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Large deviations for occupation and waiting times of infinite ergodic transformations

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Abstract. We establish large deviation estimates related to the Darling–Kac theorem and generalized arcsine laws for occupation and waiting times of ergodic transformations preserving an infinite measure, such as non-uniformly expanding interval maps with indifferent fixed points. For the proof, we imitate the study of generalized arcsine laws for occupation times of one-dimensional diffusion processes and adopt a method of double Laplace transform.

Key words: infinite ergodic theory, large deviation estimates, Darling–Kac theorem, generalized arcsine laws
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1. *Introduction*

In the study of dynamical systems with an infinite invariant measure, a variety of ergodic and probabilistic limit theorems have been established. They are often related to classical limit theorems for renewal, Markov, or diffusion processes in probability theory. Among this kind of research of dynamical systems, we are going to focus on three distributional limit theorems, the Darling–Kac theorem for occupation times in sets of finite measure, the Dynkin–Lamperti generalized arcsine law for the last time the orbit visits to sets of finite measure, and the Lamperti generalized arcsine law for occupation times in sets of infinite measure, studied by [1, 2, 11, 14, 17, 18, 24, 25, 27, 33]. The aim of the present paper is to establish large deviation estimates related to these limit theorems under similar abstract settings as in [11, 18, 27, 33]. Our abstract results can be applied to, for example, intermittent maps, that is, non-uniformly expanding interval maps with indifferent fixed points. We are motivated by the study of a large deviation estimate related to a generalized arcsine law for occupation times of one-dimensional diffusion processes [10]. We also refer the reader to [16] for another type of large deviations, which is related to the strong arcsine law for a one-dimensional Brownian motion.

In the remainder of this section, we recall known distributional limit theorems and present our large deviation estimates, using Boole's transformation as a representative example for simplicity. Nonetheless, previous studies as well as our main results are applicable to more general classes of infinite ergodic transformations.

Example 1.1. (Distributional limit theorems for Boole's transformation) We refer the reader to [3, 4, 25] for the details of Boole's transformation. The map $T : [0, 1] \to [0, 1]$ given by

$$Tx = \begin{cases} x(1-x)/(1-x-x^2), & x \in [0, 1/2], \\ 1 - T(1-x), & x \in (1/2, 1], \end{cases}$$

is conjugated to Boole's transformation $\widetilde{T}x = x - x^{-1} (x \in \mathbb{R} \setminus \{0\})$. Indeed, let $\phi(x) = (1-x)^{-1} - x^{-1} (x \in (0, 1))$, then $\widetilde{T} = \phi \circ T \circ \phi^{-1}$ on $\mathbb{R} \setminus \{0\}$. It is easy to see that $T0 = 0$, $T1 = 1$, $T'(0) = T'(1) = 1$, $T'' > 0$ on $(0, 1/2)$ and $T'' < 0$ on $(1/2, 1)$. In addition, we have $Tx - x = 1 - x - T(1-x) \sim x^3 (x \to 0)$. Thus, T is a special case of Thaler's maps, which will be explained in §8. The map T admits the invariant density h given by

$$h(x) = \frac{1}{x^2} + \frac{1}{(1-x)^2}, \quad x \in (0, 1).$$

Therefore, the invariant measure μ given by $d\mu(x) = h(x)\,dx (x \in [0, 1])$ is an infinite measure. Set $\gamma = \sqrt{2} - 1 \in (0, 1/2)$, which is a 2-periodic point of T. Indeed, $T\gamma = 1 - \gamma \in (1/2, 1)$ and hence, $T^2\gamma = \gamma$. Let

$$A_0 = [0, \gamma), \quad Y = [\gamma, T\gamma], \quad A_1 = (T\gamma, 1].$$

Then, $\mu(Y) = \sqrt{2}$ and $\mu(A_0) = \mu(A_1) = \infty$. In addition, Y dynamically separates A_0 and A_1, that is, $A_i \cap T^{-1}A_j = \emptyset (i \neq j)$. For a non-negative integer n, a Borel subset $A \subset [0, 1]$, and $x \in [0, 1]$, set

$$S_n^A(x) = \sum_{k=1}^{n} 1_A(T^k x), \quad Z_n^A(x) = \max\{k \leq n : T^k x \in A\}.$$

Here, it is understood that $\max \emptyset = 0$. In other words, $S_n^A(x)$ denotes the occupation time in A of the orbit $\{T^k x\}_{k \geq 0}$ between time 1 and n, and $Z_n^A(x)$ denotes the last time the orbit arrives in A until time n. Fix any Borel probability measure $\nu(dx)$ absolutely continuous with respect to the Lebesgue measure on $[0, 1]$. We interpret x as the initial point of the orbit $\{T^k x\}_{k \geq 0}$ and $\nu(dx)$ as the initial distribution of the orbit. Then, the Darling–Kac theorem [1, 2] yields that, as $n \to \infty$,

$$\nu\bigg(\frac{\pi S_n^Y}{2\sqrt{n}} \leq t\bigg) \to \frac{2}{\pi} \int_0^t e^{-y^2/\pi}\,dy, \quad t \geq 0. \tag{1.1}$$

Next, the Dynkin–Lamperti generalized arcsine law for waiting times [24] shows that, as $n \to \infty$,

$$\nu\bigg(\frac{Z_n^Y}{n} \leq t\bigg) \to \frac{2}{\pi} \arcsin\sqrt{t}, \quad t \in [0, 1]. \tag{1.2}$$

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Finally, the Lamperti generalized arcsine law for occupation times [25] implies that, as $n \to \infty$,

$$\nu\bigg(\frac{S_n^{A_i}}{n} \le t\bigg) \to \frac{2}{\pi} \arcsin \sqrt{t}, \quad t \in [0,1],\ i = 0, 1. \tag{1.3}$$

We also remark that convergence rates of (1.1) and (1.2) were also studied in [12, 19–21], and a large deviation estimate for the Perron–Frobenius operator related to (1.2) can be found in [26].

We now illustrate our main results. Our aim is to estimate the left-hand sides of (1.1), (1.2), and (1.3) as $t \to 0$.

Example 1.2. (Large deviation estimates for Boole's transformation) Under the setting of Example 1.1, we further assume that ν is a probability measure supported on $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0, 1/2)$ and admits a Riemann integrable density. Then, there exists some constants $0 < C_1 \le C_2 < \infty$ such that, for any positive sequence $\{c(n)\}_{n \ge 0}$ with $c(n) \to 0$ and $c(n)n \to \infty (n \to \infty)$, the following estimates hold:

$$C_1 \le \liminf_{n\to\infty} \frac{p(n)}{\sqrt{c(n)}} \le \limsup_{n\to\infty} \frac{p(n)}{\sqrt{c(n)}} \le C_2 \tag{1.4}$$

for

$$p(n) = \nu\bigg(\frac{\pi S_n^Y}{2\sqrt{n}} \le \sqrt{c(n)}\bigg), \quad \nu\bigg(\frac{Z_n^Y}{n} \le c(n)\bigg) \quad \text{and} \quad \nu\bigg(\frac{S_n^{A_i}}{n} \le c(n)\bigg).$$

Note that C_1 and C_2 may depend on ν. These estimates are compatible with (1.1), (1.2), and (1.3), respectively, since the right-hand side of (1.1) with $t = \sqrt{c(n)}$ and those of (1.2) and (1.3) with $t = c(n)$ are asymptotically equal to $(2/\pi)\sqrt{c(n)}$, as $n \to \infty$. Nevertheless, (1.1), (1.2), and (1.3) do not imply (1.4) directly.

For the proof, we adopt a method of double Laplace transform as in [18], imitating the study of generalized arcsine laws for occupation times of one-dimensional diffusion processes [5, 10, 16, 28, 29]. Although moment methods were used in [11, 24, 25, 27, 33], double Laplace transform is more adequate for our large deviation estimates. For example, the probability $\nu(Z_n^Y/n \le c(n))$ in Example 1.2 has a negligibly small contribution to the kth moment $\int_{[0,1]} (Z_n^Y/(c(n)n))^k \, d\nu (k = 1, 2, \ldots)$, while it has large contributions to the Laplace transform

$$\int_{[0,1]} \exp\bigg(-\frac{\lambda Z_n^Y}{c(n)n}\bigg) \, d\nu \quad (\lambda > 0)$$

and the double Laplace transform

$$\int_0^\infty e^{-qu} \bigg(\int_{[0,1]} \exp\bigg(-\frac{\lambda Z_{[un]}^Y}{c(n)n}\bigg) \, d\nu\bigg) \, du \quad (q, \lambda > 0).$$

This is why we adopt a method of double Laplace transform rather than moment methods to estimate $\nu(Z_n^Y/n \le c(n))$.

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This paper is organized as follows. In §2, we recall some basic notions of infinite ergodic theory and the theory of regular variation. In §3, we formulate large deviation estimates related to the Darling–Kac theorem and generalized arcsine laws in abstract settings. Section 4 is devoted to introduce some lemmas needed to calculate double Laplace transform. In §§5, 6, and 7, we prove the large deviation estimates by using double Laplace transform. In §8, we apply our abstract results to Thaler's maps.

2. *Preliminaries*

Before presenting our main results, let us recall basic concepts of infinite ergodic theory. We basically follow the settings of [**11**, **18**, **27**, **33**]. We also refer the reader to [**3**] for the foundations of infinite ergodic theory.

Throughout this paper, except in §§1 and 8, we assume the following condition.

- Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space with $\mu(X) = \infty$, and $T : X \to X$ be a conservative, ergodic, measure-preserving transformation on $(X, \mathcal{A}, \mu)$, which is abbreviated as *CEMPT*. In addition, let $Y \in \mathcal{A}$ with $\mu(Y) \in (0, \infty)$.

Let $\mathbb{N}$ denote the set of all positive integers and set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $A \in \mathcal{A}$, we write 1_A for the indicator function of A. Since T is a CEMPT, we have $\sum_{n\geq 0} 1_A \circ T^n = \infty$, almost everywhere (a.e.) for any $A \in \mathcal{A}$ with $\mu(A) > 0$. In other words, the orbit $\{T^n x\}_{n\geq 0}$ visits A infinitely often for μ-almost every initial point x. For $u \in L^1(\mu)$, define the signed measure μ_u on $(X, \mathcal{A})$ as $\mu_u(A) = \int_A u \, d\mu (A \in \mathcal{A})$. The *transfer operator* $\widehat{T} : L^1(\mu) \to L^1(\mu)$ is defined by $\widehat{T}u = d(\mu_u \circ T^{-1})/d\mu (u \in L^1(\mu))$. This operator is characterized by the equation $\int_X (v \circ T)u \, d\mu = \int_X v(\widehat{T}u) \, d\mu$ for any $v \in L^\infty(\mu)$ and $u \in L^1(\mu)$. The domain of $\widehat{T}$ can be extended to all non-negative, measurable functions $u : X \to [0, \infty)$. Then, $\int_X \widehat{T}u \, d\mu = \int_X u \, d\mu$ for any non-negative, measurable function u.

We need to extend the concept of uniform sweeping of [**27**, **33**] slightly. If, for non-negative measurable functions H and G on $(X, \mathcal{A}, \mu)$, there is some $C > 0$ and $K \in \mathbb{N}_0$ such that $C \sum_{k=0}^{K} \widehat{T}^k H \geq G$ a.e., then H will be called *uniformly sweeping (in K steps) for G*. Let $\mathfrak{H} \cup \{G\}$ be a family of measurable functions $H : X \to [0, \infty)$. We say $\mathfrak{H}$ is *uniformly sweeping (in K steps) for G* if the following condition holds: there exist some constants $C > 0$ and $K \in \mathbb{N}_0$ such that, for any $H \in \mathfrak{H}$, we have $C \sum_{k=0}^{K} \widehat{T}^k H \geq G$ a.e.

Let us recall regularly and slowly varying functions. We refer the reader to [**6**] for the details. Let $f, g : (0, \infty) \to (0, \infty)$ be positive, measurable functions. If $f(t)/g(t) \to 1 (t \to t_0)$, then we write $f(t) \sim g(t)$ $(t \to t_0)$. We say f is *regularly varying of index* $\rho \in \mathbb{R}$ *at* ∞ (respectively *at* 0) if, for any $\lambda > 0$,

$$f(\lambda t) \sim \lambda^\rho f(t) \quad (t \to \infty) \text{ (respectively } t \to 0+).$$

In the case where $\rho = 0$, we say f is *slowly varying at* ∞ (respectively *at* 0). A positive sequence $\{a(n)\}_{n\geq 0}$ is called regularly varying of index ρ if the function $a([t])$ is regularly varying of index ρ at ∞. Here, $[t]$ denotes the greatest integer that is less than or equal to t.

Let $\varphi : X \to \mathbb{N} \cup \{\infty\}$ be the *first return time* to Y, that is,

$$\varphi(x) = \min\{k \geq 1 : T^k x \in Y\} \quad (x \in X).$$

Here, it is understood that $\min \emptyset = \infty$. Define disjoint sets $Y_0, Y_1, Y_2, \ldots \in \mathcal{A}$ as

$$Y_0 = Y, \quad Y_n = (T^{-n}Y) \setminus \bigg(\bigcup_{k=0}^{n-1} T^{-k}Y \bigg) = Y^c \cap \{\varphi = n\} \quad (n \in \mathbb{N}).$$

As proved in [27, equation (2.3)],

$$1_{Y_n} = \sum_{k>n} \widehat{T}^{k-n} 1_{Y\cap\{\varphi=k\}} \quad \text{a.e.} \ \ (n \in \mathbb{N}_0), \tag{2.1}$$

and $\mu(Y_n) = \mu(Y \cap \{\varphi > n\})$. Let $\{w_n^Y\}_{n\geq 0}$ denote the *wandering rate* of Y, which is given by

$$\begin{aligned} w_n^Y &= \mu\bigg(\bigcup_{k=0}^{n-1} T^{-k}Y \bigg) = \sum_{k=0}^{n-1} \mu(Y_k) = \sum_{k=0}^{n-1} \int_Y \widehat{T}^k 1_{Y_k} \, d\mu \\ &= \sum_{k=0}^{n-1} \mu(Y \cap \{\varphi > k\}) \quad (n \in \mathbb{N}_0). \end{aligned} \tag{2.2}$$

Since T is a CEMPT, we see $\bigcup_{n\geq 0} T^{-n}Y = X$, a.e. and hence, $w_n^Y \to \infty$ $(n \to \infty)$. For $s > 0$, let $Q^Y(s)$ be a Laplace transform of $\{w_{n+1}^Y - w_n^Y\}_{n\geq 0}$:

$$Q^Y(s) = \sum_{n\geq 0} e^{-ns}(w_{n+1}^Y - w_n^Y) = \sum_{n\geq 0} e^{-ns} \mu(Y \cap \{\varphi > n\}) \quad (s > 0).$$

Then, $0 < Q^Y(s) < \infty$ and $Q^Y(s) \to \infty$ $(s \to 0+)$. Let $\alpha \in (0, 1)$ and let $\ell : (0, \infty) \to (0, \infty)$ be a positive, measurable function slowly varying at ∞. By Karamata's Tauberian theorem [6, Theorem 1.7.1], the condition

$$w_n^Y \sim n^{1-\alpha}\ell(n) \quad (n \to \infty)$$

is equivalent to

$$Q^Y(s) \sim \Gamma(2-\alpha)s^{-1+\alpha}\ell(s^{-1}) \quad (s \to 0+). \tag{2.3}$$

Here, $\Gamma(z) = \int_0^\infty e^{-t}t^{-1+z}\,dt$ $(z > 0)$ denotes the gamma function.

If $\{(w_n^Y)^{-1} \sum_{k=0}^{n-1} \widehat{T}^k 1_{Y_k}\}_{n\geq 1}$ converges in $L^\infty(\mu)$ as $n \to \infty$, then we call the limit function $H \in L^\infty(\mu)$ as the *asymptotic entrance density* of Y. Since $(w_n^Y)^{-1} \sum_{k=0}^{n-1} \widehat{T}^k 1_{Y_k}$ is a μ-probability density function supported on Y, so is H. Let $G \in \{u \in L^1(\mu) : u \geq 0\}$. Then, H is uniformly sweeping in K steps for G if and only if there exists $N \in \mathbb{N}$ such that $\{(w_n^Y)^{-1} \sum_{k=0}^{n-1} \widehat{T}^k 1_{Y_k}\}_{n\geq N}$ is uniformly sweeping in K steps for G.

For non-negative sequences $(a(n))_{n\geq 0}$ and $(b(n))_{n\geq 0}$, we write $a(n) = o(b(n))$ $(n \to \infty)$ if, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}_0$ such that $a(n) \leq \varepsilon b(n)$ for any $n > n_0$.

LEMMA 2.1. *Assume that there exists $H \in L^\infty(\mu)$ such that*

$$\|\widehat{T}^n 1_{Y\cap\{\varphi=n\}} - \mu(Y \cap \{\varphi = n\})H\|_{L^\infty(\mu)} = o(\mu(Y \cap \{\varphi = n\})) \quad (n \to \infty).$$

Then, H is the asymptotic entrance density of Y.

Proof. Fix $\varepsilon > 0$ arbitrarily. Take $n_0 \in \mathbb{N}_0$ large enough so that

$$\|\widehat{T}^n 1_{Y\cap\{\varphi=n\}} - \mu(Y\cap\{\varphi=n\})H\|_{L^\infty(\mu)} \le \varepsilon\mu(Y\cap\{\varphi=n\}) \quad \text{for any } n > n_0. \tag{2.4}$$

By (2.1) and (2.2),

$$\begin{aligned}
&\left\| \sum_{k=0}^{n-1} \widehat{T}^k 1_{Y_k} - w_n^Y H \right\|_{L^\infty(\mu)} \\
&\quad = \left\| \sum_{k=0}^{n-1} \sum_{m=k+1}^{\infty} (\widehat{T}^m 1_{Y\cap\{\varphi=m\}} - \mu(Y\cap\{\varphi=m\})H) \right\|_{L^\infty(\mu)} \\
&\quad \le \sum_{k=0}^{n_0-1} \sum_{m=k+1}^{n_0} \|\widehat{T}^m 1_{Y\cap\{\varphi=m\}} - \mu(Y\cap\{\varphi=m\})H\|_{L^\infty(\mu)} \\
&\qquad + \sum_{k=0}^{n-1} \sum_{m=\max\{k,n_0\}+1}^{\infty} \|\widehat{T}^m 1_{Y\cap\{\varphi=m\}} - \mu(Y\cap\{\varphi=m\})H\|_{L^\infty(\mu)}.
\end{aligned} \tag{2.5}$$

It follows from (2.4) and (2.2) that

$$\begin{aligned}
&\sum_{k=0}^{n-1} \sum_{m=\max\{k,n_0\}+1}^{\infty} \|\widehat{T}^m 1_{Y\cap\{\varphi=m\}} - \mu(Y\cap\{\varphi=m\})H\|_{L^\infty(\mu)} \\
&\qquad \le \varepsilon \sum_{k=0}^{n-1} \sum_{m=k+1}^{\infty} \mu(Y\cap\{\varphi=m\}) = \varepsilon w_n^Y.
\end{aligned} \tag{2.6}$$

Since $w_n^Y \to \infty (n \to \infty)$, we use (2.5) and (2.6) to obtain

$$\limsup_{n\to\infty} \left\| \frac{1}{w_n^Y} \sum_{k=0}^{n-1} \widehat{T}^k 1_{Y_k} - H \right\|_{L^\infty(\mu)} \le \varepsilon.$$

The proof is complete, since $\varepsilon > 0$ was arbitrary. □

3. *Main results*

In the following, we are going to formulate three types of large deviation estimates, which are related to already-known distributional limit theorems.

3.1. *Large deviation estimates related to the Dynkin–Lamperti generalized arcsine law.*

Let $u : X \to [0, \infty)$ be a non-negative, μ-integrable function. Recall μ_u is defined as the μ-absolutely continuous finite measure on X with density function u with respect to μ, that is,

$$\mu_u(A) = \int_A u(x)\, d\mu(x) \quad (A \in \mathcal{A}).$$

Let $Z_n^Y(x)$ denote the last time the orbit $\{T^k x\}_{k\ge 0}$ arrives in Y until time n, that is,

$$Z_n^Y(x) = \max\{k \le n : T^k x \in Y\} \quad (n \in \mathbb{N}_0,\ x \in X).$$

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THEOREM 3.1. *Suppose the following conditions are satisfied:*

(A1) $w_n^Y \sim n^{1-\alpha}\ell(n)(n \to \infty)$ *for some* $\alpha \in (0, 1)$ *and some positive, measurable function* $\ell : (0, \infty) \to (0, \infty)$ *slowly varying at* ∞*;*

(A2) *there exists* $N \in \mathbb{N}$ *such that* $\{(1/w_n^Y) \sum_{k=0}^{n-1} \widehat{T}^k 1_{Y_k}\}_{n \geq N}$ *is uniformly sweeping for* 1_Y*;*

(A3) *there exists* $H \in L^\infty(\mu)$ *such that*

$$\lim_{n \to \infty} \frac{1}{w_n^Y} \sum_{k=0}^{n-1} \widehat{T}^k 1_{Y_k} = H \quad \textit{in } L^\infty(\mu).$$

Let $\{c(n)\}_{n \geq 0}$ *be a positive sequence satisfying*

$$c(n) \to 0, \quad c(n)n \to \infty \quad (n \to \infty). \tag{3.1}$$

Then,

$$\mu_H\left(\frac{Z_n^Y}{n} \leq c(n)\right) \sim \frac{\sin(\pi\alpha)}{\pi\alpha} \frac{c(n)^\alpha \ell(n)}{\ell(c(n)n)} \quad (n \to \infty). \tag{3.2}$$

The proof of Theorem 3.1 will be given in §5.

Remark 3.2. Under the setting of Theorem 3.1, fix $\varepsilon \in (0, \alpha)$ arbitrarily. Then, the Potter bounds for slowly varying functions [6, Theorem 1.5.6] implies that there exist $C_\varepsilon \geq 1$ and $N_\varepsilon \in \mathbb{N}$ such that, for any $n \geq N_\varepsilon$, we have $c(n) \leq 1$ and

$$C_\varepsilon^{-1} c(n)^\varepsilon \leq \frac{\ell(n)}{\ell(c(n)n)} \leq C_\varepsilon c(n)^{-\varepsilon}.$$

Thus, the right-hand side of (3.2) converges to 0 as $n \to \infty$.

Remark 3.3. If we further assume

$$\frac{\ell(n)}{\ell(c(n)n)} \to 1 \quad (n \to \infty), \tag{3.3}$$

then we obtain

$$\mu_H\left(\frac{Z_n^Y}{n} \leq c(n)\right) \sim \frac{\sin(\pi\alpha)}{\pi\alpha} c(n)^\alpha \quad (n \to \infty). \tag{3.4}$$

Remark 3.4. Fix any positive, measurable function $\ell : (0, \infty) \to (0, \infty)$ slowly varying at ∞. Then, there exists a non-increasing, positive sequence $\{c(n)\}_{n \geq 0}$ satisfying (3.1) and (3.3). Indeed, we use the uniform convergence theorem for slowly varying functions [6, Theorem 1.2.1] to take a strictly increasing sequence $\{M_N\}_{N \geq 1} \subset \mathbb{N}$ so that

$$\sup\left\{\left|\frac{\ell(t)}{\ell(\lambda t)} - 1\right| : \lambda \in [N^{-1}, 1],\ t \geq M_N\right\} \leq \frac{1}{N} \quad (N \in \mathbb{N}).$$

Set $c(n) = 1$ for $0 \leq n < M_1$ and $c(n) = N^{-1/2}$ for $M_N \leq n < M_{N+1} (N \in \mathbb{N})$. It is easy to check that $\{c(n)\}_{n \geq 0}$ satisfies (3.1) and (3.3).

Remark 3.5. (Comparison with the Dynkin–Lamperti generalized arcsine law) Let us recall the Dynkin–Lamperti generalized arcsine law for waiting times. Assume conditions

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(A1) and (A3) of Theorem 3.1 are fulfilled. Then, for any μ-absolutely continuous probability measure ν on $(X, \mathcal{A})$ and any $0 \leq t \leq 1$, we have

$$\lim_{n\to\infty} \nu\left(\frac{Z_n^Y}{n} \leq t\right) = \frac{\sin(\pi\alpha)}{\pi} \int_0^t \frac{ds}{s^{1-\alpha}(1-s)^\alpha}, \tag{3.5}$$

which follows from [33, Theorem 2.3]. See also [27, Theorem 3.3] and [11, Theorem 2.1]. The limit is the distribution function of the Beta$(\alpha, 1-\alpha)$-distribution. In the case where $\alpha = 1/2$, this distribution is the usual arcsine distribution. We emphasize that the right-hand side of (3.5) does not depend on the choice of ν because of the ergodicity of T. Note that the right-hand side of (3.4) is asymptotically the same as the right-hand side of (3.5) with $t = c(n)$, as $n \to \infty$. Nevertheless, (3.4) does not follow from (3.5) directly. We do not know whether (3.2) remains valid in the case μ_H is replaced by other suitable probability measures ν except for $\mu_{\widehat{T}^k H}$ (see also Corollaries 3.6 and 3.7, Theorem 3.8, and Remark 3.9). The difficulty is that the L^1-characterization of the ergodicity [33, Theorem 3.1] is inadequate for this purpose, although it is significant for (3.5).

In the following two corollaries, we will consider what happens when we replace μ_H in the left-hand side of (3.2) by other finite measures.

COROLLARY 3.6. *Let $k \in \mathbb{N}_0$. Under the setting of Theorem 3.1,*

$$\mu_H\left(\frac{Z_n^Y \circ T^k}{n} \leq c(n)\right)\left(= \mu_{\widehat{T}^k H}\left(\frac{Z_n^Y}{n} \leq c(n)\right)\right) \sim \frac{\sin(\pi\alpha)}{\pi\alpha}\frac{c(n)^\alpha \ell(n)}{\ell(c(n)n)} \quad (n \to \infty).$$

Proof of Corollary 3.6 by using Theorem 3.1. Note that $Z_n^Y \circ T^k = \max\{0,\ Z_{n+k}^Y - k\}$ and hence, $\{Z_n^Y \circ T^k \leq nc(n)\} = \{Z_{n+k}^Y \leq nc(n) + k\}$. In addition,

$$\frac{nc(n)+k}{n+k} \to 0 \quad \text{and} \quad nc(n)+k \to \infty (n \to \infty).$$

In other words, (3.1) is satisfied with n and $c(n)$ replaced by $n+k$ and $(nc(n)+k)/(n+k)$, respectively. Therefore, Theorem 3.1 yields

$$\begin{aligned}
\mu_H\left(\frac{Z_n^Y \circ T^k}{n} \leq c(n)\right) &= \mu_H\left(\frac{Z_{n+k}^Y}{n} \leq c(n) + \frac{k}{n}\right) \\
&= \mu_H\left(\frac{Z_{n+k}^Y}{n+k} \leq \frac{nc(n)+k}{n+k}\right) \\
&\sim \frac{\sin(\pi\alpha)}{\pi\alpha}\left(\frac{c(n)n+k}{n+k}\right)^\alpha \frac{\ell(n+k)}{\ell(c(n)n+k)} \\
&\sim \frac{\sin(\pi\alpha)}{\pi\alpha}\frac{c(n)^\alpha \ell(n)}{\ell(c(n)n)} \quad (n \to \infty).
\end{aligned}$$

Here, we used the uniform convergence theorem for slowly varying functions. This completes the proof. □

COROLLARY 3.7. *Suppose that conditions (A1), (A2), (A3) in Theorem 3.1 are fulfilled. Let $G \in \{u \in L^1(\mu) : u \geq 0\}$. Then, the following assertions hold.*

(1) *Assume that G is uniformly sweeping for 1_Y. Then, there exists $C_1 \in (0, \infty)$ such that, for any positive sequence $\{c(n)\}_{n\geq 0}$ satisfying (3.1), we have*

$$C_1 \leq \liminf_{n\to\infty} \frac{\ell(c(n)n)}{c(n)^\alpha \ell(n)} \mu_G\bigg(\frac{Z_n^Y}{n} \leq c(n)\bigg). \tag{3.6}$$

(2) *Assume that 1_Y is uniformly sweeping for G. Then, there exists $C_2 \in (0, \infty)$ such that, for any positive sequence $\{c(n)\}_{n\geq 0}$ satisfying (3.1), we have*

$$\limsup_{n\to\infty} \frac{\ell(c(n)n)}{c(n)^\alpha \ell(n)} \mu_G\bigg(\frac{Z_n^Y}{n} \leq c(n)\bigg) \leq C_2. \tag{3.7}$$

Proof of Corollary 3.7 by using Theorem 3.1. (1) By the assumption, G is also uniformly sweeping for H. Take $K \in \mathbb{N}$ so large that $\sum_{k=0}^{K-1} \widehat{T}^k G \geq K^{-1} H$, a.e. Let $k \in \{0, 1, \ldots, K\}$. Note that $Z_n^Y \circ T^k + K \geq Z_n^Y$ and hence,

$$\mu_G\bigg(\frac{Z_n^Y}{n} \leq c(n)\bigg) \geq \mu_G\bigg(\frac{Z_n^Y \circ T^k}{n} \leq c(n) - \frac{K}{n}\bigg).$$

Therefore, Theorem 3.1 yields that

$$\begin{aligned} \mu_G\bigg(\frac{Z_n^Y}{n} \leq c(n)\bigg) &\geq \frac{1}{K} \sum_{k=0}^{K-1} \mu_G\bigg(\frac{Z_n^Y \circ T^k}{n} \leq c(n) - \frac{K}{n}\bigg) \\ &\geq \frac{1}{K^2} \mu_H\bigg(\frac{Z_n^Y}{n} \leq c(n) - \frac{K}{n}\bigg) \\ &\sim \frac{\sin(\pi\alpha)}{K^2\pi\alpha} \frac{c(n)^\alpha \ell(n)}{\ell(c(n)n)} \quad (n \to \infty), \end{aligned}$$

which implies the desired result.

(2) By the assumption, H is also uniformly sweeping for G. Take $K \in \mathbb{N}$ so large that $G \leq K \sum_{k=0}^{K-1} \widehat{T}^k H$, a.e. Then, we use Corollary 3.6 to obtain

$$\begin{aligned} \mu_G\bigg(\frac{Z_n^Y}{n} \leq c(n)\bigg) &\leq K \sum_{k=0}^{K-1} \mu_H\bigg(\frac{Z_n^Y \circ T^k}{n} \leq c(n)\bigg) \\ &\sim K^2 \frac{\sin(\pi\alpha)}{\pi\alpha} \frac{c(n)^\alpha \ell(n)}{\ell(c(n)n)} \quad (n \to \infty), \end{aligned}$$

as desired. □

We will also give the proof of the following theorem in §5.

THEOREM 3.8. *Suppose that conditions (A1) and (A2) of Theorem 3.1 are fulfilled. Let $G \in \{u \in L^1(\mu) : u \geq 0\}$. Then, assertion (2) of Corollary 3.7 holds.*

In other words, assertion (2) of Corollary 3.7 remains valid without assuming the existence of the asymptotic entrance density H. The reader may expect assertion (1) also remains valid under a similar setting, but we do not know whether it is true. The reason is that $\mu_G(Z_n^Y/n \leq c(n))$ can be bounded above but not below by double Laplace transform of Z_n^Y, as we shall see in the proof of Theorem 3.8.

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Remark 3.9. Let $\alpha \in (0, 1)$ and let $\{c(n)\}_{n\geq 0}$ be a non-increasing, $(0, 1]$-valued sequence satisfying $c(0) = 1$ and $c(n) \to 0 (n \to \infty)$. Then, there exists a μ-probability density function G such that

$$\limsup_{n\to\infty} \frac{1}{c(n)^\alpha} \mu_G\left(\frac{Z_n^Y}{n} \leq c(n)\right) = \infty, \tag{3.8}$$

as we shall see below. Indeed, let

$$N_0 = 0 \quad \text{and} \quad N_k = \min\{n > N_{k-1} : \mu(Y \cap \{\varphi = n\}) > 0\} \quad (k \in \mathbb{N}).$$

Then, $\{N_k\}_{k\geq 0} \subset \mathbb{N}_0$ is strictly increasing. We define the μ-probability density function $G : X \to [0, \infty)$ as

$$G = \begin{cases} (c(N_{k-1})^{\alpha/2} - c(N_k)^{\alpha/2})/\mu(Y \cap \{\varphi = N_k\}) & \text{on } Y \cap \{\varphi = N_k\} (k \in \mathbb{N}), \\ 0 & \text{otherwise.} \end{cases}$$

Then, $\mu_G(\varphi > N_k) = c(N_k)^{\alpha/2}$ and hence,

$$\frac{1}{c(N_k)^\alpha} \mu_G\left(\frac{Z_{N_k}^Y}{N_k} \leq c(N_k)\right) \geq \frac{1}{c(N_k)^\alpha} \mu_G(\varphi > N_k) = c(N_k)^{-\alpha/2} \to \infty \quad (k \to \infty),$$

which implies (3.8).

3.2. *Large deviation estimates related to the Darling–Kac theorem.* For $A \in \mathcal{A}$, let $S_n^A(x)$ denote the occupation time in A of the orbit $\{T^k x\}_{k\geq 0}$ from time 1 to time n, that is,

$$S_n^A(x) = \sum_{k=1}^{n} 1_A(T^k x) \quad (n \in \mathbb{N}_0,\ x \in X).$$

In the following, we consider occupation times in a set of finite measure.

THEOREM 3.10. *Suppose the following conditions are satisfied:*

(B1) $w_n^Y \sim n^{1-\alpha}\ell(n) (n \to \infty)$ *for some* $\alpha \in (0, 1)$ *and some positive, measurable function* $\ell : (0, \infty) \to (0, \infty)$ *slowly varying at* ∞*;*

(B2) $\{(1/w_n^Y) \sum_{k=0}^{n-1} \widehat{T}^k 1_{Y_k}\}_{n\geq 1}$ *is* $L^\infty(\mu)$*-bounded.*

For $t > 0$*, set*

$$a(t) = \frac{t}{\Gamma(1 + \alpha) Q^Y(t^{-1})} \sim \frac{t^\alpha}{\Gamma(1 + \alpha)\Gamma(2 - \alpha)\ell(t)} \quad (t \to \infty).$$

Let $\{\widetilde{c}(n)\}_{n\geq 0}$ *be a positive sequence satisfying*

$$\widetilde{c}(n) \to 0 \quad \textit{and} \quad \widetilde{c}(n)a(n) \to \infty \quad (n \to \infty). \tag{3.9}$$

Then,

$$\mu_{1_Y}\left(\frac{S_n^Y}{a(n)} \leq \widetilde{c}(n)\right) \sim \frac{\sin(\pi\alpha)}{\pi\alpha} \widetilde{c}(n) \quad (n \to \infty). \tag{3.10}$$

The proof of Theorem 3.10 will be given in §6.

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Remark 3.11. (Comparison with the Darling–Kac theorem) Let us recall the Darling–Kac theorem. Set

$$F(t) = \frac{1}{\pi\alpha}\int_0^t \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} \sin(\pi\alpha k)\Gamma(1+\alpha k)s^{k-1}\,ds \quad (t \geq 0),$$

which is the distribution function of the Mittag–Leffler distribution of order α with Laplace transform

$$\int_0^{\infty} e^{-\lambda t}\,dF(t) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{\Gamma(1+\alpha k)} \quad (\lambda \in \mathbb{R}).$$

See [7, 15] for the details. As a special case, the Mittag–Leffler distribution of order $1/2$ is the half-normal distribution with mean $2/\sqrt{\pi}$. Suppose conditions (A1), (A2), (A3) of Theorem 3.1 are satisfied. Then, for any μ-absolutely continuous probability measure ν on X and for any $t \geq 0$, we have

$$\lim_{n\to\infty} \nu\biggl(\frac{S_n^Y}{a(n)} \leq t\biggr) = F\biggl(\frac{t}{\Gamma(1+\alpha)\mu(Y)}\biggr), \tag{3.11}$$

which follows from [27, Theorem 3.1]. See also [33, Theorem 2.1] and [11, Theorem 2.1]. Note that

$$F\biggl(\frac{\widetilde{c}(n)}{\Gamma(1+\alpha)\mu(Y)}\biggr) \sim \frac{\sin(\pi\alpha)}{\pi\alpha}\frac{\widetilde{c}(n)}{\mu(Y)} \quad (n \to \infty).$$

Nevertheless, (3.10) does not follow from (3.11) directly.

COROLLARY 3.12. *Let $k \in \mathbb{N}_0$. Under the setting of Theorem 3.10,*

$$\mu_{1_Y}\biggl(\frac{S_n^Y \circ T^k}{a(n)} \leq \widetilde{c}(n)\biggr)\biggl(= \mu_{\widehat{T}^k 1_Y}\biggl(\frac{S_n^Y}{a(n)} \leq \widetilde{c}(n)\biggr)\biggr) \sim \frac{\sin(\pi\alpha)}{\pi\alpha}\widetilde{c}(n) \quad (n \to \infty).$$

Proof of Corollary 3.12 by using Theorem 3.10. Since $|S_n^Y - S_n^Y \circ T^k| \leq k$, we see that

$$\mu_{1_Y}\biggl(\frac{S_n^Y}{a(n)} \leq \widetilde{c}(n) - \frac{k}{a(n)}\biggr) \leq \mu_{1_Y}\biggl(\frac{S_n^Y \circ T^k}{a(n)} \leq \widetilde{c}(n)\biggr) \leq \mu_{1_Y}\biggl(\frac{S_n^Y}{a(n)} \leq \widetilde{c}(n) + \frac{k}{a(n)}\biggr).$$

By Theorem 3.10,

$$\mu_{1_Y}\biggl(\frac{S_n^Y}{a(n)} \leq \widetilde{c}(n) \pm \frac{k}{a(n)}\biggr) \sim \frac{\sin(\pi\alpha)}{\pi\alpha}\widetilde{c}(n) \quad (n \to \infty),$$

as desired. □

COROLLARY 3.13. *Suppose that conditions (B1) and (B2) of Theorem 3.10 are fulfilled. Let $G \in \{u \in L^1(\mu) : u \geq 0\}$. Then, the following assertions hold.*

(1) *Assume that G is uniformly sweeping for 1_Y. Then, there exists $C_1 \in (0, \infty)$ such that, for any positive sequence $\{\widetilde{c}(n)\}_{n\geq 0}$ satisfying (3.9), we have*

$$C_1 \leq \liminf_{n\to\infty} \frac{1}{\widetilde{c}(n)}\mu_G\biggl(\frac{S_n^Y}{a(n)} \leq \widetilde{c}(n)\biggr). \tag{3.12}$$

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(2) *Assume that* 1_Y *is uniformly sweeping for G. Then, there exists* $C_2 \in (0, \infty)$ *such that, for any positive sequence* $\{\widetilde{c}(n)\}_{n\geq 0}$ *satisfying (3.9), we have*

$$\limsup_{n\to\infty} \frac{1}{\widetilde{c}(n)} \mu_G\bigg(\frac{S_n^Y}{a(n)} \leq \widetilde{c}(n)\bigg) \leq C_2. \tag{3.13}$$

Proof of Corollary 3.13 by using Theorem 3.10. (1) Take $K \in \mathbb{N}$ so large that $\sum_{k=0}^{K-1} \widehat{T}^k G \geq K^{-1} 1_Y$, a.e. Let $k \in \{0, 1, 2, \ldots, K\}$. Then, $S_n^Y \leq S_n^Y \circ T^k + k \leq S_n^Y \circ T^k + K$ and hence,

$$\mu_G\bigg(\frac{S_n^Y}{a(n)} \leq \widetilde{c}(n)\bigg) \geq \mu_G\bigg(\frac{S_n^Y \circ T^k}{a(n)} \leq \widetilde{c}(n) - \frac{K}{a(n)}\bigg).$$

Note that

$$\widetilde{c}(n) - \frac{K}{a(n)} \to 0 \quad \text{and} \quad \bigg(\widetilde{c}(n) - \frac{K}{a(n)}\bigg)a(n) \to \infty \quad (n \to \infty).$$

Hence, Theorem 3.10 with $\widetilde{c}(n)$ replaced by $\widetilde{c}(n) - (K/a(n))$ implies

$$\begin{aligned}
\mu_G\bigg(\frac{S_n^Y}{a(n)} \leq \widetilde{c}(n)\bigg) &\geq K^{-1} \sum_{k=0}^{K-1} \mu_G\bigg(\frac{S_n^Y \circ T^k}{a(n)} \leq \widetilde{c}(n) - \frac{K}{a(n)}\bigg) \\
&\geq K^{-2} \mu_{1_Y}\bigg(\frac{S_n^Y}{a(n)} \leq \widetilde{c}(n) - \frac{K}{a(n)}\bigg) \\
&\sim \frac{\sin(\pi\alpha)}{K^2\pi\alpha}\widetilde{c}(n) \quad (n \to \infty),
\end{aligned}$$

which implies the desired result.

(2) Take $K \in \mathbb{N}$ large enough that $G \leq K \sum_{k=0}^{K-1} \widehat{T}^k 1_Y$, a.e. Then, we use Corollary 3.12 to obtain

$$\begin{aligned}
\mu_G\bigg(\frac{S_n^Y}{a(n)} \leq \widetilde{c}(n)\bigg) &\leq K \sum_{k=0}^{K-1} \mu_{1_Y}\bigg(\frac{S_n^Y \circ T^k}{a(n)} \leq \widetilde{c}(n)\bigg) \\
&\sim K^2 \frac{\sin(\pi\alpha)}{\pi\alpha}\widetilde{c}(n) \quad (n \to \infty),
\end{aligned}$$

as desired. □

3.3. *Large deviation estimates related to the Lamperti generalized arcsine law.* In the following, we consider occupation times in sets of infinite measure under certain additional assumptions. Fix disjoint sets $Y, A_1, A_2, \ldots, A_d \in \mathcal{A}$ with $d \in \mathbb{N}$, $d \geq 2$, $X = Y \cup \bigcup_{i=1}^d A_i$, $0 < \mu(Y) < \infty$, and $\mu(A_i) = \infty (i = 1, 2, \ldots, d)$. We assume Y *dynamically separates* $A_1, A_2, \ldots, A_d$ *(under the action of* T*)*, that is, $A_i \cap T^{-1}A_j = \emptyset$ whenever $i \neq j$. Then, the condition $[x \in A_i$ and $T^n x \in A_j (i \neq j)]$ implies $n \geq 2$ and the existence of $k = k(x) \in \{1, \ldots, n-1\}$ for which $T^k x \in Y$. As shown in [27, (6.6)],

$$1_{Y_n \cap A_i} = \sum_{k>n} \widehat{T}^{k-n} 1_{Y \cap (T^{-1}A_i) \cap \{\varphi = k\}} \quad \text{a.e.} \quad (n \in \mathbb{N},\ i = 1, 2, \ldots, d), \tag{3.14}$$

and $\mu(Y_n \cap A_i) = \mu(Y \cap (T^{-1}A_i) \cap \{\varphi > n\})$ $(n \in \mathbb{N}, i = 1, 2, \ldots, d)$. Let $\{w_n^{Y,A_i}\}_{n\geq 0}$ denote the *wandering rate of Y starting from* A_i, which is given by

$$\begin{aligned} w_n^{Y,A_i} &= \mu\bigg(\bigcup_{k=0}^{n-1}(T^{-k}Y) \cap A_i\bigg) = \sum_{k=0}^{n-1} \mu(Y_k \cap A_i) = \sum_{k=0}^{n-1} \int_Y \widehat{T}^k 1_{Y_k \cap A_i}\, d\mu \\ &= \sum_{k=1}^{n-1} \mu(Y \cap (T^{-1}A_i) \cap \{\varphi > k\}) \quad (n \in \mathbb{N}_0,\ i = 1, 2, \ldots, d). \end{aligned} \tag{3.15}$$

Since T is a CEMPT, we see $\bigcup_{n\geq 0}(T^{-n}Y) \cap A_i = A_i$, a.e. and hence, $w_n^{Y,A_i} \to \infty (n \to \infty,\ i = 1, \ldots, d)$. We write $Q^{Y,A_i}(s)$ for Laplace transform of $\{w_{n+1}^{Y,A_i} - w_n^{Y,A_i}\}_{n\geq 0}$:

$$\begin{aligned} Q^{Y,A_i}(s) &= \sum_{n\geq 0} e^{-ns}(w_{n+1}^{Y,A_i} - w_n^{Y,A_i}) \\ &= \sum_{n\geq 1} e^{-ns}\mu(Y \cap (T^{-1}A_i) \cap \{\varphi > n\}) \quad (s > 0,\ i = 1, 2, \ldots, d). \end{aligned}$$

Then, $w_n^Y = \mu(Y) + \sum_{i=1}^d w_n^{Y,A_i}$ and $Q^Y(s) = \mu(Y) + \sum_{i=1}^d Q^{Y,A_i}(s)$. In addition, $0 < Q^{Y,A_i}(s) < \infty$ and $Q^{Y,A_i}(s) \to \infty (s \to 0+,\ i = 1, \ldots, d)$. Let $\alpha, \beta_1, \beta_2, \ldots, \beta_d \in (0, 1)$ with $\sum_{i=1}^d \beta_i = 1$ and let $\ell : (0, \infty) \to (0, \infty)$ be a positive, measurable function slowly varying at ∞. By Karamata's Tauberian theorem, the condition

$$w_n^{Y,A_i} \sim \beta_i n^{1-\alpha}\ell(n) \quad (n \to \infty,\ i = 1, 2, \ldots, d)$$

is equivalent to

$$Q^{Y,A_i}(s) \sim \Gamma(2 - \alpha)\beta_i s^{-1+\alpha}\ell(s^{-1}) \quad (s \to 0+,\ i = 1, 2, \ldots, d). \tag{3.16}$$

The following lemma will be used in §8.

LEMMA 3.14. *Fix $i \in \{1, 2, \ldots, d\}$. Assume that there exists $H^{(i)} \in L^\infty(\mu)$ such that*

$$\begin{aligned} &\|\widehat{T}^n 1_{Y\cap(T^{-1}A_i)\cap\{\varphi=n\}} - \mu(Y \cap (T^{-1}A_i) \cap \{\varphi = n\})H^{(i)}\|_{L^\infty(\mu)} \\ &\quad = o(\mu(Y \cap (T^{-1}A_i) \cap \{\varphi = n\})) \quad (n \to \infty). \end{aligned}$$

Then,

$$\lim_{n\to\infty} \bigg(\frac{1}{w_n^{Y,A_i}} \sum_{k=0}^{n-1} \widehat{T}^k 1_{Y_k\cap A_i}\bigg) = H^{(i)} \quad \text{in } L^\infty(\mu).$$

We omit the proof of Lemma 3.14, since it is almost the same as that of Lemma 2.1.

THEOREM 3.15. *Suppose the following conditions are satisfied.*

(C1) *For $d \in \mathbb{N}$ with $d \geq 2$, let $A_1, \ldots, A_d \in \mathcal{A}$ with $X = Y \cup \bigcup_{i=1}^d A_i$ and $\mu(A_i) = \infty (i = 1, 2, \ldots, d)$. In addition, $Y, A_1, A_2, \ldots, A_d$ are disjoint sets, and Y dynamically separates $A_1, A_2, \ldots, A_d$.*

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(C2) $w_n^{Y,A_i} \sim \beta_i n^{1-\alpha}\ell(n)(n \to \infty,\ i = 1, 2, \ldots, d)$ *for some* $\alpha, \beta_1, \beta_2, \ldots, \beta_d \in (0, 1)$ *with* $\sum_{i=1}^{d} \beta_i = 1$ *and for some positive, measurable function* $\ell : (0, \infty) \to (0, \infty)$ *slowly varying at* ∞.

(C3) $\{(1/w_n^{Y,A_d}) \sum_{k=0}^{n-1} \widehat{T}^k 1_{Y_k \cap A_d}\}_{n\geq 2}$ *is* $L^\infty(\mu)$*-bounded.*

(C4) *There exists* $N \geq 2$ *such that* $\{1/w_n^{Y,A_i} \sum_{k=0}^{n-1} \widehat{T}^k 1_{Y_k \cap A_i}\}_{n\geq N,\, i=1,\ldots,d-1}$ *is uniformly sweeping for* 1_Y.

(C5) *There exist* $H^{(1)}, \ldots, H^{(d-1)} \in L^\infty(\mu)$ *such that*

$$\lim_{n\to\infty} \left(\frac{1}{w_n^{Y,A_i}} \sum_{k=0}^{n-1} \widehat{T}^k 1_{Y_k\cap A_i} \right) = H^{(i)} \quad \text{in } L^\infty(\mu) \quad (i = 1, \ldots, d-1).$$

Let $\lambda_1, \ldots, \lambda_{d-1} \in (0, \infty)$ *and let* $\{c(n)\}_{n\geq 0}$ *be a positive sequence satisfying (3.1). Set* $\widetilde{H} = \sum_{i=1}^{d-1} \beta_i \lambda_i^\alpha H^{(i)}$. *Then,*

$$\mu_{\widetilde{H}}\left(\frac{1}{n} \sum_{i=1}^{d-1} \lambda_i S_n^{A_i} \leq c(n) \right) \sim \beta_d \frac{\sin(\pi\alpha)}{\pi\alpha} \frac{c(n)^\alpha \ell(n)}{\ell(c(n)n)} \quad (n \to \infty). \tag{3.17}$$

The proof of Theorem 3.15 will be given in §7.

Remark 3.16. Under the setting of Theorem 3.15 with $d = 2$ and $\lambda_1 = 1$, let us assume (3.3). Then, we obtain

$$\mu_{H^{(1)}}\left(\frac{S_n^{A_1}}{n} \leq c(n) \right) \sim \frac{\beta_2}{\beta_1} \frac{\sin(\pi\alpha)}{\pi\alpha} c(n)^\alpha \quad (n \to \infty). \tag{3.18}$$

Remark 3.17. If $\lambda_i = 0$ for some i, then (3.17) does not remain valid. For example, let $d \geq 3$, $\lambda_1, \ldots, \lambda_{d-2} \in (0, \infty)$, and $\lambda_{d-1} = 0$. Then,

$$\mu_{\widetilde{H}}\left(\frac{1}{n} \sum_{i=1}^{d-2} \lambda_i S_n^{A_i} \leq c(n) \right) \sim (\beta_{d-1} + \beta_d) \frac{\sin(\pi\alpha)}{\pi\alpha} \frac{c(n)^\alpha \ell(n)}{\ell(c(n)n)} \quad (n \to \infty),$$

which follows from Theorem 3.15.

Remark 3.18. (Comparison with the Lamperti generalized arcsine law) Let us recall the Lamperti generalized arcsine law for occupation times. Suppose conditions (C1), (C2), (C5) of Theorem 3.15 and condition (A2) of Theorem 3.1 are fulfilled with $d = 2$. Set $b = \beta_2/\beta_1$. Then, for any μ-absolutely continuous probability measure ν on $(X, \mathcal{A})$ and for any $0 \leq t \leq 1$, we have

$$\begin{aligned} \lim_{n\to\infty} \nu\left(\frac{S_n^{A_1}}{n} \leq t \right) &= \frac{b \sin(\pi\alpha)}{\pi} \int_0^t \frac{s^{\alpha-1}(1-s)^{\alpha-1}\, ds}{b^2 s^{2\alpha} + 2bs^\alpha(1-s)^\alpha \cos(\pi\alpha) + (1-s)^{2\alpha}} \\ &= \frac{1}{\pi\alpha} \operatorname{arccot}\left(\frac{(1-t)^\alpha}{b \sin(\pi\alpha) t^\alpha} + \cot(\pi\alpha) \right), \end{aligned} \tag{3.19}$$

as shown in [**33**, Theorem 2.2]. See also [**27**, Theorem 3.2] and [**18**, Theorem 2.7]. The limit is the distribution function of the Lamperti generalized arcsine distribution of parameter (α, β_1). In the case where $\alpha = \beta_1 = \beta_2 = 1/2$, this distribution is the usual arcsine distribution. Note that the right-hand side of (3.18) is asymptotically the same as the right-hand side of (3.19) with $t = c(n)$. Nevertheless, (3.18) does not follow from (3.19) directly.

The proofs of the following two corollaries are almost the same as those of Corollaries 3.6, 3.7, 3.12, and 3.13, so we omit them.

COROLLARY 3.19. *Let $k \in \mathbb{N}_0$. Under the setting of Theorem 3.15,*

$$\mu_{\widetilde{H}}\left(\frac{1}{n}\sum_{i=1}^{d-1}\lambda_i S_n^{A_i} \circ T^k \le c(n)\right) \sim \beta_d \frac{\sin(\pi\alpha)}{\pi\alpha}\frac{c(n)^\alpha \ell(n)}{\ell(c(n)n)} \quad (n \to \infty).$$

COROLLARY 3.20. *Suppose that conditions (C1)–(C5) of Theorem 3.15 are satisfied. Let $G \in \{u \in L^1(\mu) : u \ge 0\}$ and $\lambda_1, \ldots, \lambda_{d-1} \in (0, \infty)$. Then, the following assertions hold.*

(1) *Assume that G is uniformly sweeping for 1_Y. Then, there exists $C_1 \in (0, \infty)$ such that, for any positive sequence $\{c(n)\}_{n\ge 0}$ satisfying (3.1), we have*

$$C_1 \le \liminf_{n\to\infty} \frac{\ell(c(n)n)}{c(n)^\alpha \ell(n)}\mu_G\left(\frac{1}{n}\sum_{i=1}^{d-1}\lambda_i S_n^{A_i} \le c(n)\right).$$

(2) *Assume that 1_Y is uniformly sweeping for G. Then, there exists $C_2 \in (0, \infty)$ such that, for any positive sequence $\{c(n)\}_{n\ge 0}$ satisfying (3.1), we have*

$$\limsup_{n\to\infty} \frac{\ell(c(n)n)}{c(n)^\alpha \ell(n)}\mu_G\left(\frac{1}{n}\sum_{i=1}^{d-1}\lambda_i S_n^{A_i} \le c(n)\right) \le C_2.$$

Assertion (2) remains valid even if we drop condition (C5) of Theorem 3.15.

THEOREM 3.21. *Suppose that conditions (C1)–(C4) of Theorem 3.15 are satisfied. Let $G \in \{u \in L^1(\mu) : u \ge 0\}$ and $\lambda_1, \ldots, \lambda_{d-1} \in (0, \infty)$. Then, assertion (2) of Corollary 3.20 holds.*

We will give the proof of Theorem 3.21 in §7.

4. *Analytical tools*

In this section, we prove lemmas needed in the following.

LEMMA 4.1. *Let $f_n : (0, \infty) \to [0, \infty) (n \in \mathbb{N} \cup \{\infty\})$ be non-increasing functions. Assume there exists a non-empty open interval $I \subset (0, \infty)$ such that for any $q \in I$,*

$$\lim_{n\to\infty} \int_0^\infty e^{-qu} f_n(u)\, du = \int_0^\infty e^{-qu} f_\infty(u)\, du < \infty.$$

Then, $\lim_{n\to\infty} f_n(u) = f_\infty(u)$ for all continuity points $u \in (0, \infty)$ of f_∞.

See [**18**, Lemma 3.2] for the proof of Lemma 4.1.

LEMMA 4.2. *Fix a constant $C > 0$. Let $S_n : X \to [0, Cn] (n \in \mathbb{N}_0)$ be measurable functions, and let $\lambda : (0, \infty) \to [0, \infty)$ be a non-negative function with $\lambda(t) \to 0 (t \to \infty)$.*

Suppose $\nu_t (t > 0)$ are non-zero finite measures on X. Then, for any $q > 0$,

$$\sum_{n=0}^{\infty} e^{-nqt^{-1}} \int_X \exp(-\lambda(t) S_n)\, d\nu_t$$
$$\sim t \int_0^{\infty} e^{-qu} \bigg(\int_X \exp(-\lambda(t) S_{[ut]})\, d\nu_t \bigg)\, du \quad (t \to \infty). \tag{4.1}$$

Proof. For fixed $q > 0$, let $l(t)$ and $r(t)$ denote the left-hand and right-hand sides of (4.1), respectively. Note that

$$r(t) = \sum_{n=0}^{\infty} t \int_{nt^{-1}}^{(n+1)t^{-1}} e^{-qu} \bigg(\int_X \exp(-\lambda(t) S_n)\, d\nu_t \bigg)\, du, \tag{4.2}$$

and hence,

$$0 \le l(t) - r(t) \le \sum_{n=0}^{\infty} (e^{-nqt^{-1}} - e^{-(n+1)qt^{-1}}) \nu_t(X) = \nu_t(X).$$

In addition, since $0 \le S_{[ut]} \le Cut$, we have

$$r(t) \ge \nu_t(X) t \int_0^{\infty} \exp(-(q + C\lambda(t)t)u)\, du = \frac{\nu_t(X)}{qt^{-1} + C\lambda(t)}.$$

Therefore, we obtain

$$1 \le \frac{l(t)}{r(t)} \le 1 + qt^{-1} + C\lambda(t) \to 1 \quad (t \to \infty).$$

This completes the proof. □

The following three lemmas are slight extensions of [27, Lemma 4.2].

LEMMA 4.3. *Fix $t_0 > 0$ and $K \in \mathbb{N}_0$. Suppose that the following conditions are fulfilled.*

(i) $\{H_t\}_{t \ge t_0} \cup \{G\} \subset \{u \in L^1(\mu) : u \ge 0\}$. *In addition, $\{H_t\}_{t \ge t_0}$ is uniformly sweeping in K steps for G.*

(ii) $R_{n,t} : X \to (0, 1] (n \in \mathbb{N}_0,\ t > 0)$ *are measurable functions with*

$$\sup \bigg\{ \bigg\| \frac{R_{n,t} \circ T^k}{R_{n+k,t}} \bigg\|_{L^\infty(\mu)} : n, k \in \mathbb{N}_0,\ 0 \le k \le K,\ t \ge t_0 \bigg\} < \infty.$$

Then, for any $q > 0$, we have

$$\sup_{t \ge t_0} \frac{\sum_{n \ge 0} e^{-nqt^{-1}} \int_X R_{n,t} G\, d\mu}{\sum_{n \ge 0} e^{-nqt^{-1}} \int_X R_{n,t} H_t\, d\mu} < \infty.$$

Proof. By condition (i), there exists $C_0 > 0$ such that $G \le C_0 \sum_{k=0}^{K} \widehat{T}^k H_t$ a.e. for any $t \ge t_0$. Moreover, by condition (ii), we can take $C \ge C_0$ large enough so that $R_{n,t} \circ T^k \le C R_{n+k,t}$ a.e. for any $n, k \in \mathbb{N}_0$ with $0 \le k \le K$ and for any $t \ge t_0$. Then,

$$\int_X R_{n,t} G\, d\mu \le C \int_X R_{n,t} \bigg(\sum_{k=0}^{K} \widehat{T}^k H_t \bigg)\, d\mu = C \sum_{k=0}^{K} \int_X (R_{n,t} \circ T^k) H_t\, d\mu$$
$$\le C^2 \sum_{k=0}^{K} \int_X R_{n+k,t} H_t\, d\mu \quad (n \in \mathbb{N}_0,\ t \ge t_0).$$

This implies

$$\sum_{n\ge 0} e^{-nqt^{-1}} \int_X R_{n,t} G\, d\mu \le C^2 \sum_{k=0}^{K} \sum_{n \ge k} e^{-(n-k)qt^{-1}} \int_X R_{n,t} H_t\, d\mu$$
$$\le \bigg(C^2 \sum_{k=0}^{K} e^{kqt_0^{-1}} \bigg) \bigg(\sum_{n\ge 0} e^{-nqt^{-1}} \int_X R_{n,t} H_t\, d\mu \bigg) \quad (t \ge t_0),$$

which completes the proof. □

LEMMA 4.4. (Integrating transforms) *Under the assumptions of Lemma 4.3 with $G = 1_Y$, we further suppose the following condition.*

(iii) $\{H_t\}_{t \ge t_0} \cup \{H\} \subset \{u \in L^\infty(\mu) : u \ge 0$ *and u is supported on $Y\}$ and $H_t \to H$ in* $L^\infty(\mu) (t \to \infty)$.

Then, for any $q > 0$,

$$\sum_{n\ge 0} e^{-nqt^{-1}} \int_Y R_{n,t} H_t\, d\mu \sim \sum_{n\ge 0} e^{-nqt^{-1}} \int_Y R_{n,t} H\, d\mu \quad (t \to \infty).$$

Proof. Note that H_t is supported on Y. By Lemma 4.3 with $G = 1_Y$,

$$C := \sup_{t \ge t_0} \frac{\sum_{n\ge 0} e^{-nqt^{-1}} \int_Y R_{n,t}\, d\mu}{\sum_{n\ge 0} e^{-nqt^{-1}} \int_Y R_{n,t} H_t\, d\mu} = \sup_{t \ge t_0} \frac{\sum_{n\ge 0} e^{-nqt^{-1}} \int_X R_{n,t} 1_Y\, d\mu}{\sum_{n\ge 0} e^{-nqt^{-1}} \int_X R_{n,t} H_t\, d\mu} < \infty.$$

Therefore, for $t \ge t_0$,

$$\left| \frac{\sum_{n\ge 0} e^{-nqt^{-1}} \int_Y R_{n,t} H\, d\mu}{\sum_{n\ge 0} e^{-nqt^{-1}} \int_Y R_{n,t} H_t\, d\mu} - 1 \right| = \frac{| \sum_{n\ge 0} e^{-nqt^{-1}} \int_Y R_{n,t} (H - H_t)\, d\mu |}{\sum_{n\ge 0} e^{-nqt^{-1}} \int_Y R_{n,t} H_t\, d\mu}$$
$$\le C \| H - H_t \|_{L^\infty(\mu)} \to 0 \quad (t \to \infty),$$

as desired. □

LEMMA 4.5. *Let $\{v_n\}_{n\ge 0} \subset \{u \in L^\infty(\mu) : u \ge 0$ and u is supported on $Y\}$ with*

$$0 < \sum_{n\ge 0} e^{-ns} \int_Y v_n\, d\mu < \infty \quad (s > 0).$$

Let $\lambda : (0, \infty) \to (0, \infty)$ be a positive function with $\lambda(t) \to 0 (t \to \infty)$. We define $H_t \in L^1(\mu)$ as

$$H_t = \frac{\sum_{n\ge 0} e^{-n\lambda(t)} v_n}{\sum_{n\ge 0} e^{-n\lambda(t)} \int_Y v_n\, d\mu} \quad (t > 0). \tag{4.3}$$

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Then, the following assertions hold.

(1) H_t *can be represented as*

$$H_t = \frac{\sum_{n\ge 0} e^{-n\lambda(t)} \sum_{k=0}^{n} v_k}{\sum_{n\ge 0} e^{-n\lambda(t)} \sum_{k=0}^{n} \int_Y v_k \, d\mu} \quad (t > 0).$$

(2) *Assume there exists* $N \in \mathbb{N}_0$ *such that* $\sum_{k=0}^{N} \int_Y v_k \, d\mu > 0$ *and*

$$\left\{ \frac{\sum_{k=0}^{n} v_k}{\sum_{k=0}^{n} \int_Y v_k \, d\mu} \right\}_{n\ge N}$$

is uniformly sweeping in K *steps for* 1_Y. *Then, there exists* $t_0 > 0$ *such that* $\{H_t\}_{t\ge t_0}$ *is uniformly sweeping in* K *steps for* 1_Y.

(3) *Assume there exists* $C > 0$ *such that, for any* $n \in \mathbb{N}_0$,

$$\left\| \sum_{k=0}^{n} v_k \right\|_{L^\infty(\mu)} \le C \sum_{k=0}^{n} \int_Y v_k \, d\mu. \tag{4.4}$$

Then, $\|H_t\|_{L^\infty(\mu)} \le C$ *for any* $t > 0$.

(4) *Assume there exists* $H \in L^\infty(\mu)$ *such that*

$$\frac{\sum_{k=0}^{n} v_k}{\sum_{k=0}^{n} \int_Y v_k \, d\mu} \to H \quad \text{in } L^\infty(\mu) \quad (n \to \infty).$$

Then, $H_t \to H$ *in* $L^\infty(\mu)(t \to \infty)$.

Proof. (1) By Fubini's theorem,

$$\sum_{k\ge 0} e^{-k\lambda(t)} v_k = \sum_{k\ge 0} \left((1 - e^{-\lambda(t)}) \sum_{n\ge k} e^{-n\lambda(t)} \right) v_k = (1 - e^{-\lambda(t)}) \sum_{n\ge 0} e^{-n\lambda(t)} \sum_{k=0}^{n} v_k,$$

and hence,

$$H_t = \frac{\sum_{k\ge 0} e^{-k\lambda(t)} v_k}{\sum_{k\ge 0} e^{-k\lambda(t)} \int_Y v_k \, d\mu} = \frac{\sum_{n\ge 0} e^{-n\lambda(t)} \sum_{k=0}^{n} v_k}{\sum_{n\ge 0} e^{-n\lambda(t)} \sum_{k=0}^{n} \int_Y v_k \, d\mu},$$

as desired.

(2) Take $C > 0$ large enough so that

$$C \sum_{m=0}^{K} T^m \left(\frac{\sum_{k=0}^{n} v_k}{\sum_{k=0}^{n} \int_Y v_k \, d\mu} \right) \ge 1_Y \quad \text{a.e., for any } n \ge N,$$

or, equivalently,

$$\sum_{k=0}^{n} \left(C \sum_{m=0}^{K} \widehat{T}^m v_k - \int_Y v_k \, d\mu \right) \ge 0 \quad \text{a.e. on } Y, \text{ for any } n \ge N.$$

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Then,

$$\sum_{n\geq 0} e^{-n\lambda(t)} \sum_{k=0}^{n} \bigg(C \sum_{m=0}^{K} \widehat{T}^m v_k - \int_Y v_k \, d\mu \bigg) \geq \sum_{n=0}^{N-1} e^{-n\lambda(t)} \sum_{k=0}^{n} \bigg(C \sum_{m=0}^{K} \widehat{T}^m v_k - \int_Y v_k \, d\mu \bigg)$$
$$\geq - \sum_{n=0}^{N-1} \sum_{k=0}^{n} \int_Y v_k \, d\mu \quad \text{a.e. on } Y.$$

We have used the fact that $\widehat{T}^m v_k \geq 0$, a.e. Then,

$$C \sum_{m=0}^{K} \widehat{T}^m H_t - 1 = \frac{\sum_{n\geq 0} e^{-n\lambda(t)} \sum_{k=0}^{n} (C \sum_{m=0}^{K} \widehat{T}^m v_k - \int_Y v_k \, d\mu)}{\sum_{n\geq 0} e^{-n\lambda(t)} \sum_{k=0}^{n} \int_Y v_k \, d\mu}$$
$$\geq \frac{- \sum_{n=0}^{N-1} \sum_{k=0}^{n} \int_Y v_k \, d\mu}{\sum_{n\geq 0} e^{-n\lambda(t)} \sum_{k=0}^{n} \int_Y v_k \, d\mu} \quad \text{a.e. on } Y. \tag{4.5}$$

Since $\sum_{n\geq 0} e^{-n\lambda(t)} \sum_{k=0}^{n} \int_Y v_k \, d\mu \to \infty (t \to \infty)$, we can take $t_0 > 0$ large enough so that, for any $t \geq t_0$, the right-hand side of (4.5) is greater than $-1/2$. Thus, for any $t \geq t_0$, we have $2C \sum_{m=0}^{K} \widehat{T}^m H_t \geq 1_Y$ a.e., and hence, $\{H_t\}_{t\geq t_0}$ is uniformly sweeping in K steps for 1_Y.

(3) By (4.3) and (4.4),

$$\|H_t\|_{L^\infty(\mu)} = \frac{\|\sum_{n\geq 0} e^{-n\lambda(t)} \sum_{k=0}^{n} v_k\|_{L^\infty(\mu)}}{\sum_{n\geq 0} e^{-n\lambda(t)} \sum_{k=0}^{n} \int_Y v_k \, d\mu} \leq C,$$

as desired.

(4) Fix $\varepsilon > 0$ arbitrarily. Take $N \in \mathbb{N}_0$ large enough so that $\sum_{k=0}^{N} \int_Y v_k \, d\mu > 0$ and

$$\bigg\| \frac{\sum_{k=0}^{n} v_k}{\sum_{k=0}^{n} \int_Y v_k \, d\mu} - H \bigg\|_{L^\infty(\mu)} \leq \varepsilon \quad \text{for any } n \geq N,$$

or, equivalently,

$$\bigg\| \sum_{k=0}^{n} \bigg(v_k - H \int_Y v_k \, d\mu \bigg) \bigg\|_{L^\infty(\mu)} \leq \varepsilon \sum_{k=0}^{n} \int_Y v_k \, d\mu \quad \text{for any } n \geq N.$$

Hence,

$$\bigg\| \sum_{n\geq 0} e^{-n\lambda(t)} \sum_{k=0}^{n} \bigg(v_k - H \int_Y v_k \, d\mu \bigg) \bigg\|_{L^\infty(\mu)}$$
$$\leq \sum_{n=0}^{N-1} \sum_{k=0}^{n} \bigg\| v_k - H \int_Y v_k \, d\mu \bigg\|_{L^\infty(\mu)} + \varepsilon \sum_{n\geq N} e^{-n\lambda(t)} \sum_{k=0}^{n} \int_Y v_k \, d\mu,$$

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which implies

$$\|H_t - H\|_{L^\infty(\mu)} = \frac{\|\sum_{n\ge 0} e^{-n\lambda(t)} \sum_{k=0}^{n}(v_k - H\int_Y v_k\, d\mu)\|_{L^\infty(\mu)}}{\sum_{n\ge 0} e^{-n\lambda(t)} \sum_{k=0}^{n}\int_Y v_k\, d\mu}$$
$$\le \frac{\sum_{n=0}^{N-1}\sum_{k=0}^{n}\|v_k - H\int_Y v_k\, d\mu\|_{L^\infty(\mu)}}{\sum_{n\ge 0} e^{-n\lambda(t)} \sum_{k=0}^{n}\int_Y v_k\, d\mu} + \varepsilon$$
$$\to \varepsilon \quad (t\to\infty).$$

This establishes the result, since $\varepsilon > 0$ was arbitrary. □

The following lemma ensures that condition (ii) in Lemma 4.3 is satisfied for $R_{n,t} = \exp(-\lambda(t)Z_n^Y)$.

LEMMA 4.6. *Let $\lambda : (0,\infty) \to [0,\infty)$ be a non-negative function with $\lambda(t) \to 0 (t\to\infty)$. Set*

$$R_{n,t} = \exp(-\lambda(t)Z_n^Y) \quad (n\in\mathbb{N}_0,\ t>0).$$

Then, there exists a positive constant $t_0 > 0$ such that for any $n, k \in \mathbb{N}_0$ and $t \ge t_0$, we have

$$R_{n,t}\circ T^k \le e^k R_{n+k,t}.$$

Proof. Take $t_0 > 0$ so large that $\lambda(t) \le 1$ for any $t \ge t_0$. Since $Z_n^Y \circ T^k \ge Z_{n+k}^Y - k$, we have $R_{n,t}\circ T^k \le \exp(\lambda(t)k)R_{n+k,t} \le e^k R_{n+k,t}$ for any $t \ge t_0$. □

We can also prove the following lemma in almost the same way.

LEMMA 4.7. *Let $d \in \mathbb{N}$ and $A_1, \ldots, A_d \in \mathcal{A}$. Let $\lambda_i : (0,\infty) \to [0,\infty) (i = 1,\ldots,d)$ be non-negative functions with $\lambda_i(t) \to 0 (t\to\infty,\ i = 1,\ldots,d)$. Set*

$$R_{n,t} = \exp\bigg(-\sum_{i=1}^{d}\lambda_i(t)S_n^{A_i}\bigg) \quad (n\in\mathbb{N}_0,\ t>0).$$

Then, there exists a positive constant $t_0 > 0$ such that for any $n, k \in \mathbb{N}_0$ and $t \ge t_0$, we have

$$R_{n,t}\circ T^k \le e^k R_{n+k,t}.$$

Proof. Take $t_0 > 0$ so large that $\sum_{i=1}^{d}\lambda_i(t) \le 1$ for any $t \ge t_0$. Since $S_n^{A_i}\circ T^k \ge S_{n+k}^{A_i} - k$, we have $R_{n,t}\circ T^k \le \exp(\sum_{i=1}^{d}\lambda_i(t)k)R_{n+k,t} \le e^k R_{n+k,t}$ for any $t \ge t_0$. □

5. *Proofs of Theorems 3.1 and 3.8*

In the following lemma, we give a representation of double Laplace transform of Z_n^Y in terms of $Q^Y(s)$. A similar formula can be found in [27, Lemma 7.1].

LEMMA 5.1. *Let $s_1 > 0$ and $s_2 \geq 0$. Then, we have*

$$\int_Y \bigg(\sum_{n\geq 0} e^{-ns_1} \exp(-s_2 Z_n^Y)\bigg)\bigg(\sum_{n\geq 0} e^{-n(s_1+s_2)} \widehat{T}^n 1_{Y_n}\bigg)\, d\mu = \frac{Q^Y(s_1)}{1-e^{-(s_1+s_2)}}. \tag{5.1}$$

Proof. Note that $Z_n^Y = Z_{n-k}^Y \circ T^k + k$ on $\{\varphi = k\}(1 \leq k \leq n)$ and $Z_n^Y = 0$ on $\{\varphi > n\}$, and hence,

$$\exp(-s_2 Z_n^Y) = \begin{cases} \exp(-s_2 Z_{n-k}^Y \circ T^k) e^{-ks_2} & \text{on } \{\varphi = k\}(1 \leq k \leq n), \\ 1 & \text{on } \{\varphi > n\}. \end{cases}$$

Therefore, for $n \in \mathbb{N}_0$,

$$\begin{aligned} &\int_Y (e^{-ns_1} \exp(-s_2 Z_n^Y))\, d\mu \\ &\quad = \int_Y \bigg(\sum_{k=1}^{n} e^{-ns_1} \exp(-s_2 Z_{n-k}^Y \circ T^k) e^{-ks_2} 1_{Y\cap\{\varphi=k\}}\bigg)\, d\mu + e^{-ns_1}\mu(Y \cap \{\varphi > n\}) \\ &\quad = \int_Y \sum_{k=1}^{n} (e^{-(n-k)s_1} \exp(-s_2 Z_{n-k}^Y))(e^{-k(s_1+s_2)} \widehat{T}^k 1_{Y\cap\{\varphi=k\}})\, d\mu \\ &\qquad + e^{-ns_1}\mu(Y \cap \{\varphi > n\}). \end{aligned}$$

By taking the sum over $n \in \mathbb{N}_0$, we get

$$\begin{aligned} &\int_Y \bigg(\sum_{n\geq 0} e^{-ns_1} \exp(-s_2 Z_n^Y)\bigg)\, d\mu \\ &\quad = \int_Y \bigg(\sum_{n\geq 0} e^{-ns_1} \exp(-s_2 Z_n^Y)\bigg)\bigg(\sum_{k\geq 1} e^{-k(s_1+s_2)} \widehat{T}^k 1_{Y\cap\{\varphi=k\}}\bigg)\, d\mu + Q^Y(s_1), \end{aligned}$$

and hence,

$$\int_Y \bigg(\sum_{n\geq 0} e^{-ns_1} \exp(-s_2 Z_n)\bigg)\bigg(1_Y - \sum_{k\geq 1} e^{-k(s_1+s_2)} \widehat{T}^k 1_{Y\cap\{\varphi=k\}}\bigg)\, d\mu = Q^Y(s_1). \tag{5.2}$$

As shown in [27, equation (5.3)],

$$1_Y - \sum_{k\geq 1} e^{-ks} \widehat{T}^k 1_{Y\cap\{\varphi=k\}} = (1-e^{-s}) \sum_{n\geq 0} e^{-ns} \widehat{T}^n 1_{Y_n} \quad \text{a.e.} \ \ (s > 0). \tag{5.3}$$

Combining (5.2) with (5.3) completes the proof. □

LEMMA 5.2. *Assume that conditions (A2) and (A3) of Theorem 3.1 hold. Let $q > 0$ and let $\lambda : (0,\infty) \to (0,\infty)$ be a positive function with $\lambda(t) \to 0 (t \to \infty)$. Then, we have*

$$\int_0^\infty e^{-qu} \bigg(\int_Y \exp(-\lambda(t) Z_{[ut]}^Y)\, d\mu_H\bigg)\, du \sim \frac{Q^Y(qt^{-1})}{(q+\lambda(t)t) Q^Y(qt^{-1}+\lambda(t))}, \tag{5.4}$$

as $t \to \infty$.

Proof. By substituting $s_1 = qt^{-1}$ and $s_2 = \lambda(t)$ into (5.1), we have

$$\int_Y \bigg(\sum_{n\geq 0} e^{-nqt^{-1}} \exp(-\lambda(t) Z_n^Y) \bigg) \bigg(\sum_{n\geq 0} e^{-n(qt^{-1}+\lambda(t))} \widehat{T}^n 1_{Y_n} \bigg) \, d\mu$$
$$= \frac{Q^Y(qt^{-1})}{1 - e^{-qt^{-1}-\lambda(t)}} \sim \frac{Q^Y(qt^{-1})}{qt^{-1} + \lambda(t)} \quad (t \to \infty). \tag{5.5}$$

For $t > 0$, set

$$H_t = \frac{1}{Q^Y(qt^{-1} + \lambda(t))} \sum_{n\geq 0} e^{-n(qt^{-1}+\lambda(t))} \widehat{T}^n 1_{Y_n}. \tag{5.6}$$

Note that condition (A2) implies that the assumptions of Lemma 4.5(2) hold with

$$v_n = \widehat{T}^n 1_{Y_n}, \quad \int_Y v_n \, d\mu = w_{n+1}^Y - w_n^Y, \quad \sum_{n\geq 0} e^{-ns} \int_Y v_n \, d\mu = Q^Y(s),$$

and $\lambda(t)$ replaced by $qt^{-1} + \lambda(t)$. Hence, there exists $t_1 > 0$ such that $\{H_t\}_{t\geq t_1}$ is uniformly sweeping for 1_Y. In addition, condition (A3) implies that the assumptions in Lemma 4.5(4) are satisfied, and hence, $H_t \to H$ in $L^\infty(\mu)(t \to \infty)$. Moreover, we can use Lemma 4.6 to take $t_0 \geq t_1$ large enough so that $\exp(-\lambda(t) Z_n^Y \circ T^k) \leq e^k \exp(-\lambda(t) Z_{n+k}^Y)$ for any $t \geq t_0$ and $n, k \in \mathbb{N}_0$. Consequently, the assumptions in Lemma 4.4 are fulfilled with $R_{n,t} = \exp(-\lambda(t) Z_n^Y)$. Therefore, we can apply Lemmas 4.2 and 4.4 with $S_n = Z_n^Y$ to get

$$\int_Y \bigg(\sum_{n\geq 0} e^{-nqt^{-1}} \exp(-\lambda(t) Z_n^Y) \bigg) \bigg(\sum_{n\geq 0} e^{-n(qt^{-1}+\lambda(t))} \widehat{T}^n 1_{Y_n} \bigg) \, d\mu$$
$$\sim \bigg(\int_Y \sum_{n\geq 0} e^{-nqt^{-1}} \exp(-\lambda(t) Z_n^Y) \, d\mu_H \bigg) Q^Y(qt^{-1} + \lambda(t))$$
$$\sim \bigg(t \int_0^\infty e^{-qu} \bigg(\int_Y \exp(-\lambda(t) Z_{[ut]}^Y) \, d\mu_H \bigg) du \bigg) Q^Y(qt^{-1} + \lambda(t)) \quad (t \to \infty). \tag{5.7}$$

Combining (5.5) with (5.7) completes the proof. □

We now prove Theorems 3.1 and 3.8 by using Lemmas 5.1 and 5.2. We imitate the proof of [**10**, Theorem 2].

Proof of Theorem 3.1. Set $c(t) = c([t])$ for $t > 0$. Let $q, \lambda > 0$ be positive constants. By substituting $\lambda(t) = \lambda/(c(t)t)$ into (5.4), we see that

$$\int_0^\infty e^{-qu} \bigg(\int_Y \exp\bigg(-\frac{\lambda Z_{[ut]}^Y}{c(t)t} \bigg) d\mu_H \bigg) du \sim \frac{c(t) Q^Y(qt^{-1})}{\lambda Q^Y(qt^{-1} + \lambda c(t)^{-1} t^{-1})} \quad (t \to \infty).$$

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By (2.3), (3.1), and the uniform convergence theorem for regular varying functions [6, Theorem 1.5.2], we have $Q^Y(qt^{-1}+\lambda c(t)^{-1}t^{-1}) \sim Q^Y(\lambda c(t)^{-1}t^{-1})(t\to\infty)$ and

$$\frac{Q^Y(qt^{-1})}{Q^Y(qt^{-1}+\lambda c(t)^{-1}t^{-1})} \sim \frac{Q^Y(qt^{-1})}{Q^Y(\lambda c(t)^{-1}t^{-1})} \sim \frac{(qt^{-1})^{-1+\alpha}\ell(q^{-1}t)}{(\lambda c(t)^{-1}t^{-1})^{-1+\alpha}\ell(\lambda^{-1}c(t)t)}$$
$$\sim \left(\frac{qc(t)}{\lambda}\right)^{-1+\alpha}\frac{\ell(t)}{\ell(c(t)t)} \quad (t\to\infty). \tag{5.8}$$

Hence,

$$\frac{\ell(c(t)t)}{c(t)^\alpha \ell(t)}\int_0^\infty e^{-qu}\left(\int_Y \exp\left(-\frac{\lambda Z^Y_{[ut]}}{c(t)t}\right)d\mu_H\right)du$$
$$\to q^{-1+\alpha}\lambda^{-\alpha} = \frac{1}{\Gamma(1-\alpha)}\left(\int_0^\infty e^{-qu}u^{-\alpha}\,du\right)\lambda^{-\alpha} \quad (t\to\infty).$$

We use Lemma 4.1 to get, for $0<u<\infty$,

$$\frac{\ell(c(t)t)}{c(t)^\alpha \ell(t)}\int_Y \exp\left(-\frac{\lambda Z^Y_{[ut]}}{c(t)t}\right)d\mu_H$$
$$\to \frac{1}{\Gamma(1-\alpha)}u^{-\alpha}\lambda^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi}u^{-\alpha}\int_0^\infty e^{-\lambda s}s^{-1+\alpha}\,ds \quad (t\to\infty).$$

Here, we used Euler's reflection formula $\Gamma(\alpha)\Gamma(1-\alpha)=\pi/\sin(\pi\alpha)$. By the extended continuity theorem for Laplace transforms of locally finite measures [8, Ch. XIII.1, Theorem 2a], for $0\le s_0<\infty$,

$$\frac{\ell(c(t)t)}{c(t)^\alpha \ell(t)}\mu_H\left(\frac{Z^Y_{[ut]}}{c(t)t}\le s_0\right)$$
$$\to \frac{\sin(\pi\alpha)}{\pi}u^{-\alpha}\int_0^{s_0} s^{-1+\alpha}\,ds = \frac{\sin(\pi\alpha)}{\pi\alpha}\left(\frac{s_0}{u}\right)^\alpha \quad (t\to\infty). \tag{5.9}$$

Therefore, we substitute $t=n$ and $u=s_0=1$ into (5.9), and then obtain

$$\mu_H\left(\frac{Z^Y_n}{n}\le c(n)\right) \sim \frac{\sin(\pi\alpha)}{\pi\alpha}\frac{c(n)^\alpha\ell(n)}{\ell(c(n)n)} \quad (n\to\infty),$$

which is the desired result. □

Proof of Theorem 3.8. Set $c(t)=c([t])$ for $t>0$. By Chebyshev's inequality,

$$\mu_G\left(\frac{Z^Y_{[t]}}{t}\le c(t)\right)\le e\int_X \exp\left(-\frac{Z^Y_{[t]}}{c(t)t}\right)d\mu_G. \tag{5.10}$$

For each $t>0$, the map $(0,\infty)\ni u\mapsto \int_X \exp(-Z_{[ut]}/(c(t)t))\,d\mu_G\in[0,\infty)$ is non-increasing. Hence, we have

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$$\int_X \exp\left(-\frac{Z^Y_{[t]}}{c(t)t}\right) d\mu_G \le \int_0^1 \left(\int_X \exp\left(-\frac{Z^Y_{[ut]}}{c(t)t}\right) d\mu_G\right) du$$
$$\le e\int_0^\infty e^{-u}\left(\int_X \exp\left(-\frac{Z^Y_{[ut]}}{c(t)t}\right) d\mu_G\right) du$$
$$\le et^{-1}\sum_{n\ge 0} e^{-nt^{-1}}\int_X \exp\left(-\frac{Z^Y_n}{c(t)t}\right) G\, d\mu. \qquad (5.11)$$

Here, we also used (4.2). Define $H_t (t > 0)$ as in (5.6) with $q = 1$. As shown in the proof of Lemma 5.2, there exists $t_1 > 0$ such that $\{H_t\}_{t\ge t_1}$ is uniformly sweeping for 1_Y. Since 1_Y is uniformly sweeping for G, so is $\{H_t\}_{t\ge t_1}$. Moreover, by Lemma 4.6, we can take $t_0 \ge t_1$ large enough so that $\exp(-Z^Y_n \circ T^k/(c(t)t)) \le e^k \exp(-Z^Y_{n+k}/(c(t)t))$ for any $t \ge t_0$ and $n, k \in \mathbb{N}_0$. Consequently, the assumptions in Lemma 4.3 are fulfilled with $R_{n,t} = \exp(-Z^Y_n/(c(t)t))$. Therefore, Lemma 4.3 implies

$$\sup_{t\ge t_0} \frac{\sum_{n\ge 0} e^{-nt^{-1}}\int_X \exp(-Z^Y_n/(c(t)t))G\, d\mu}{\sum_{n\ge 0} e^{-nt^{-1}}\int_X \exp(-Z^Y_n/(c(t)t))H_t\, d\mu} < \infty. \qquad (5.12)$$

By substituting $q = 1$ and $\lambda(t) = c(t)^{-1}t^{-1}$ into (5.5) and making a similar estimate as in (5.8), we see that

$$t^{-1}\sum_{n\ge 0} e^{-nt^{-1}}\int_X \exp\left(-\frac{Z^Y_n}{c(t)t}\right) H_t\, d\mu$$
$$= \frac{t^{-1}}{1-\exp(t^{-1}+c(t)^{-1}t^{-1})}\cdot\frac{Q^Y(t^{-1})}{Q^Y(t^{-1}+c(t)^{-1}t^{-1})} \sim \frac{c(t)^\alpha \ell(t)}{\ell(c(t)t)} \quad (t\to\infty). \qquad (5.13)$$

Combining (5.10) with (5.11), (5.12), (5.13), we obtain

$$\limsup_{t\to\infty} \frac{\ell(c(t)t)}{c(t)^\alpha \ell(t)}\mu_G\left(\frac{Z^Y_{[t]}}{t} \le c(t)\right) < \infty,$$

as desired. □

6. *Proof of Theorem 3.10*

Let us represent double Laplace transform of S^Y_n in terms of $Q^Y(s)$. We also refer the reader to [**27**, Lemma 5.1] for a similar formula.

LEMMA 6.1. *Let $s_1 > 0$ and $s_2 \ge 0$. Then, we have*

$$(1-e^{-s_2})\int_Y \left(\sum_{n\ge 0} e^{-ns_1}\exp(-s_2 S^Y_n)\right) d\mu$$
$$+ (1-e^{-s_1})e^{-s_2}\int_Y \left(\sum_{n\ge 0} e^{-ns_1}\exp(-s_2 S^Y_n)\right)\left(\sum_{n\ge 0} e^{-ns_1}\widehat{T}^n 1_{Y_n}\right) d\mu$$
$$= Q^Y(s_1). \qquad (6.1)$$

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Proof. It is easy to see that $S_n^Y = S_{n-k}^Y \circ T^k + 1$ on $\{\varphi = k\}(1 \le k \le n)$ and $S_n^Y = 0$ on $\{\varphi > n\}$, which implies

$$\exp(-s_2 S_n^Y) = \begin{cases} \exp(-s_2 S_{n-k}^Y \circ T^k) e^{-s_2} & \text{on } \{\varphi = k\},\ 1 \le k \le n, \\ 1 & \text{on } \{\varphi > n\}. \end{cases}$$

Thus, for $n \in \mathbb{N}_0$,

$$\begin{aligned} &\int_Y (e^{-ns_1} \exp(-s_2 S_n^Y))\, d\mu \\ &\quad = \int_Y \sum_{k=1}^n (e^{-ns_1} \exp(-s_2 S_{n-k}^Y \circ T^k) e^{-s_2} 1_{Y\cap\{\varphi=k\}})\, d\mu + e^{-ns_1}\mu(Y \cap \{\varphi > n\}) \\ &\quad = e^{-s_2} \int_Y \sum_{k=1}^n (e^{-(n-k)s_1} \exp(-\lambda(t) S_{n-k}^Y))(e^{-ks_2} \widehat{T}^k 1_{Y\cap\{\varphi=k\}})\, d\mu \\ &\qquad + e^{-ns_1}\mu(Y \cap \{\varphi > n\}). \end{aligned}$$

By taking the sum over $n \in \mathbb{N}_0$, we get

$$\begin{aligned} &\int_Y \bigg(\sum_{n\ge 0} e^{-ns_1} \exp(-s_2 S_n^Y) \bigg)\, d\mu \\ &\quad = e^{-s_2} \int_Y \bigg(\sum_{n\ge 0} e^{-ns_1} \exp(-s_2 S_n^Y) \bigg) \bigg(\sum_{k\ge 0} e^{-ks_1} \widehat{T}^k 1_{Y\cap\{\varphi=k\}} \bigg)\, d\mu + Q^Y(s_1), \end{aligned}$$

and hence,

$$\begin{aligned} &(1 - e^{-s_2}) \int_Y \bigg(\sum_{n\ge 0} e^{-ns_1} \exp(-s_2 S_n^Y) \bigg)\, d\mu \\ &\qquad + e^{-s_2} \int_Y \bigg(\sum_{n\ge 0} e^{-ns_1} \exp(-s_2 S_n^Y) \bigg) \bigg(1_Y - \sum_{k\ge 0} e^{-ks_1} \widehat{T}^k 1_{Y\cap\{\varphi=k\}} \bigg)\, d\mu \\ &\quad = Q^Y(s_1). \end{aligned} \tag{6.2}$$

The lemma follows from (6.2) and (5.3). $\square$

LEMMA 6.2. *Assume that condition (B2) of Theorem 3.10 holds. Let $q > 0$ and let $\lambda : (0,\infty) \to (0,\infty)$ be a positive function with*

$$\lambda(t) \to 0 \quad \textit{and} \quad \frac{Q^Y(qt^{-1})}{\lambda(t)t} \to 0 \quad (t \to \infty). \tag{6.3}$$

Then, we have

$$\int_0^\infty e^{-qu} \bigg(\int_Y \exp(-\lambda(t) S_{[ut]}^Y)\, d\mu \bigg)\, du \sim \frac{Q^Y(qt^{-1})}{\lambda(t)t} \quad (t \to \infty). \tag{6.4}$$

Proof. By substituting $s_1 = qt^{-1}$ and $s_2 = \lambda(t)$ into (6.1), we have

$$\begin{aligned}(1 - e^{-\lambda(t)}) \int_Y \bigg(\sum_{n\geq 0} e^{-nqt^{-1}} \exp(-\lambda(t) S_n^Y)\bigg)\, d\mu \\ + (1 - e^{-qt^{-1}})e^{-\lambda(t)} \int_Y \bigg(\sum_{n\geq 0} e^{-nqt^{-1}} \exp(-\lambda(t) S_n^Y)\bigg)\bigg(\sum_{n\geq 0} e^{-nqt^{-1}} \widehat{T}^n 1_{Y_n}\bigg)\, d\mu \\ = Q^Y(qt^{-1}). \qquad (6.5)\end{aligned}$$

Let $l_1(t)$ and $l_2(t)$ denote the first and second terms of the left-hand side of (6.5), respectively. Let us prove $l_2(t)$ is negligibly small as $t \to \infty$. By condition (B2), the assumptions of Lemma 4.5(3) are satisfied with

$$v_n = \widehat{T}^n 1_{Y_n}, \quad \sum_{n\geq 0} e^{-ns} \int_Y v_n\, d\mu = Q^Y(s), \quad H_t = \frac{1}{Q^Y(qt^{-1})} \sum_{n\geq 0} e^{-nqt^{-1}} \widehat{T}^n 1_{Y_n}.$$

Hence, by Lemma 4.5(3), there exists $C > 0$ such that

$$\bigg\| \sum_{n\geq 0} e^{-nqt^{-1}} \widehat{T}^n 1_{Y_n} \bigg\|_{L^\infty(\mu)} \leq C Q^Y(qt^{-1}). \qquad (6.6)$$

By using (6.3) and (6.6),

$$0 \leq \frac{l_2(t)}{l_1(t)} \leq \frac{C(1 - e^{-qt^{-1}}) Q^Y(qt^{-1})}{e^{\lambda(t)} - 1} \leq \frac{Cq Q^Y(qt^{-1})}{\lambda(t) t} \to 0 \quad (t \to \infty). \qquad (6.7)$$

However, Lemma 4.2 yields

$$l_1(t) \sim \lambda(t) t \int_0^\infty e^{-qu} \bigg(\int_Y \exp(-\lambda(t) S_{[ut]}^Y)\, d\mu\bigg)\, du \quad (t \to \infty). \qquad (6.8)$$

The lemma follows from (6.5), (6.7), and (6.8). □

We now prove Theorem 3.10 by using Lemma 6.2.

Proof of Theorem 3.10. Set $\widetilde{c}(t) = \widetilde{c}([t])$ for $t > 0$. Let $q, \lambda > 0$ be positive constants. By substituting $\lambda(t) = \lambda/(\widetilde{c}(t) a(t))$ into (6.4), we have

$$\frac{1}{\widetilde{c}(t)} \int_0^\infty e^{-qu} \bigg(\int_Y \exp\bigg(-\frac{\lambda S_{[ut]}^Y}{\widetilde{c}(t) a(t)}\bigg)\, d\mu\bigg)\, du \sim \frac{1}{\Gamma(1+\alpha)} \frac{Q^Y(qt^{-1})}{\lambda Q^Y(t^{-1})} \quad (t \to \infty). \qquad (6.9)$$

Let $r(t)$ denote the right-hand side of (6.9). By (2.3),

$$r(t) \to \frac{q^{-1+\alpha} \lambda^{-1}}{\Gamma(1+\alpha)} = \frac{\sin(\pi\alpha)}{\pi\alpha} \bigg(\int_0^\infty e^{-qu} u^{-\alpha}\, du\bigg)\bigg(\int_0^\infty e^{-\lambda s}\, ds\bigg) \quad (t \to \infty).$$

Hence, we use Lemma 4.1 to get, for $0 < u < \infty$,

$$\frac{1}{\widetilde{c}(t)} \int_Y \exp\bigg(-\frac{\lambda S_{[ut]}^Y}{\widetilde{c}(t) a(t)}\bigg)\, d\mu \to \frac{\sin(\pi\alpha)}{\pi\alpha} u^{-\alpha} \int_0^\infty e^{-\lambda s}\, ds \quad (t \to \infty).$$

By the extended continuity theorem for Laplace transforms, for $0 \le s_0 < \infty$,

$$\frac{1}{\widetilde{c}(t)}\mu_{1_Y}\bigg(\frac{S^Y_{[ut]}}{\widetilde{c}(t)a(t)} \le s_0\bigg) \to \frac{\sin(\pi\alpha)}{\pi\alpha}u^{-\alpha}\int_0^{s_0} ds = \frac{\sin(\pi\alpha)}{\pi\alpha}\frac{s_0}{u^\alpha} \quad (t \to \infty).$$

Substituting $t = n$ and $u = s_0 = 1$ completes the proof. □

7. *Proofs of Theorems 3.15 and 3.21*

We can also represent double Laplace transform of $S_n^{A_i}(i = 1, \ldots, d)$ in terms of $Q^{Y,A_i}(s)(i = 1, \ldots, d)$. We refer the reader to [27, Lemma 6.1] and [18, Proposition 5.1] for similar formulas.

LEMMA 7.1. *Suppose condition (C1) of Theorem 3.15 is satisfied. Let $s > 0$ and $s_1, s_2, \ldots, s_d \ge 0$. Then, we have*

$$\begin{aligned}(1 - e^{-s})&\int_Y \bigg(\sum_{n\ge0} e^{-ns}\exp\bigg(-\sum_{j=1}^d s_j S_n^{A_j}\bigg)\bigg)\,d\mu \\ &+ \sum_{i=1}^d (e^{s_i} - e^{-s})\int_Y \bigg(\sum_{n\ge0} e^{-ns}\exp\bigg(-\sum_{j=1}^d s_j S_n^{A_j}\bigg)\bigg)\bigg(\sum_{n\ge1} e^{-n(s+s_i)}\widehat{T}^n 1_{Y_n\cap A_i}\bigg)\,d\mu \\ &= \mu(Y) + \sum_{i=1}^d Q^{Y,A_i}(s+s_i). \end{aligned} \tag{7.1}$$

Proof. Set

$$R_n = \exp\bigg(-\sum_{i=1}^d s_i S_n^{A_i}\bigg), \quad n \in \mathbb{N}_0.$$

Note that, for $n \in \mathbb{N}$,

$$R_n = \begin{cases} R_{n-1}\circ T & \text{on } \{\varphi = 1\} = T^{-1}Y, \\ (R_{n-k}\circ T^k)e^{-(k-1)s_i} & \text{on } (T^{-1}A_i)\cap\{\varphi = k\}(1 \le i \le d \text{ and } 2 \le k \le n), \\ e^{-ns_i} & \text{on } (T^{-1}A_i)\cap\{\varphi > n\}(1 \le i \le d). \end{cases}$$

Hence, $\int_Y R_0\,d\mu = \mu(Y)$ and, for $n \in \mathbb{N}$,

$$\begin{aligned}&\int_Y e^{-ns}R_n\,d\mu \\ &\quad= \int_Y e^{-ns}R_{n-1}\,\widehat{T}1_{Y\cap T^{-1}Y}\,d\mu \\ &\qquad+ e^{-s}\sum_{i=1}^d\int_Y\sum_{k=2}^n (e^{-(n-k)s}R_{n-k})\,(e^{-(k-1)(s+s_i)}\widehat{T}^k 1_{Y\cap(T^{-1}A_i)\cap\{\varphi=k\}})\,d\mu \\ &\qquad+ \sum_{i=1}^d e^{-n(s+s_i)}\mu(Y\cap(T^{-1}A_i)\cap\{\varphi > n\}).\end{aligned}$$

By taking the sum over $n \in \mathbb{N}_0$, we get

$$
\begin{aligned}
&\int_Y \left(\sum_{n\ge 0} e^{-ns} R_n \right) d\mu \\
&\quad = \mu(Y) + e^{-s} \int_Y \left(\sum_{n\ge 0} e^{-ns} R_n \right) \widehat{T} 1_{Y\cap T^{-1}Y}\, d\mu \\
&\qquad + e^{-s} \sum_{i=1}^{d} \int_Y \left(\sum_{n\ge 0} e^{-ns} R_n \right) \left(\sum_{k\ge 2} e^{-(k-1)(s+s_i)} \widehat{T}^k 1_{Y\cap (T^{-1}A_i)\cap\{\varphi=k\}} \right) d\mu \\
&\qquad + \sum_{i=1}^{d} Q^{Y,A_i}(s+s_i).
\end{aligned}
$$

By (3.14), we have

$$
\begin{aligned}
1_Y = \widehat{T} 1_{T^{-1}Y} &= \widehat{T} 1_{Y\cap T^{-1}Y} + \sum_{i=1}^{d} \widehat{T} 1_{A_i\cap T^{-1}Y} \\
&= \widehat{T} 1_{Y\cap T^{-1}Y} + \sum_{i=1}^{d} \sum_{k\ge 2} \widehat{T}^k 1_{Y\cap (T^{-1}A_i)\cap\{\varphi=k\}},
\end{aligned}
$$

which implies

$$
\begin{aligned}
&(1-e^{-s}) \int_Y \left(\sum_{n\ge 0} e^{-ns} R_n \right) d\mu \\
&\quad + \sum_{i=1}^{d} e^{-s} \int_Y \left(\sum_{n\ge 0} e^{-ns} R_n \right) \left(\sum_{k\ge 2} (1 - e^{-(k-1)(s+s_i)}) \widehat{T}^k 1_{Y\cap (T^{-1}A_i)\cap\{\varphi=k\}} \right) d\mu \\
&= \mu(Y) + \sum_{i=1}^{d} Q^{Y,A_i}(s+s_i).
\end{aligned} \tag{7.2}
$$

In addition, we use (3.14) to get, for $t > 0$,

$$
\begin{aligned}
\sum_{k\ge 2} (1 - e^{-(k-1)t}) \widehat{T}^k 1_{Y\cap (T^{-1}A_i)\cap\{\varphi=k\}} &= \sum_{k\ge 2} \left((e^t - 1) \sum_{n=1}^{k-1} e^{-nt} \right) \widehat{T}^k 1_{Y\cap (T^{-1}A_i)\cap\{\varphi=k\}} \\
&= (e^t - 1) \sum_{n\ge 1} e^{-nt} \sum_{k>n} \widehat{T}^k 1_{Y\cap (T^{-1}A_i)\cap\{\varphi=k\}} \\
&= (e^t - 1) \sum_{n\ge 1} e^{-nt}\, \widehat{T}^n 1_{Y_n\cap A_i}.
\end{aligned} \tag{7.3}
$$

Combining (7.2) with (7.3) completes the proof. □

LEMMA 7.2. *Assume that conditions (C1), (C3), and (C4) of Theorem 3.15 hold. Let $q > 0$ be a positive constant and let $\lambda_i : (0,\infty) \to (0,\infty) (i = 1, \ldots, d-1)$ be positive*

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functions with

$$\lambda_i(t) \to 0, \quad \lambda_i(t)t \to \infty \quad (i = 1, \ldots, d-1), \tag{7.4}$$

$$\frac{Q^{Y,A_d}(qt^{-1})}{\sum_{i=1}^{d-1} Q^{Y,A_i}(qt^{-1} + \lambda_i(t))} \to \infty, \quad \frac{Q^{Y,A_d}(qt^{-1})}{\sum_{i=1}^{d-1} \lambda_i(t)t\, Q^{Y,A_i}(qt^{-1} + \lambda_i(t))} \to 0, \tag{7.5}$$

as $t \to \infty$. *Then,*

$$\sum_{i=1}^{d-1} \lambda_i(t) \int_Y \bigg(\sum_{n \geq 0} e^{-nqt^{-1}} \exp\bigg(-\sum_{j=1}^{d-1} \lambda_j(t) S_n^{A_j} \bigg) \bigg) \bigg(\sum_{n \geq 1} e^{-n(qt^{-1} + \lambda_i(t))} \widehat{T}^n 1_{Y_n \cap A_i} \bigg) \, d\mu$$
$$\sim Q^{Y,A_d}(qt^{-1}) \quad (t \to \infty).$$

Proof. Set

$$R_{n,t} = \exp\bigg(-\sum_{j=1}^{d-1} \lambda_j(t) S_n^{A_j} \bigg) \quad (n \in \mathbb{N}_0, \ t > 0), \tag{7.6}$$

$$H_t^{(i)} = \frac{\sum_{n \geq 1} e^{-n(qt^{-1} + \lambda_i(t))} \widehat{T}^n 1_{Y_n \cap A_i}}{Q^{Y,A_i}(qt^{-1} + \lambda_i(t))} \quad (t > 0, \ i = 1, \ldots, d-1), \tag{7.7}$$

$$H_t^{(d)} = \frac{\sum_{n \geq 1} e^{-nqt^{-1}} \widehat{T}^n 1_{Y_n \cap A_d}}{Q^{Y,A_d}(qt^{-1})} \quad (t > 0).$$

By substituting $s = qt^{-1}$, $s_i = \lambda_i(t) (i = 1, \ldots, d-1)$, and $s_d = 0$ into (7.1), we have

$$\begin{aligned}
&(1 - e^{-qt^{-1}}) \int_Y \bigg(\sum_{n \geq 0} e^{-nqt^{-1}} R_{n,t} \bigg) \, d\mu \\
&\quad + \sum_{i=1}^{d-1} (e^{\lambda_i(t)} - e^{-qt^{-1}}) Q^{Y,A_i}(qt^{-1} + \lambda_i(t)) \int_Y \bigg(\sum_{n \geq 0} e^{-nqt^{-1}} R_{n,t} \bigg) H_t^{(i)} \, d\mu \\
&\quad + (1 - e^{-qt^{-1}}) Q^{Y,A_d}(qt^{-1}) \int_Y \bigg(\sum_{n \geq 0} e^{-nqt^{-1}} R_{n,t} \bigg) H_t^{(d)} \, d\mu \\
&= \mu(Y) + \sum_{i=1}^{d-1} Q^{Y,A_i}(qt^{-1} + \lambda_i(t)) + Q^{Y,A_d}(qt^{-1}).
\end{aligned} \tag{7.8}$$

Let $l_1(t)$, $l_2(t)$, and $l_3(t)$ denote the first, second, and third terms of the left-hand side of (7.8), respectively, and $r(t)$ denote the right-hand side of (7.8). Note that

$$r(t) \sim Q^{Y,A_d}(qt^{-1}) \quad (t \to \infty), \tag{7.9}$$

since $Q^{Y,A_i}(s) \to \infty (s \to 0+, \ i = 1, \ldots, d)$ and the assumption (7.5).

Let us prove $l_2(t)$ is the leading term of the left-hand side of (7.8), and $l_1(t)$ and $l_3(t)$ are negligible as $t \to \infty$. Note that condition (C4) implies that the assumptions in Lemma 4.5(2) are satisfied with

$$v_n = \widehat{T}^n 1_{Y_n \cap A_i}, \ \int_Y v_n \, d\mu = w_{n+1}^{Y,A_i} - w_n^{Y,A_i}, \ \sum_{n \geq 0} e^{-ns} \int_Y v_n \, d\mu = Q^{Y,A_i}(s), \tag{7.10}$$

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$$\lambda(t) = qt^{-1} + \lambda_i(t), \quad H_t = H_t^{(i)} \quad (i = 1, \ldots, d-1). \tag{7.11}$$

Hence, by Lemma 4.5(2), there exists $t_1 > 0$ such that $\{H_t^{(i)}\}_{t \geq t_1,\, i=1,\ldots,d-1}$ is uniformly sweeping for 1_Y. In addition, we use Lemma 4.7 to take $t_0 \geq t_1$ large enough so that $R_{n,t} \circ T^k \leq e^k R_{n+k,t}$ for any $n, k \in \mathbb{N}_0$ and $t \geq t_0$. Consequently, the assumptions in Lemma 4.3 are fulfilled with $H_t = H_t^{(i)} (i = 1, \ldots, d-1)$ and $G = 1_Y$. Therefore, Lemma 4.3 implies

$$C = \limsup_{t \to \infty} \max_{1 \leq i \leq d-1} \frac{\int_Y (\sum_{n \geq 0} e^{-nqt^{-1}} R_{n,t})\, d\mu}{\int_Y (\sum_{n \geq 0} e^{-nqt^{-1}} R_{n,t}) H_t^{(i)}\, d\mu} < \infty.$$

Hence,

$$0 \leq \limsup_{t \to \infty} \frac{l_1(t)}{l_2(t)} \leq \limsup_{t \to \infty} \frac{qC}{\sum_{i=1}^{d-1} \lambda_i(t) t\ Q^{Y,A_i}(qt^{-1} + \lambda_i(t))} = 0.$$

In addition, $\{H_t^{(d)}\}_{t>0}$ is $L^\infty(\mu)$-bounded, which follows from condition (C3) and Lemma 4.5(3) with

$$v_n = \widehat{T}^n 1_{Y_n \cap A_d}, \quad \sum_{n \geq 0} e^{-ns} \int_Y v_n\, d\mu = Q^{Y,A_d}(s), \quad H_t = H_t^{(d)}.$$

We use assumption (7.5) to see

$$\begin{aligned} 0 &\leq \limsup_{t \to \infty} \frac{l_3(t)}{l_2(t)} \\ &\leq \Big(\sup_{t>0} \|H_t^{(d)}\|_{L^\infty(\mu)} \Big) \Big(\limsup_{t \to \infty} \frac{qC\, Q^{Y,A_d}(qt^{-1})}{\sum_{i=1}^{d-1} \lambda_i(t) t\ Q^{Y,A_i}(qt^{-1} + \lambda_i(t))} \Big) = 0. \end{aligned}$$

Therefore, we get

$$\begin{aligned} &l_1(t) + l_2(t) + l_3(t) \\ &\sim \sum_{i=1}^{d-1} \lambda_i(t) Q^{Y,A_i}(qt^{-1} + \lambda_i(t)) \int_Y \Big(\sum_{n \geq 0} e^{-nqt^{-1}} R_{n,t} \Big) H_t^{(i)}\, d\mu \quad (t \to \infty). \end{aligned} \tag{7.12}$$

The lemma follows from (7.8), (7.9), and (7.12). □

We now prove Theorems 3.15 and 3.21 by using Lemma 7.2.

Proof of Theorem 3.15. Set $c(t) = c([t])$ for $t > 0$. Let $q, \lambda, \lambda_1, \ldots, \lambda_{d-1} > 0$ and $\lambda_i(t) = \lambda\lambda_i/(c(t)t)$. By (3.1), (3.16), and the uniform convergence theorem for regular varying functions, we see $Q^{Y,A_i}(qt^{-1} + \lambda_i(t)) \sim Q^{Y,A_i}(\lambda_i(t)) (t \to \infty,\ i = 1, \ldots, d-1)$. By the Potter bounds for slowly varying functions, we see that $c(t)^{-1+\alpha}\ell(t)/\ell(c(t)t) \to \infty$ and $c(t)^\alpha \ell(t)/\ell(c(t)t) \to 0 (t \to \infty)$. Thus, for $i = 1, \ldots, d-1$,

$$\frac{Q^{Y,A_d}(qt^{-1})}{Q^{Y,A_i}(qt^{-1} + \lambda_i(t))} \sim \frac{Q^{Y,A_d}(qt^{-1})}{Q^{Y,A_i}(\lambda_i(t))} \sim \frac{\beta_d q^{-1+\alpha}}{\beta_i (\lambda\lambda_i)^{-1+\alpha}} \frac{c(t)^{-1+\alpha}\ell(t)}{\ell(c(t)t)} \to \infty \quad (t \to \infty)$$

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and

$$\frac{Q^{Y,A_d}(qt^{-1})}{\lambda_i(t)tQ^{Y,A_i}(qt^{-1}+\lambda_i(t))} \sim \frac{\beta_d q^{-1+\alpha}}{\beta_i(\lambda\lambda_i)^\alpha}\frac{c(t)^\alpha \ell(t)}{\ell(c(t)t)} \to 0 \quad (t\to\infty). \tag{7.13}$$

Therefore, (7.4) and (7.5) are fulfilled. Define $R_{n,t}(n\in\mathbb{N}_0,\ t>0)$ and $H_t^{(i)}(t>0, i = 1,\ldots,d-1)$ as in (7.6) and (7.7), respectively. By Lemma 7.2,

$$\sum_{i=1}^{d-1}\lambda_i(t)Q^{Y,A_i}(qt^{-1}+\lambda_i(t))\int_Y\bigg(\sum_{n\geq 0}e^{-nqt^{-1}}R_{n,t}\bigg)H_t^{(i)}\,d\mu \\ \sim Q^{Y,A_d}(qt^{-1}) \quad (t\to\infty). \tag{7.14}$$

Set $\widetilde{H}_t = \sum_{i=1}^{d-1}\beta_i\lambda_i^\alpha H_t^{(i)}(t>0)$. Then, we use (7.13) and (7.14) to get

$$\frac{\ell(c(t)t)}{c(t)^\alpha\ell(t)}t^{-1}\int_Y\bigg(\sum_{n\geq 0}e^{-nqt^{-1}}R_{n,t}\bigg)\widetilde{H}_t\,d\mu \to \beta_d q^{-1+\alpha}\lambda^{-\alpha} \quad (t\to\infty). \tag{7.15}$$

As shown in the proof of Lemma 7.2, there exists $t_0>0$ such that the assumptions of Lemma 4.3 are satisfied with $H_t = H_t^{(i)}(i=1,\ldots,d-1)$ and $G = 1_Y$, and hence, with $H_t = \widetilde{H}_t$ and $G = 1_Y$. Similarly, condition (C5) implies that the assumptions in Lemma 4.5(4) are satisfied with (7.10), (7.11), and $H = H^{(i)}(i=1,\ldots,d-1)$. Therefore, Lemma 4.5(4) implies that $H_t^{(i)} \to H^{(i)}$ in $L^\infty(\mu)(t\to\infty, i=1,\ldots,d-1)$, and hence, $\widetilde{H}_t \to \widetilde{H} = \sum_{i=1}^{d-1}\beta_i\lambda_i^\alpha H^{(i)}$ in $L^\infty(\mu)(t\to\infty)$. Consequently, the assumptions of Lemma 4.4 are fulfilled with $H_t = \widetilde{H}_t$ and $H = \widetilde{H}$. Therefore, we use Lemmas 4.2 and 4.4 to get

$$t^{-1}\int_Y\bigg(\sum_{n\geq 0}e^{-nqt^{-1}}R_{n,t}\bigg)\widetilde{H}_t\,d\mu \sim t^{-1}\int_Y\bigg(\sum_{n\geq 0}e^{-nqt^{-1}}R_{n,t}\bigg)\widetilde{H}\,d\mu \\ \sim \int_0^\infty e^{-qu}\bigg(\int_Y R_{[ut],t}\,d\mu_{\widetilde{H}}\bigg)\,du \quad (t\to\infty). \tag{7.16}$$

By (7.15) and (7.16),

$$\frac{\ell(c(t)t)}{c(t)^\alpha\ell(t)}\int_0^\infty e^{-qu}\bigg(\int_Y R_{[ut],t}\,d\mu_{\widetilde{H}}\bigg)\,du \to \beta_d q^{-1+\alpha}\lambda^{-\alpha} \quad (t\to\infty).$$

Thus, we use similar arguments as in the proof of Theorem 3.1 to obtain

$$\frac{\ell(c(t)t)}{c(t)^\alpha\ell(t)}\mu_{\widetilde{H}}\bigg(\frac{\sum_{j=1}^{d-1}\lambda_j S_{[ut]}^{A_j}}{c(t)t}\leq s_0\bigg) \to \beta_d\frac{\sin(\pi\alpha)}{\pi\alpha}\bigg(\frac{s_0}{u}\bigg)^\alpha \quad (t\to\infty,\ s_0,u>0).$$

Substituting $t=n$ and $s_0=u=1$ completes the proof. □

Proof of Theorem 3.21. Set $c(t)=c([t])(t>0)$ and $\lambda_i(t)=\lambda_i/(c(t)t)(t>0,\ i = 1,\ldots,d-1)$. Define $R_{n,t}(n\in\mathbb{N}_0, t>0)$ as in (7.6). By Chebyshev's inequality,

$$\mu_G\bigg(\frac{\sum_{j=1}^{d-1}\lambda_j S_{[t]}^{A_j}}{t}\leq c(t)\bigg) \leq e\int_X R_{[t],t}\,d\mu_G. \tag{7.17}$$

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For each $t > 0$, the map $(0, \infty) \ni u \mapsto \int_X R_{[ut],t}\, d\mu_G \in [0, \infty)$ is non-increasing. Hence, we have

$$\begin{aligned}\int_X R_{[t],t}\, d\mu_G &\le \int_0^1 \bigg(\int_X R_{[ut],t}\, d\mu_G \bigg)\, du \\ &\le e \int_0^\infty e^{-u} \bigg(\int_X R_{[ut],t}\, d\mu_G \bigg)\, du \\ &\le e t^{-1} \sum_{n\ge 0} e^{-nt^{-1}} \int_X R_{n,t} G\, d\mu. \end{aligned} \tag{7.18}$$

Define $H_t^{(i)}$ $(t > 0,\ i = 1, \dots, d-1)$ by (7.7) with $q = 1$, and set $\widetilde{H}_t = \sum_{i=1}^{d-1} \beta_i \lambda_i^\alpha H_t^{(i)}$. Recall from the proofs of Lemma 7.2 and Theorem 3.15 that there exists $t_1 > 0$ such that $\{\widetilde{H}_t\}_{t\ge t_1}$ is uniformly sweeping for 1_Y, and hence, for G. In addition, by Lemma 4.7, we can take $t_0 \ge t_1$ large enough so that $R_{n,t} \circ T^k \le e^k R_{n+k,t}$ for any $n, k \in \mathbb{N}_0$ and $t \ge t_0$. Consequently, the assumptions in Lemma 4.3 are fulfilled with $H_t = \widetilde{H}_t$. Therefore, Lemma 4.3 implies

$$\sup_{t\ge t_0} \frac{\sum_{n\ge 0} e^{-nt^{-1}} \int_X R_{n,t} G\, d\mu}{\sum_{n\ge 0} e^{-nt^{-1}} \int_X R_{n,t} \widetilde{H}_t\, d\mu} < \infty. \tag{7.19}$$

By substituting $q = \lambda = 1$ into (7.15), we get

$$t^{-1} \sum_{n\ge 0} e^{-nt^{-1}} \int_X R_{n,t} \widetilde{H}_t\, d\mu \sim \beta_d \frac{c(t)^\alpha \ell(t)}{\ell(c(t)t)} \quad (t \to \infty). \tag{7.20}$$

The result follows from (7.17), (7.18), (7.19), and (7.20). □

8. *Applications to Thaler's maps*

Our abstract results in §3 are applicable to a variety of classes of ergodic transformations. Indeed, the assumptions of Theorems 3.1, 3.10, and 3.15 are milder than those of [**27**, Theorems 3.3, 3.1, and 3.2], respectively, which are verified for interval maps with indifferent fixed points (see [**27**, §8] and [**18**, §2.4]), Markov chains on multiray [**18**, §2.5], and random iterations of piecewise linear maps (as summarized in [**9**, Theorem 1.1, the subsequent paragraph, and Lemma 3.5] and [**13**, Theorem 1.2, Remark 1.4, and §4.2]) under suitable settings. The assumptions of Theorems 3.8, 3.10, and 3.21 are also verified for random walks driven by Gibbs–Markov maps, as shown in [**33**, §7.3]. For simplicity, we are going to focus only on Thaler's maps with two indifferent fixed points [**25**] in this section.

Assumption 8.1. (Thaler's map) Suppose that the map $T : [0, 1] \to [0, 1]$ satisfies the following conditions.

(i) For some $c \in (0, 1)$, the restrictions $T : [0, c) \to [0, 1)$ and $T : (c, 1] \to (0, 1]$ are strictly increasing, onto, and can be extended to C^2 maps $T_0 : [0, c] \to [0, 1]$ and $T_1 : [c, 1] \to [0, 1]$, respectively.

(ii) $T_0' > 1$ and $T_0'' > 0$ on $(0, c]$, $T_1' > 1$ and $T_1'' < 0$ on $[c, 1)$, and $T'(0) = T'(1) = 1$.

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(iii) For some $p \in (1, \infty)$, $a \in (0, \infty)$, and some positive, measurable function $\ell^* : (0, \infty) \to (0, \infty)$ slowly varying at 0 such that

$$Tx - x \sim a^{-p}(1 - x - T(1 - x)) \sim x^{p+1}\ell^*(x) \quad (x \to 0+).$$

In the following, we always impose Assumption 8.1. Let us summarize the facts that are shown in [22, 23, 25, 27, 30, 31]. After that, we will explain applications of our abstract results to Thaler's maps.

Let f_i denote the inverse function of $T_i (i = 0, 1)$. Then, T admits an invariant density h of the form

$$h(x) = h_0(x)\frac{x(1-x)}{(x - f_0(x))(f_1(x) - x)} \quad (x \in (0, 1)),$$

where h_0 is continuous and positive on $[0, 1]$. In addition, h has bounded variation on $[\varepsilon, 1 - \varepsilon]$ for any $\varepsilon \in (0, 1/2)$. Define the σ-finite measure μ as $d\mu(x) = h(x)\, dx$, $x \in [0, 1]$. Then, $\mu([0, \varepsilon]) = \mu([1 - \varepsilon, 1]) = \infty$ for any $\varepsilon \in (0, 1)$, and T is a CEMPT on the σ-finite measure space $([0, 1], \mathcal{B}([0, 1]), \mu)$.

Since $f_0(x) \sim x (x \to 0+)$, we use the uniform convergence theorem for slowly varying functions [6, Theorem 1.2.1] to get $\ell^*(f_0(x)) \sim \ell^*(x) (x \to 0+)$ and

$$\begin{aligned} x - f_0(x) &= T_0(f_0(x)) - f_0(x) \\ &\sim f_0(x)^{p+1}\ell^*(f_0(x)) \sim x^{p+1}\ell^*(x) \quad (x \to 0+). \end{aligned} \tag{8.1}$$

Similarly, it is easily seen that $1 - f_1(1 - x) \sim x (x \to 0+)$ and

$$\begin{aligned} f_1(1 - x) - (1 - x) &= 1 - (1 - f_1(1 - x)) - T_1(1 - (1 - f_1(1 - x))) \\ &\sim a^p x^{p+1}\ell^*(x) \quad (x \to 0+). \end{aligned} \tag{8.2}$$

Let $\gamma \in (0, c)$ be a 2-periodic point of T. Then, $T\gamma \in (c, 1)$. Take $c_0 \in (0, \gamma]$ and $c_1 \in [T\gamma, 1)$ arbitrarily, and set

$$A_0 = [0, c_0), \quad Y = [c_0, c_1], \quad A_1 = (c_1, 1]. \tag{8.3}$$

Then, $\mu(Y) \in (0, \infty)$, $\mu(A_i) = \infty (i = 0, 1)$, and Y dynamically separates A_0 and A_1.

LEMMA 8.2. *For $i = 0, 1$, there exists a μ-probability density function $H^{(i)}$ such that $H^{(i)}$ is positive, continuous, supported, and has bounded variation on $(TA_i) \setminus A_i$, and satisfies*

$$\lim_{n\to\infty} \frac{\widehat{T}^n 1_{Y\cap(T^{-1}A_i)\cap\{\varphi=n\}}}{\mu(Y \cap (T^{-1}A_i) \cap \{\varphi = n\})} = H^{(i)} \quad \text{in } L^\infty(\mu) \quad (i = 0, 1).$$

In addition,

$$\begin{aligned} \mu(Y \cap (T^{-1}A_i) \cap \{\varphi = n\}) &= \int_{(TA_i)\setminus A_i} \widehat{T}^n 1_{Y\cap(T^{-1}A_i)\cap\{\varphi=n\}}\, d\mu \\ &\sim \begin{cases} h(c) f_1'(0)(f_0^n(1) - f_0^{n+1}(1)) & (n \to \infty,\ i = 0), \\ h(c) f_0'(1)(f_1^{n+1}(0) - f_1^n(0)) & (n \to \infty,\ i = 1). \end{cases} \end{aligned} \tag{8.4}$$

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Remark 8.3. By Lemmas 3.14 and 8.2, we have

$$\lim_{n\to\infty}\left(\frac{1}{w_n^{Y,A_i}}\sum_{k=0}^{n-1}\widehat{T}^k 1_{Y_k\cap A_i}\right)=H^{(i)}\quad\text{in } L^\infty(\mu)\quad(i=0,1). \tag{8.5}$$

Proof of Lemma 8.2. We proceed as in the proofs of [25, Lemma 3] and [27, Theorem 8.1]. As in the calculations in [25, p. 1301] and [27, p. 46], for $n\ge 2$,

$$\widehat{T}^n 1_{Y\cap(T^{-1}A_0)\cap\{\varphi=n\}}=1_{(TA_0)\setminus A_0}(h\circ f_1\circ f_0^{n-1})\cdot(f_1'\circ f_0^{n-1})\cdot(f_0^{n-1})'/h\quad\text{a.e.,} \tag{8.6}$$

$$\widehat{T}^n 1_{Y\cap(T^{-1}A_1)\cap\{\varphi=n\}}=1_{(TA_1)\setminus A_1}(h\circ f_0\circ f_1^{n-1})\cdot(f_0'\circ f_1^{n-1})\cdot(f_1^{n-1})'/h\quad\text{a.e.} \tag{8.7}$$

It is easily seen that $f_0^{n-1}\to 0$ and $f_1^{n-1}\to 1 (n\to\infty)$ uniformly on $[0,1]$. By applying [25, Lemma 2] to $f(x)=f_0(x)$ and $f(x)=1-f_1(1-x)$, respectively, we see that there exist continuous functions $g_0:(0,1]\to(0,\infty)$ and $g_1:[0,1)\to(0,\infty)$ such that

$$\frac{(f_0^n)'(x)}{f_0^n(1)-f_0^{n+1}(1)}\to g_0(x)\quad(n\to\infty),\ \text{uniformly on compact subsets on (0,1]}, \tag{8.8}$$

$$\frac{(f_1^n)'(x)}{f_1^{n+1}(0)-f_1^n(0)}\to g_1(x)\quad(n\to\infty),\ \text{uniformly on compact subsets on [0,1)}, \tag{8.9}$$

and

$$\int_{f_0(x)}^{x} g_0(y)\,dy=\int_x^{f_1(x)} g_1(y)\,dy=1\quad\text{for any } x\in(0,1). \tag{8.10}$$

It follows from the concavity of f_0^n and (8.8) that g_0 is non-increasing. Similarly, g_1 is non-decreasing, which follows from the convexity of f_1^n and (8.9). Set

$$H^{(i)}=1_{(TA_i)\setminus A_i}g_i/h\quad(i=0,1),$$

which is positive, continuous, supported, and has bounded variation on $(TA_i)\setminus A_i (i=0,1)$. By (8.6), (8.7) (8.8), and (8.9),

$$\lim_{n\to\infty}\frac{\widehat{T}^n 1_{Y\cap(T^{-1}A_0)\cap\{\varphi=n\}}}{h(c)f_1'(0)(f_0^n(1)-f_0^{n+1}(1))}=H^{(0)}\quad\text{in } L^\infty(\mu), \tag{8.11}$$

$$\lim_{n\to\infty}\frac{\widehat{T}^n 1_{Y\cap(T^{-1}A_1)\cap\{\varphi=n\}}}{h(c)f_0'(1)(f_1^{n+1}(0)-f_1^n(0))}=H^{(1)}\quad\text{in } L^\infty(\mu). \tag{8.12}$$

By using $d\mu(x)=h(x)\,dx$ and (8.10), we see that

$$\int_{(TA_0)\setminus A_0} H^{(0)}\,d\mu=\int_{c_0}^{T(c_0)} g_0(y)\,dy=1, \tag{8.13}$$

$$\int_{(TA_1)\setminus A_1} H^{(1)}\,d\mu=\int_{T(c_1)}^{c_1} g_1(y)\,dy=1. \tag{8.14}$$

Hence, $H^{(0)}$ and $H^{(1)}$ are μ-probability density functions. We use (8.11), (8.12), (8.13), and (8.14) to obtain the desired result. □

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LEMMA 8.4. *There exists a function $\ell_0(x)$ slowly varying at ∞ such that*

$$f_0^n(1) - f_0^{n+1}(1) \sim p^{-1} n^{-1-(1/p)} \ell_0(n) \quad (n \to \infty), \tag{8.15}$$

$$f_1^{n+1}(0) - f_1^n(0) \sim (ap)^{-1} n^{-1-(1/p)} \ell_0(n) \quad (n \to \infty). \tag{8.16}$$

Proof. We follow the arguments of [25, Lemma 5] and [32, Remark 1]. Let

$$u_0(x) = \int_x^1 \frac{dy}{y - f_0(y)}, \quad u_1(x) = \int_x^1 \frac{dy}{f_1(1-y) - (1-y)} \quad (x \in (0, 1]). \tag{8.17}$$

We use (8.1), (8.2), and Karamata's theorem [6, Theorem 1.5.11] to see

$$u_0(x) \sim \frac{1}{p x^p \ell^*(x)}, \quad u_1(x) \sim \frac{1}{p a^p x^p \ell^*(x)} \quad (x \to 0+). \tag{8.18}$$

By (8.18), the map $x \mapsto u_0(x^{-1})$ is regularly varying at ∞ of index p, and therefore, [6, Theorem 1.5.12] implies that its inverse function $1/u_0^{-1}(x)$ is regularly varying at ∞ of index $1/p$. Hence, there exists a function $\ell_0(x)$ slowly varying at ∞ such that

$$u_0^{-1}(x) \sim x^{-1/p} \ell_0(x) \quad (x \to \infty).$$

Since $u_1(x) \sim a^{-p} u_0(x) (x \to \infty)$, we have

$$u_1^{-1}(x) \sim (a^{-p} u_0)^{-1}(x) = u_0^{-1}(a^p x) \sim a^{-1} x^{-1/p} \ell_0(x) \quad (x \to \infty).$$

Using [23, Lemma 2] with $f(x) = f_0(x)$, $a_n = f_0^n(1)$ (respectively $f(x) = 1 - f_1(1 - x)$ and $a_n = 1 - f_1^n(0)$), and $g(x) \equiv 1$, we see $u_0(f_0^n(1)) \sim n (n \to \infty)$ (respectively $u_1(1 - f_1^n(0)) \sim n (n \to \infty)$). Hence, it follows from the uniform convergence theorem for regular varying functions [6, Theorem 1.5.2] that

$$f_0^n(1) = u_0^{-1}(u_0(f_0^n(1))) \sim u_0^{-1}(n) \sim n^{-1/p} \ell_0(n) \quad (n \to \infty), \tag{8.19}$$

$$1 - f_1^n(0) = u_1^{-1}(u_1(1 - f_1^n(0))) \sim u_1^{-1}(n) \sim a^{-1} n^{-1/p} \ell_0(n) \quad (n \to \infty). \tag{8.20}$$

Note that

$$f_0^n(1) = \sum_{k \geq n} (f_0^k(1) - f_0^{k+1}(1)), \quad 1 - f_1^n(0) = \sum_{k \geq n} (f_1^{k+1}(0) - f_1^k(0)).$$

In addition, $(f_0^n(1) - f_0^{n+1}(1))_{n \geq 0}$ and $(f_1^{n+1}(0) - f_1^n(0))_{n \geq 0}$ are decreasing sequences, since f_0 and f_1 are C^2 and $0 < f_0'(x), f_1'(y) < 1 (x \in (0, 1], y \in [0, 1))$. Hence, the desired result follows from (8.19), (8.20), and the monotone density theorem [6, Theorem 1.7.2]. □

LEMMA 8.5. *Define $\alpha, \beta_0, \beta_1 \in (0, 1)$ by*

$$\alpha = \frac{1}{p}, \quad \beta_0 = \frac{f_1'(0)}{f_1'(0) + f_0'(1) a^{-1}} = \frac{T'(c-)}{T'(c-) + T'(c+) a^{-1}}, \quad \beta_1 = 1 - \beta_0.$$

Set

$$\ell(x) = h(c)(f_1'(0) + f_0'(1) a^{-1})(1 - p^{-1})^{-1} \ell_0(x),$$

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which is slowly varying at ∞. *Then,*

$$w_n^{Y,A_i} \sim \beta_i n^{1-\alpha}\ell(n) \quad (n\to\infty,\ i=0,1). \tag{8.21}$$

Remark 8.6. For example, if $\ell^*(x)\sim C^*(x\to 0+)$ for some constant $C^*>0$ in Assumption 8.1(iii), then $\ell(x)\sim C(x\to\infty)$ for some constant $C>0$.

Proof of Lemma 8.5. Combining (8.4) with (8.15) and (8.16), we have

$$\begin{aligned}
&\mu(Y\cap(T^{-1}A_i)\cap\{\varphi=n\})\\
&\sim \begin{cases} h(c)f_1'(0)p^{-1}n^{-1-(1/p)}\ell_0(n) & (n\to\infty,\ i=0)\\ h(c)f_0'(1)(ap)^{-1}n^{-1-(1/p)}\ell_0(n) & (n\to\infty,\ i=1)\end{cases}\\
&\sim \alpha(1-\alpha)\beta_i n^{-1-\alpha}\ell(n) \qquad (n\to\infty,\ i=0,1).
\end{aligned} \tag{8.22}$$

We use (3.15), (8.22), and apply Karamata's theorem [6, Theorem 1.5.11] twice to obtain

$$w_n^{Y,A_i}=\sum_{k=1}^{n-1}\mu(Y\cap(T^{-1}A_i)\cap\{\varphi>k\})\sim\beta_i n^{1-\alpha}\ell(n) \quad (n\to\infty,\ i=0,1),$$

as desired. □

By (8.5) and (8.21), we get

$$\lim_{n\to\infty}\left(\frac{1}{w_n^Y}\sum_{k=0}^{n-1}\widehat{T}^k 1_{Y_k}\right)=\beta_0H^{(0)}+\beta_1H^{(1)}=:H \quad \text{in } L^\infty(\mu),$$

and

$$w_n^Y\sim w_n^{Y,A_0}+w_n^{Y,A_1}\sim n^{1-\alpha}\ell(n) \quad (n\to\infty).$$

Moreover, if $G:[0,1]\to[0,\infty)$ is Riemann integrable on $[0,1]$ with $\int_0^1 G(x)\,dx>0$, then G is uniformly sweeping for $1_{[\varepsilon,1-\varepsilon]}$ for any $\varepsilon\in(0,1/2)$, which follows from [27, Theorem 8.1]. Therefore, H, $H^{(0)}$, $H^{(1)}$ are uniformly sweeping for $1_{[\varepsilon,1-\varepsilon]}$ and hence, for 1_Y. So we use our main results in §3 to obtain the following theorems.

THEOREM 8.7. *Let* $\{c(n)\}_{n\ge0}$ *and* $\{\widetilde{c}(n)\}_{n\ge0}$ *be positive sequences satisfying (3.1) and (3.9), respectively. Then, we have (3.2), (3.10), and*

$$\mu_{H^{(i)}}\left(\frac{S_n^{A_i}}{n}\le c(n)\right)\sim\frac{1-\beta_i}{\beta_i}\frac{\sin(\pi\alpha)}{\pi\alpha}\frac{c(n)^\alpha\ell(n)}{\ell(c(n)n)} \quad (n\to\infty,\ i=0,1).$$

THEOREM 8.8. *Assume* $G\in\{u\in L^1(\mu):u\ge0\}$ *admits a version that is Riemann integrable on* $[0,1]$ *with* $\int_0^1 G(x)\,dx>0$. *Then, there exists some constant* $C_1\in(0,\infty)$ *such that, for any positive sequences* $\{c(n)\}_{n\ge0}$ *and* $\{\widetilde{c}(n)\}_{n\ge0}$ *satisfying (3.1) and (3.9), we have (3.6), (3.12), and*

$$C_1\le\liminf_{n\to\infty}\frac{\ell(c(n)n)}{c(n)^\alpha\ell(n)}\mu_G\left(\frac{S_n^{A_i}}{n}\le c(n)\right) \quad (i=0,1).$$

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THEOREM 8.9. *Assume $G \in \{u \in L^\infty(\mu) : u \geq 0\}$ is supported on $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0, 1/2)$. Then, there exists some constant $C_2 \in (0, \infty)$ such that, for any positive sequences $\{c(n)\}_{n\geq 0}$ and $\{\widetilde{c}(n)\}_{n\geq 0}$ satisfying (3.1) and (3.9), we have (3.7), (3.13), and*

$$\limsup_{n\to\infty} \frac{\ell(c(n)n)}{c(n)^\alpha \ell(n)} \mu_G\bigg(\frac{S_n^{A_i}}{n} \leq c(n)\bigg) \leq C_2 \quad (i = 0, 1).$$

Remark 8.10. Let ν be a probability measure on $[0, 1]$ that is supported on $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0, 1/2)$ and admits a Riemann integrable density u with respect to the Lebesgue measure. Then, $G = u/h$ is also supported on $[\varepsilon, 1-\varepsilon]$ and Riemann integrable, and hence, Theorems 8.8 and 8.9 can be applied to $\nu = \mu_G$.

Remark 8.11. Let $0 < \alpha < 1$ and let $\ell_i : (0, \infty) \to (0, \infty) (i = 0, 1)$ be slowly varying at ∞. As shown below, there exists $T : [0, 1] \to [0, 1]$ satisfying conditions (i) and (ii) of Assumption 8.1 with $c = 1/2$ and

$$w_n^{Y, A_i} \sim d_i n^{1-\alpha} \ell_i(n) \quad (n \to \infty,\ i = 0, 1) \tag{8.23}$$

for some constant $d_i > 0$, where Y and A_i are chosen as in (8.3). Let us construct such a map T. Let $\phi_i(x) = x^{-\alpha}\ell_i(x)$. We may assume that $\phi_i(x)$ is bounded below on $(0, R)$ for any $R > 0$. Applying [6, Theorem 1.5.12] to $f_i(x) = 1/\phi_i(x)$, we can see that the right-continuous inverse $f_i^{-1}(y) := \sup\{y \in (0, \infty) : f_i(y) > x\}$ is a regularly varying function at ∞ of index $1/\alpha$ satisfying $f_i(f_i^{-1}(x)) \sim f_i^{-1}(f_i(x)) \sim x (x \to \infty)$. By [6, Theorem 1.8.2], we can take a C^∞ function $\psi_i : (0, \infty) \to (0, \infty)$ such that $\psi_i(x) \sim f_i^{-1}(x^{-1}) (x \to 0+)$. Then, $\psi_i(x)$ is a regularly varying function at 0 of index $-1/\alpha$ satisfying $\phi_i(\psi_i(x)) \sim x (x \to 0+)$ and $\psi_i(\phi_i(x)) \sim x (x \to \infty)$. Set

$$\Psi_i(x) = \int_0^x \bigg(\int_0^y \frac{dt}{t\psi_i(t)}\bigg)\, dy, \quad x \geq 0.$$

Karamata's theorem [6, Theorem 1.5.11] implies that $\Psi_i(x) \sim \alpha^2(1+\alpha)^{-1} x/\psi_i(x)$ $(x \to 0+)$. Take a constant $b_i > 0$ so that $\Psi_i(b_i/2) = 1/2$. We now define $T : [0, 1] \to [0, 1]$ by

$$Tx = \begin{cases} x + \Psi_0(b_0 x), & x \in [0, 1/2], \\ x - \Psi_1(b_1(1-x)), & x \in (1/2, 1]. \end{cases}$$

It is easily seen that T satisfies conditions (i) and (ii) of Assumption 8.1 with $c = 1/2$. In addition,

$$Tx - x \sim \frac{\alpha^2 b_0^{1+1/\alpha}}{1+\alpha} \frac{x}{\psi_0(x)}, \quad (1-x) - T(1-x) \sim \frac{\alpha^2 b_1^{1+1/\alpha}}{1+\alpha} \frac{x}{\psi_1(x)} \quad (x \to 0+).$$

Define $u_0(x)$ and $u_1(x)$ by (8.17). Then,

$$u_i(x) \sim \frac{1+\alpha}{\alpha b_i^{1+1/\alpha}} \psi_i(x) \quad (x \to 0+,\ i = 0, 1),$$

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and hence,

$$u_i^{-1}(x) \sim \left(\frac{1+\alpha}{\alpha b_i^{1+1/\alpha}}\right)^{\alpha} \phi_i(x) \quad (x \to \infty,\ i = 0, 1).$$

Therefore, as in (8.21), we obtain (8.23) for some $d_i > 0$.

Remark 8.12. Following [33, Example 7.1], let us construct $T : [0, 1] \to [0, 1]$ and $Y \subset [0, 1]$ satisfying conditions (A1) and (A2) of Theorem 3.1 and condition (B2) of Theorem 3.10, but violating condition (A3) of Theorem 3.1. We can apply Theorems 3.8 and 3.10, but cannot apply Theorem 3.1 to such T and Y. Let $\ell_0(x)$ be a slowly varying function at ∞ satisfying

$$\liminf_{x\to\infty} \ell_0(x) = 0, \quad \limsup_{x\to\infty} \ell_0(x) = \infty. \tag{8.24}$$

An example of such a function can be found in [6, §1.3.3]. Let $\ell_1(x) \equiv 1$. By Remark 8.11, there exists a map $T : [0, 1] \to [0, 1]$ satisfying conditions (i) and (ii) of Assumption 8.1 with $c = 1/2$ and (8.23) for some constant $d_i > 0$, where Y and A_i are chosen as in (8.3) with $Tc_0 < Tc_1$. Then, $(TA_0) \cap (TA_1) = \emptyset$. In addition,

$$w_n^Y \sim w_n^{Y,A_0} + w_n^{Y,A_1} \sim n^{1-\alpha}(d_0\ell_0(n) + d_1) \quad (n \to \infty),$$

and $d_0\ell_0(x) + d_1$ is slowly varying at ∞, since, for any $\lambda > 0$,

$$\left|\frac{d_0\ell_0(\lambda x) + d_1}{d_0\ell_0(x) + d_1} - 1\right| = \left|\frac{d_0(\ell_0(\lambda x) - \ell_0(x))}{d_0\ell_0(x) + d_1}\right| \le \left|\frac{\ell_0(\lambda x)}{\ell_0(x)} - 1\right| \to 0 \quad (x \to \infty).$$

Therefore, condition (A1) is verified. Moreover, there exist μ-probability density functions $H^{(0)}, H^{(1)} : [0, 1] \to [0, \infty)$ such that $H^{(i)}$ is supported and has bounded variation on $(TA_i) \setminus A_i \subset Y (i = 0, 1)$, and (8.5) holds. Note that condition (A3) does not hold because $H^{(0)} \neq H^{(1)}$ and (8.24). By [27, Theorem 8.1], H_0 and H_1 are uniformly sweeping for 1_Y. Hence, there exists $N \in \mathbb{N}$ such that

$$\left\{\frac{1}{w_n^{Y,A_i}} \sum_{k=0}^{n-1} \widehat{T}^k 1_{Y_k \cap A_i}\right\}_{n \ge N;\, i=0,1}$$

is $L^\infty(\mu)$-bounded and uniformly sweeping for 1_Y. Therefore, conditions (A2) and (B2) are verified, as desired.

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