1-3\gamma)}{4\gamma}\alpha\beta\\&+A_2(\beta-\alpha)-(B_2-A_2)\alpha.\end{aligned}\tag{4.31}$$

Recalling (4.4) yields $B_i-A_i\ge 0$ $(i=1,2)$.

We verify the conclusion of the lemma by utilizing the contradiction argument again. As in the proof of Lemma 4.3, we may assume that there exists a characteristic triangle tip at $(\bar r,T)$ such that $\alpha=M$ or $\beta=M$ at $(\bar r,T)$, but $0\le\alpha<M$ and $0\le\beta<M$ in the characteristic triangle. Without loss of generality, we assume $0\le\alpha<M$ and $\beta=M$ at $(\bar r,T)$. Then it is clear by (4.30) that

$$\begin{aligned}&\partial_1\beta(\bar r,T)\\&<\Big\{-\frac{1-\varepsilon u^2}{(1-\varepsilon au)^2}\Big(\frac{(\gamma+1)(\gamma-\varepsilon a^2)}{4\gamma}\beta^2+\frac{\gamma(3-\gamma)+\varepsilon a^2(1-3\gamma)}{4\gamma}\alpha\beta\Big)\Big\}\Big|_{(\bar r,T)}\\&<0,\end{aligned}$$

which contradicts to the assumption that $\beta<M$ in the characteristic triangle. A similar contradiction when $0\le\beta<M$ and $\alpha=M$ at $(\bar r,T)$ can be obtained by (4.31). This finishes the proof of the lemma. ■

At the end of this section, we derive an only time-dependent density positive lower bound, which directly leads to $w_-<w_+$ by (2.4).

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Lemma 4.5 *Let the assumption in Lemma 4.4 hold. Moreover, suppose that the initial data satisfy*

$$\underline{\rho} := \min_{r\in[b,\infty)} \rho(r,0) > 0. \tag{4.32}$$

Then the smooth solution for Problem 1 satisfies

$$\rho(r,t) \geq \underline{\rho}\Big(\frac{4b\sqrt{\varepsilon}}{4b\sqrt{\varepsilon}+(3-\gamma)t}\Big)^{d-1} e^{-M_b t}, \tag{4.33}$$

for $(r,t)\in \Omega_b \cap \{t\leq T_0\}$, *where* M_b *is a positive constant depending on* b.

Proof Set $\partial_0 = \partial_t + u\partial_r$. Thanks to the definitions of (∂_1,∂_2), we attain

$$\partial_0 = \frac{(1-\varepsilon au)\partial_1 + (1+\varepsilon au)\partial_2}{2}. \tag{4.34}$$

Next we derive the equation for $\partial_0 a$. Applying the definitions of (∂_1,∂_2) again, we calculate by (3.13) and (3.14),

$$\begin{aligned}
&\partial_2 a = \partial_t a + \frac{u+a}{1+\varepsilon au}\partial_r a\\
=&-\frac{\gamma-1}{2\gamma}\frac{\gamma+\varepsilon a^2}{1-\varepsilon^2a^2u^2}\Big(\frac{d-1}{r}ua + \frac{2\gamma}{\gamma-1}\frac{u(1-\varepsilon a^2)}{\gamma+\varepsilon a^2}\partial_r a + a\partial_r u\Big) + \frac{u+a}{1+\varepsilon au}\partial_r a\\
=&\frac{\gamma-1}{4\gamma}\frac{(u+a)(\gamma+\varepsilon a^2)}{1+\varepsilon au}\Big(\alpha-\beta-\frac{d-1}{r}\frac{2u^2a}{(u-a)(u+a)}\Big)\\
&-\frac{d-1}{r}\frac{\gamma-1}{2\gamma}\frac{ua(\gamma+\varepsilon a^2)}{1-\varepsilon^2a^2u^2}\\
&-\frac{\gamma-1}{4\gamma}\frac{u(1-\varepsilon a^2)(\gamma+\varepsilon a^2)}{1-\varepsilon^2a^2u^2}\Big(\alpha-\beta-\frac{d-1}{r}\frac{2u^2a}{(u-a)(u+a)}\Big)\\
&-\frac{\gamma-1}{2\gamma}\frac{a(\gamma+\varepsilon a^2)(1-\varepsilon u^2)}{2(1-\varepsilon^2a^2u^2)}\Big(\alpha+\beta+\frac{d-1}{r}\frac{2ua^2}{(u-a)(u+a)}\Big).
\end{aligned} \tag{4.35}$$

Doing a simplification for (4.35) obtains

$$\partial_2 a = -\frac{\gamma-1}{2\gamma}\frac{\gamma+\varepsilon a^2}{1-\varepsilon^2a^2u^2}\Big(\frac{d-1}{r}\frac{u^2a(1-\varepsilon au)}{u-a} + a(1-\varepsilon u^2)\beta\Big). \tag{4.36}$$

A similar computations gives

$$\begin{aligned}
\partial_1 a =&\partial_t a + \frac{u-a}{1-\varepsilon au}\partial_r a\\
=&-\frac{\gamma-1}{2\gamma}\frac{\gamma+\varepsilon a^2}{1-\varepsilon^2a^2u^2}\Big(\frac{d-1}{r}\frac{u^2a(1+\varepsilon au)}{u+a} + a(1-\varepsilon u^2)\alpha\Big).
\end{aligned} \tag{4.37}$$

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Inserting (4.36) and (4.37) into (4.34) leads to

$$\begin{aligned}\partial_0 a =&\frac{(1-\varepsilon au)\partial_1 a+(1+\varepsilon au)\partial_2 a}{2}\\ =&-\frac{\gamma-1}{4\gamma}\Big\{\frac{\gamma+\varepsilon a^2}{1+\varepsilon au}\Big(\frac{d-1}{r}\frac{u^2a(1+\varepsilon au)}{u+a}+a(1-\varepsilon u^2)\alpha\Big)\\ &+\frac{\gamma+\varepsilon a^2}{1-\varepsilon au}\Big(\frac{d-1}{r}\frac{u^2a(1-\varepsilon au)}{u-a}+a(1-\varepsilon u^2)\beta\Big)\Big\}\\ =&-\frac{\gamma-1}{4\gamma}(\gamma+\varepsilon a^2)a\\ &\times\Big\{\frac{d-1}{r}\frac{u^2}{u-a}+\frac{d-1}{r}\frac{u^2}{u+a}+\frac{1-\varepsilon u^2}{1-\varepsilon au}\beta+\frac{1-\varepsilon u^2}{1+\varepsilon au}\alpha\Big\}.\end{aligned}\tag{4.38}$$

Then we employ the relationship between ρ and a to acquire

$$\begin{aligned}&\partial_0\Big(\frac{1}{r^{d-1}\rho}\Big)=-\frac{1}{r^{d-1}\rho}\frac{d-1}{r}\partial_0 r+\frac{1}{r^{d-1}\rho^2}\partial_0\rho\\ =&\frac{1}{r^{d-1}\rho}\Big(-\frac{2}{\gamma-1}\frac{1}{a}\partial_0 a-\frac{d-1}{r}\partial_0 r\Big)\\ =&\frac{1}{r^{d-1}\rho}\Big\{\frac{\gamma+\varepsilon a^2}{2\gamma}\Big(\frac{d-1}{r}\frac{u^2}{u-a}+\frac{d-1}{r}\frac{u^2}{u+a}+\frac{1-\varepsilon u^2}{1-\varepsilon au}\beta+\frac{1-\varepsilon u^2}{1+\varepsilon au}\alpha\Big)\\ &-\frac{d-1}{r}u\Big\},\end{aligned}$$

from which one derives

$$\begin{aligned}&\partial_0\ln\Big(\frac{1}{r^{d-1}\rho}\Big)\\ =&\frac{\gamma+\varepsilon a^2}{2\gamma}\Big(\frac{d-1}{r}\frac{u^2}{u-a}+\frac{d-1}{r}\frac{u^2}{u+a}+\frac{1-\varepsilon u^2}{1-\varepsilon au}\beta+\frac{1-\varepsilon u^2}{1+\varepsilon au}\alpha\Big)-\frac{d-1}{r}u\\ =&\frac{(d-1)(\gamma+\varepsilon u^2)}{2r\gamma}\Big(\frac{a^2}{u-a}+\frac{a^2}{u+a}\Big)\\ &+\frac{(\gamma+\varepsilon a^2)(1-\varepsilon u^2)}{2\gamma}\Big(\frac{\beta}{1-\varepsilon au}+\frac{\alpha}{1+\varepsilon au}\Big).\end{aligned}\tag{4.39}$$

Recalling (4.3) and (4.17) and applying the fact $r\geq b$ for any point $(r,t)\in\Omega_b$, there hold

$$\begin{aligned}&\tfrac{1}{1+\varepsilon au}\leq\tfrac{1}{1-\varepsilon au}\leq\tfrac{1}{1-\varepsilon u^2}, &&\tfrac{\gamma+\varepsilon a^2}{2\gamma}\leq\tfrac{\gamma+1}{2\gamma}\leq 1,\\ &\tfrac{(d-1)(\gamma+\varepsilon u^2)}{2r\gamma}\leq\tfrac{(d-1)(\gamma+1)}{2b\gamma}, &&\tfrac{a^2}{u+a}\leq\tfrac{a^2}{u-a}\leq\tfrac{(\gamma-1)(\sqrt{2}+\sqrt{\gamma+1})}{4\sqrt{2\varepsilon}}.\end{aligned}\tag{4.40}$$

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Putting (4.39) into (4.39) achieves

$$\begin{aligned}\partial_0 \ln\Big(\frac{1}{r^{d-1}\rho}\Big) &\le \frac{(d-1)(\gamma^2-1)(\sqrt{2}+\sqrt{\gamma+1})}{4b\gamma\sqrt{2\varepsilon}} + \beta + \alpha \\ &\le \frac{(d-1)(\gamma^2-1)(\sqrt{2}+\sqrt{\gamma+1})}{4b\gamma\sqrt{2\varepsilon}} + 2M =: M_b.\end{aligned} \tag{4.41}$$

For any point $(\nu,\mu) \in \Omega_b \cap \{t \le T_0\}$, we define the curve $r = r_0(t) = r_0(t;\nu,\mu)$ $(t \le \mu)$ by

$$\frac{\mathrm{d}r_0(t)}{\mathrm{d}t} = u(r_0(t),t), \quad r_0(\mu) = \nu. \tag{4.42}$$

Denote the intersection point of $r = r_0(t)$ and the initial line $t = 0$ by $(r_0, 0)$. Then one has

$$r_0 \le r_0(t) \le r_0 + \int_0^t u(r_0(\tau),\tau)\mathrm{d}\tau \le r_0 + \frac{3-\gamma}{4\sqrt{\varepsilon}}t, \tag{4.43}$$

which leads to

$$\frac{r_0}{r_0(t)} \ge \frac{4b\sqrt{\varepsilon}}{4b\sqrt{\varepsilon}+(3-\gamma)t}. \tag{4.44}$$

Now we integrate (4.41) along $r_0(t)$ to conclude that

$$\begin{aligned}\rho(r,t) &\ge \Big(\frac{r_0(t_0)}{r_0(t)}\Big)^{d-1}\rho_0(r_0)e^{-M_b t} \\ &\ge \rho_0(r_0)\Big(\frac{4b\sqrt{\varepsilon}}{4b\sqrt{\varepsilon}+(3-\gamma)t}\Big)^{d-1}e^{-M_b t} \\ &\ge \underline{\rho}\Big(\frac{4b\sqrt{\varepsilon}}{4b\sqrt{\varepsilon}+(3-\gamma)t}\Big)^{d-1}e^{-M_b t},\end{aligned} \tag{4.45}$$

which is the desired estimate (4.33). The proof of the lemma is ended. ■

Remark 9 A density positive lower bound that is only linear about time was previously established by Cai, Chen and Wang [6] for the classical radially symmetric isentropic Euler equations (1.5). However, their lower bound also depends on the spatial variable r, especially when $r \to \infty$, this lower bound tends to zero. In other words, their lower bound is not uniform with respect to spatial variable.

5 Global existence of smooth solutions

In this section, we complete the proof of Theorem 2.1 by establishing the global existence of smooth solutions in Ω_b for Problem 1 and then letting $b \to 0$. These results are based on the L^∞ bound in (4.3), the density lower bound in (4.33) and C^1 bound for w_+ and w_-.

We first have the following existence theorem in Ω_b.

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Theorem 5.1 *Let the initial data* $(\rho_0(r), u_0(r)) \in C^1([b, \infty))$ *satisfy Assumptions 1 and 2. We further suppose that* $\underline{\rho} = \min_{r\in[b,\infty)} \rho_0(r) > 0$. *Then Problem 1 admits a global* C^1 *solution on the domain of dependence* Ω_b *on the* (r, t) *plane with base* $t = 0$ *and* $r \in (b, \infty)$ *for any* $b > 0$. *Moreover, the solution* $(\rho, u)(r, t)$ *satisfies*

$$\begin{aligned} &\sqrt{\frac{2A\gamma}{\gamma-1}}\rho^{\frac{\gamma-1}{2}}(r,t) \le u(r,t) \le \frac{3-\gamma}{4\sqrt{\varepsilon}},\\ &\rho(r,t) \ge \underline{\rho}\Big(\frac{4b\sqrt{\varepsilon}}{4b\sqrt{\varepsilon}+(3-\gamma)t}\Big)^{d-1} e^{-M_b t}, \end{aligned} \tag{5.1}$$

for some positive constant M_b, *and*

$$\min_{\Omega_b}(\alpha, \beta) \ge 0, \quad \max_{\Omega_b}(\alpha, \beta) < M. \tag{5.2}$$

Proof The local existence of smooth solutions for Problem 1 can be obtained by the classical theory, see for example Li and Yu [41]. We next use the framework by Li [40] to extend the local smooth solution to a global domain. According to the extension framework in [40], it suffices to establish the a priori C^1 estimates of the solution on the domain Ω_b. Actually, the local existence time δ of smooth solution in [41] depends only on the norm $||\Gamma^*||$ and C^1 norm $||(w_-(r, 0), w_+(r, 0))||_{C^1([b,\infty))}$, where Γ^* is the following set of functions

$$\begin{aligned}\Gamma^* =\Big\{&\lambda_1,\ \lambda_2,\ \frac{1}{\lambda_2-\lambda_1},\ \frac{\partial\lambda_1}{\partial w_-},\ \frac{\partial\lambda_1}{\partial w_+},\ \frac{\partial\lambda_2}{\partial w_-},\ \frac{\partial\lambda_2}{\partial w_+},\ \frac{ua}{r(1+\varepsilon au)},\\ &\frac{\partial}{\partial r}\Big(\frac{ua}{r(1+\varepsilon au)}\Big), \frac{\partial}{\partial w_-}\Big(\frac{ua}{r(1+\varepsilon au)}\Big),\ \frac{\partial}{\partial w_+}\Big(\frac{ua}{r(1+\varepsilon au)}\Big),\ \frac{ua}{r(1-\varepsilon au)},\\ &\frac{\partial}{\partial r}\Big(\frac{ua}{r(1-\varepsilon au)}\Big), \frac{\partial}{\partial w_-}\Big(\frac{ua}{r(1-\varepsilon au)}\Big),\ \frac{\partial}{\partial w_+}\Big(\frac{ua}{r(1-\varepsilon au)}\Big)\Big\}.\end{aligned} \tag{5.3}$$

The above conclusion can be found in Remark 4.1. in Chapter 1 in [41]. From the above analysis and the a priori estimates established in Section 4, we know that, for any number $b > 0$ and any time $T > 0$, the norm $||\Gamma^*||$ depends only on b and T. This means that, for fixed b and T, the local existence time δ is a constant. Therefore, by solving a finite number of local existence problems, we can extend the solution in the region $\Omega_b \cap \{0 \le t \le \delta\}$ to the global region $\Omega_b \cap \{0 \le t \le T\}$. Due to the arbitrariness of T, we thus obtain the smooth solution on the global region Ω_b for any fixed $b > 0$. Finally, we can directly attain the properties of solution in (5.1) and (5.2) by the results in Section 4. This completes the proof of the theorem. ■

Proof of Theorem 2.1. Let us start establishing the global existence of smooth solutions on the entire half line $r \in [0, \infty)$. We further suppose that the boundary condition $u(0, t) = 0$ at origin, which is a reasonable physical assumption. In order to match the condition (2.8), we specify the boundary

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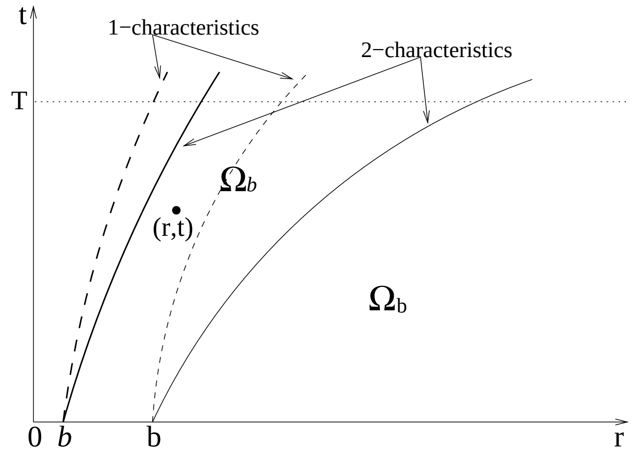

Figure 2: The regions Ω_{b} and Ω_b..

data of density at the origin as $\rho(0,t)=0$. Thus Assumption 1 is fulfilled on the entire half line $r\in[0,\infty)$.

Assume that the conditions in Theorem 2.1 hold. Then for any $b>0$, all the conditions in Theorem 5.1 are satisfied and then the global smooth solution exists in domain Ω_b. Now for any point (r,t) with $r>0$, we claim that one can find a number $b>0$ small enough such that (r,t) is in the domain Ω_b, see Figure 2 for illustration. To show this assertion, it is only necessary to verify that the characteristic starting from $(r,t)=(0,0)$ must be $r=0$. We also use the contradiction argument to demonstrate it. Assume that, without lose of generality, the 2-characteristic $r=\tilde{r}(t)$ starting from the point $(r,t)=(0,0)$ goes away from $r=0$ at $t=\check{t}$, that is $\tilde{r}(\check{t})=0$ and $\tilde{r}(t)>0$ when $\check{t}<t<\check{t}+\bar{\delta}$, where $\bar{\delta}$ is a small positive number. And we only consider smooth solutions before gradient blowup on $r=\tilde{r}(t)$ with $\check{t}\le t\le\hat{t}<\check{t}+\bar{\delta}$. According to the differential mean value theorem with respect to r, it suggests that

$$\lambda_2(r,t)=\lambda_2(r,t)-\lambda_2(0,t)\le\widetilde{K}r, \tag{5.4}$$

for some positive constant $\widetilde{K}$. Then we see that on $r=\tilde{r}(t)$

$$\frac{\mathrm{d}\tilde{r}(t)}{\mathrm{d}t}=\lambda_2\le\widetilde{K}\tilde{r}(t), \tag{5.5}$$

from which one has

$$\tilde{r}(\hat{t})\le e^{\widetilde{K}(\hat{t}-\check{t})}\tilde{r}(\check{t})=0, \tag{5.6}$$

which contradicts to the fact $\tilde{r}(\hat{t})>0$. Therefore, the characteristic starting from $(r,t)=(0,0)$ must be $r=0$. Hence we obtain the smooth solution on the entire domain $r\ge0$, $t\ge0$. The properties of solution in (2.12) and (2.13) follow directly by Theorem 5.1. The proof of Theorem 2.1 is complete.

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6 Singularity formation

In this section, we present the proof of Theorem 2.2, that is, the singularity forms in finite time when the compressive initial data include strong compression somewhere.

We first establish the invariant domain for the compressive initial data.

Lemma 6.1 *Consider a smooth solution of (1.4) with (1.2) on $\Omega_b \cap \{t \le T_0\}$, satisfying Assumption 1 on (b, ∞) with $b > 0$. We further assume that the functions $\alpha_0(r)$ and $\beta_0(r)$ fulfill (2.14), that is they are compressed. Then the smooth solution satisfies*

$$\alpha(r,t) \le 0, \quad \beta(r,t) \le 0, \quad \forall\ (r,t) \in \Omega_b \cap \{t \le T_0\}. \tag{6.1}$$

Proof We note that the inequality (4.21) still holds now by the fact that the proof of Lemmas 4.1 and 4.2 only needs to use Assumption 1. Let K_b and $\widehat{K}_b = 2K_b + 2$ be the constants defined in (4.21) and (4.22), respectively, and $\eta > 0$ be a small number. We introduce two new variables

$$\widetilde{X} = \alpha + \eta e^{-\widehat{K}_b t}, \quad \widetilde{Y} = \beta + \eta e^{-\widehat{K}_b t}. \tag{6.2}$$

Let $R > b$ be any fixed real number. Denote

$$\kappa_R = -\max_{r \in [b,R]} \{\alpha_0(r),\ \beta_0(r)\}. \tag{6.3}$$

According to (2.14), we know that the number κ_R is a positive constant. Now we choose η small enough such that

$$\eta \le \min\left\{\kappa_R, \frac{16 - (3-\gamma)^2}{16}\right\}. \tag{6.4}$$

Thus it follows by (6.2)-(6.4) that

$$\widetilde{X}(r,0) < 0,\ \widetilde{Y}(r,0) < 0, \quad \forall\ r \in [b,R]. \tag{6.5}$$

Moreover, in view of Lemma 3.1, we can obtain the governing system of $(\widetilde{X}, \widetilde{Y})$ as follows

$$\begin{aligned}
\partial_2 \widetilde{X} = &\bigg\{ - \frac{1-\varepsilon u^2}{(1+\varepsilon a u)^2} \frac{(\gamma+1)(\gamma - \varepsilon a^2)}{4\gamma} (\widetilde{X} - 2\eta e^{-\widehat{K}_b t}) \\
&- \frac{1-\varepsilon u^2}{(1+\varepsilon a u)^2} \frac{\gamma(3-\gamma) + \varepsilon a^2(1-3\gamma)}{4\gamma} (\widetilde{Y} - \eta e^{-\widehat{K}_b t}) - B_2 \bigg\} \widetilde{X} \\
&+ \bigg\{ \frac{1-\varepsilon u^2}{(1+\varepsilon a u)^2} \frac{\gamma(3-\gamma) + \varepsilon a^2(1-3\gamma)}{4\gamma} \eta e^{-\widehat{K}_b t} + A_2 \bigg\} \widetilde{Y} \\
&- \eta e^{-\widehat{K}_b t} \bigg\{ \widehat{K}_b + A_2 - B_2 + \frac{(1-\varepsilon u^2)(1-\varepsilon a^2)}{(1+\varepsilon a u)^2} \eta e^{-\widehat{K}_b t} \bigg\},
\end{aligned} \tag{6.6}$$

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and

$$
\begin{aligned}
\partial_1 \widetilde{Y} =& \Big\{ - \frac{1-\varepsilon u^2}{(1-\varepsilon a u)^2} \frac{(\gamma+1)(\gamma-\varepsilon a^2)}{4\gamma} (\widetilde{Y} - 2\eta e^{-\widehat{K}_b t}) \\
& - \frac{1-\varepsilon u^2}{(1-\varepsilon a u)^2} \frac{\gamma(3-\gamma)+\varepsilon a^2(1-3\gamma)}{4\gamma} (\widetilde{X} - \eta e^{-\widehat{K}_b t}) - B_1 \Big\} \widetilde{Y} \\
& + \Big\{ \frac{1-\varepsilon u^2}{(1-\varepsilon a u)^2} \frac{\gamma(3-\gamma)+\varepsilon a^2(1-3\gamma)}{4\gamma} \eta e^{-\widehat{K}_b t} + A_1 \Big\} \widetilde{X} \\
& - \eta e^{-\widehat{K}_b t} \Big\{ \widehat{K}_b + A_1 - B_1 + \frac{(1-\varepsilon u^2)(1-\varepsilon a^2)}{(1-\varepsilon a u)^2} \eta e^{-\widehat{K}_b t} \Big\}.
\end{aligned}
\tag{6.7}
$$

We first consider the smooth solution in the region $\Omega_b \cap \{(r,t)|\ r \leq R,\ t \leq T_0\}$. If $(\bar{r},\bar{t}) \in \Omega_b \cap \{(r,t)|\ r \leq R,\ t \leq T_0\}$ is the first time such that $\widetilde{X}(\bar{r},\bar{t}) = 0, \widetilde{Y}(\bar{r},\bar{t}) \leq 0$, then one has

$$\partial_2 \widetilde{X}|_{(\bar{r},\bar{t})} \geq 0. \tag{6.8}$$

On the other hand, we observe from (6.6) that

$$
\begin{aligned}
& \partial_2 \widetilde{X}|_{(\bar{r},\bar{t})} \\
\leq & - \eta e^{-\widehat{K}_b t} \Big\{ \widehat{K}_b + A_2 - B_2 + \frac{(1-\varepsilon u^2)(1-\varepsilon a^2)}{(1+\varepsilon a u)^2} \eta e^{-\widehat{K}_b t} \Big\} < 0,
\end{aligned}
\tag{6.9}
$$

by (4.25). This yields a contradiction. If $(\bar{r},\bar{t}) \in \Omega_b \cap \{(r,t)|\ r \leq R,\ t \leq T_0\}$ is the first time such that $\widetilde{X}(\bar{r},\bar{t}) \leq 0, \widetilde{Y}(\bar{r},\bar{t}) = 0$, the proof is similar. Thus we have

$$\widetilde{X}(r,t) < 0,\ \widetilde{Y}(r,t) < 0, \quad \forall\ (r,t) \in \Omega_b \cap \{(r,t)|\ r \leq R,\ t \leq T_0\}, \tag{6.10}$$

and subsequently

$$\alpha(r,t) < -\eta e^{-\widehat{K}_b t} < 0, \quad \beta(r,t) < -\eta e^{-\widehat{K}_b t} < 0, \tag{6.11}$$

for any $(r,t) \in \Omega_b \cap \{(r,t)|\ r \leq R,\ t \leq T_0\}$. Due to the arbitrariness of R, we obtain

$$\alpha(r,t) \leq 0,\ \beta(r,t) \leq 0, \quad \forall\ (r,t) \in \Omega_b \cap \{t \leq T_0\}. \tag{6.12}$$

The proof of the lemma is complete. ■

To proceed, we need derive an only time-dependent density positive lower bound in the current conditions.

Lemma 6.2 *Let the assumptions in Lemma 6.1 hold. Moreover, suppose that* (4.32) *holds, that is,* $\underline{\rho} = \min_{r \in [b,\infty)} \rho(r,0) > 0$. *Then the smooth solution satisfies*

$$\rho(r,t) \geq \underline{\rho} \Big(\frac{4b\sqrt{\varepsilon}}{4b\sqrt{\varepsilon} + (3-\gamma)t} \Big)^{d-1} e^{-\widetilde{M}_b t}, \tag{6.13}$$

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for $(r,t)\in\Omega_b\cap\{t\le T_0\}$, *where*

$$\widetilde{M}_b=\frac{(d-1)(\gamma^2-1)(\sqrt{2}+\sqrt{\gamma+1})}{4b\gamma\sqrt{2\varepsilon}}. \tag{6.14}$$

Proof Recalling the first inequality in (4.41), we have by (6.1) and (6.14)

$$\partial_0\ln\Big(\frac{1}{r^{d-1}\rho}\Big)\le\frac{(d-1)(\gamma^2-1)(\sqrt{2}+\sqrt{\gamma+1})}{4b\gamma\sqrt{2\varepsilon}}+\beta+\alpha\le\widetilde{M}_b. \tag{6.15}$$

Recalling the definition of the curve $r=r_0(t)$ in (4.42) and using the estimate (4.44), we integrate (6.15) along $r=r_0(t)$ to obtain

$$\begin{aligned}\rho(r,t)\ge&\Big(\frac{r_0(t_0)}{r_0(t)}\Big)^{d-1}\rho_0(r_0)e^{-\widetilde{M}_bt}\\ \ge&\underline{\rho}\Big(\frac{4b\sqrt{\varepsilon}}{4b\sqrt{\varepsilon}+(3-\gamma)t}\Big)^{d-1}e^{-\widetilde{M}_bt},\end{aligned} \tag{6.16}$$

for any $(r,t)\in\Omega_b\cap\{t\le T_0\}$. The proof of the lemma is completed. ■

Proof of Theorem 2.2. Now we are going to show Theorem 2.2. Recalling the coefficients of β^2 and α^2 in (3.5) and (3.7), respectively, one has by (4.19)

$$\begin{aligned}&-\frac{1-\varepsilon u^2}{(1\pm\varepsilon au)^2}\frac{(\gamma+1)(\gamma-\varepsilon a^2)}{4\gamma}\\ \le&-\frac{16-(3-\gamma)^2}{2^2\cdot16}\cdot\frac{\gamma-1}{4}=-\frac{(\gamma-1)[16-(3-\gamma)^2]}{256},\end{aligned} \tag{6.17}$$

which, together with Lemmas 3.1, 6.1 and 6.2, yields

$$\begin{aligned}\partial_1\beta\le&-\frac{(\gamma-1)[16-(3-\gamma)^2]}{256}\beta^2-K_b\beta\\ =&-\frac{(\gamma-1)[16-(3-\gamma)^2]}{512}\beta^2-\Big(\frac{(\gamma-1)[16-(3-\gamma)^2]}{512}\beta+K_b\Big)\beta,\\ \partial_2\alpha\le&-\frac{(\gamma-1)[16-(3-\gamma)^2]}{256}\alpha^2-K_b\alpha\\ =&-\frac{(\gamma-1)[16-(3-\gamma)^2]}{512}\alpha^2-\Big(\frac{(\gamma-1)[16-(3-\gamma)^2]}{512}\alpha+K_b\Big)\alpha.\end{aligned} \tag{6.18}$$

Set

$$N(b,T)=\max\Big\{\frac{512K_b}{(\gamma-1)[16-(3-\gamma)^2]},\frac{512}{(\gamma-1)[16-(3-\gamma)^2]T}\Big\}. \tag{6.19}$$

If $\alpha_0(r^*)\le-N(b,T)$, we consider the 2-characteristic curve $r=r_2(t;r^*,0)$ for $t\in[0,T)$ to observe that

$$\frac{(\gamma-1)[16-(3-\gamma)^2]}{512}\alpha(r_2(t;r^*,0),t)+K_b\le0,$$

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and then

$$\partial_2 \alpha \leq -\frac{(\gamma-1)[16-(3-\gamma)^2]}{512}\alpha^2, \tag{6.20}$$

along $r = r_2(t; r^*, 0)$. Integrating (6.20) yields

$$\frac{1}{-\alpha(r,t)} \leq \frac{1}{-\alpha_0(r^*)} - \frac{(\gamma-1)[16-(3-\gamma)^2]}{512}t, \tag{6.21}$$

which implies that blowup happens not later than

$$T^* = \frac{512}{-\alpha_0(r^*)(\gamma-1)[16-(3-\gamma)^2]} \leq T. \tag{6.22}$$

If $\beta_0(r^*) \leq -N(b,T)$ at some point r^*, we can similarly show the blowup time is before T^*. The proof of Theorem 2.2 is complete.

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