A CENSUS OF PLANAR TRIANGULATIONS

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1. Triangulations. Let P be a closed region in the plane bounded by a simple closed curve, and let S be a simplicial dissection of P. We may say that S is a dissection of P into a finite number α of triangles so that no vertex of any one triangle is an interior point of an edge of another. The triangles are "topological" triangles and their edges are closed arcs which need not be straight segments. No two distinct edges of the dissection join the same two vertices, and no two triangles have more than two vertices in common.

There are $k \geqslant 3$ vertices of S in the boundary of P, and they subdivide this boundary into k edges of S. We call these edges *external* and the remaining edges of S, if any, *internal*. If r is the number of internal edges we have

$$3\alpha = 2r + k, \tag{1.1}$$

$$r \equiv k \pmod 3. \tag{1.2}$$

Let us call S a *triangulation* of P if it satisfies the following condition: *no internal edge of S has both its ends in the boundary of P.* We note that in the case $k = 3$ every simplicial dissection is a triangulation.

Let T_1 and T_2 be triangulations of P having the same external edges. We call them *isomorphic* if there is a $1-1$ mapping f of the vertices of T_1 onto those of T_2 which satisfies the following conditions.

(i) *Each vertex in the boundary of P is mapped by f onto itself.*

(ii) *Two distinct vertices v and w of T_1 are joined by an edge of T_1 if and only if $f(v)$ and $f(w)$ are joined by an edge of T_2.*

(iii) *Three distinct vertices u, v, and w of T_1 define a triangle of T_1 if and only if $f(u)$, $f(v)$, and $f(w)$ define a triangle of T_2.*

The triangulations of the polygon $abcd$ shown in Figures I A and I B are isomorphic, but those of Figures I B and I C are not.

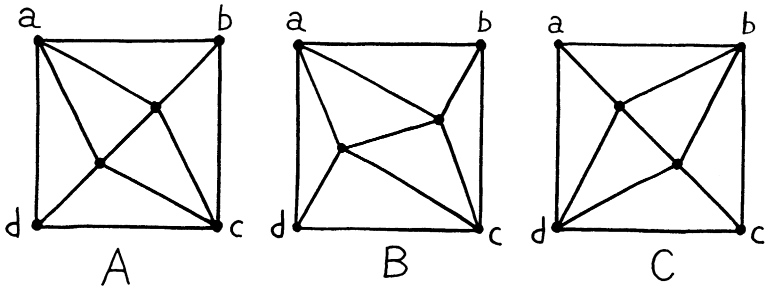

FIGURE 1

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The number of non-isomorphic triangulations of P with k given external edges and r internal edges depends only on k and r, since all possible regions P are homeomorphic. We write $\psi_{n,m}$ for the number of such triangulations where

$$k = m + 3, \tag{1.3}$$

$$r = 3n + m. \tag{1.4}$$

It is clear that m and n must be non-negative integers (see **1.2**). We observe that $\psi_{0,0} = 1$, this being the number of triangulations with three given external edges and no internal ones. It can be verified that $\psi_{1,0} = 1$ and $\psi_{2,0} = 3$. In the latter case the triangulations are given by the three rotations of the pattern of Figure 2. We find also that $\psi_{3,0} = 13$. The corresponding triangulations are represented by the four diagrams of Figure 3. The first diagram gives rise, by rotation and reflection, to six triangulations. The next two give rise to three each and the last one to one only.

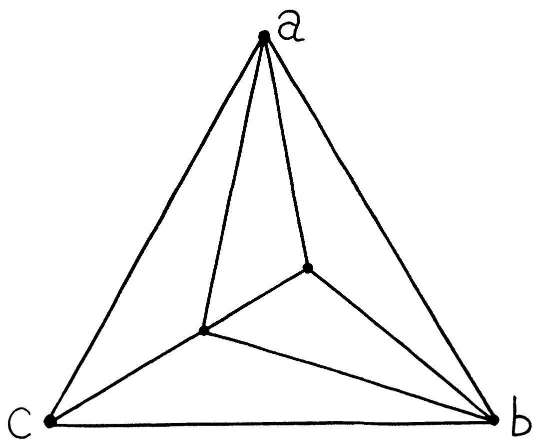

FIGURE 2

We shall prove that

$$\psi_{n,0} = \frac{2}{(n+1)!}\,(3n+3)\,(3n+4)\ldots(4n+1) \tag{1.5}$$

when $n \geqslant 2$. Our main objective in this paper is the complete evaluation of the function $\psi_{n,m}$ (§ 5).

Let us call a triangulation *simple* if no three edges, not all external, form a simple closed curve enclosing a region which is subdivided into three or more triangles. Thus the triangulation satisfying $n = 1$ and $m = 0$ is simple. As another example we may take the last diagram in Figure 3.

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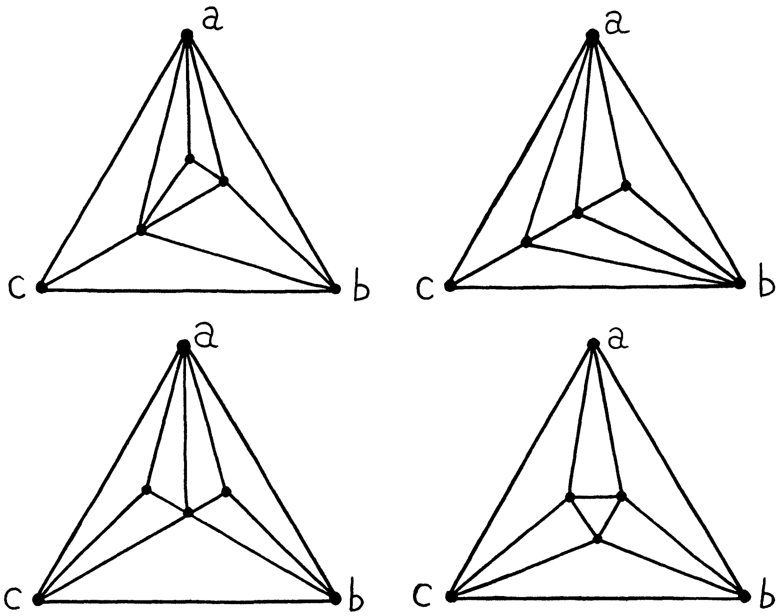

FIGURE 3

We write $\phi_{n,m}$ for the number of simple triangulations with specified values of m and n. In § 7 we obtain an explicit expression for $\phi_{n,0}$. In § 8 we discuss the behaviour of $\psi_{n,0}$ and $\phi_{n,0}$ as $n \to \infty$.

It can be shown that if T is a triangulation of a convex polygon, and if the vertices of T in the boundary are the vertices of the polygon, then T has an isomorphic triangulation in which every edge is a straight segment. This follows from **(1)**, together with Hassler Whitney's theorem that a triply connected planar graph can be represented in the plane in essentially only one way **(2)**. But we use general topological triangulations in this paper because the argument of § 3 requires us to consider triangulations of regions which are not convex.

In § 9 we discuss the behaviour as $n \to \infty$ of the number of simplicial dissections of the 2-sphere into $2n$ triangles.

2. Generating functions. In what follows we shall use the following formal power series:

2.1 $$\psi(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \psi_{n,m} x^n y^m,$$

2.2 $$g(x) = \sum_{n=0}^{\infty} \psi_{n,0} x^n,$$

2.3 $$\phi(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \phi_{n,m} x^n y^m,$$

2.4 $$h(x) = \sum_{n=0}^{\infty} \phi_{n,0} x^n.$$

In this section we obtain some formulae which make possible the computation of ϕ and h when ψ and g are known.

It is clear that to each triangulation T with $r > 0$ there corresponds a unique simple triangulation T' (with $r > 0$) such that either $T = T'$ or T can be derived from T' by further subdivision of the triangles. A triangle of T' may be characterized as a closed region bounded by three edges of T, not all external, which is not contained in any other such region.

Let T' be a simple triangulation with r internal edges, k external ones, and $\alpha > 1$ triangles. The triangulations derivable from T' are enumerated by the series

$$\{g(x)\}^{\alpha}$$

The coefficient of x^p in this is the number of such triangulations with k external edges and

$$3p + \tfrac{1}{2}(3\alpha - k)$$

internal ones. Let us define m and n in terms of T' by (1.3) and (1.4) and let us write

$$3p + \tfrac{1}{2}(3\alpha - k) = 3s + m,$$
$$s = p + \tfrac{1}{2}\alpha - \tfrac{1}{2}m - \tfrac{1}{2}.$$

Let the number of triangulations derivable from T' and with a given value of s be M_s. Then

$$\{g(x)\}^{\alpha} = \sum_{s=0}^{\infty} M_s x^{s-\frac{1}{2}\alpha+\frac{1}{2}m+\frac{1}{2}},$$
$$\{x^{\frac{1}{2}}g(x)\}^{\alpha} = \sum_{s=0}^{\infty} M_s x^s x^{\frac{1}{2}m+\frac{1}{2}}.$$

Let $q(\alpha, m)$ be the number of simple triangulations with α triangles and $m + 3$ given external edges. Then

$$\sum_{\alpha=3}^{\infty} q(\alpha, m)\{x^{\frac{1}{2}}g(x)\}^{\alpha} = x^{\frac{1}{2}} \sum_{s=1}^{\infty} \psi_{s,m} x^s x^{\frac{1}{2}m}.$$

By (1.1), (1.3), and (1.4) we have $\alpha = 2n + m + 1$. We may therefore rewrite the above equation as

$$g(x) \sum_{n=1}^{\infty} \phi_{n,m}\{x^{\frac{1}{2}}g(x)\}^{2n}\{g(x)\}^{m} = \sum_{s=1}^{\infty} \psi_{s,m} x^s.$$

Multiplying by y^m and summing over m we obtain

$$g(x)\{\phi(xg^2(x), yg(x)) - 1\} = \psi(x, y) - 1. \tag{2.5}$$

We observe that the coefficients of ϕ can be deduced from this identity when those of ψ are known. In this paper we attempt the calculation only in the special case $m = 0$. Putting $y = 0$ in (2.5) we obtain

$$h(xg^2(x)) = 2 - \{g(x)\}^{-1}. \tag{2.6}$$

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So far we have not discussed the convergence of our generating functions. It is not necessary to do so if we regard an equation such as (2.5) as merely asserting the equality of the coefficients of $x^n y^m$ in the expanded functions on the two sides. It will be seen that all such coefficients are finite expressions in terms of the coefficients of ψ, g, ϕ and h. However, in what follows we shall express the generating series in terms of simple analytic functions and so establish their convergence for appropriate non-zero values of x and y.

3. An Equation for ψ. Let T be a triangulation of P with k external and r internal edges, r being non-zero. Let A be an external edge, with ends x and y, and let z be the third vertex of the triangle incident with A. By removing from P the interior of the triangle xyz and all the points of A except x and y we obtain another region P' bounded by a simple closed curve. If L is the complementary arc of A in the boundary of P then the boundary of P' is made up of L and the two edges xz and yz (see Figure 4).

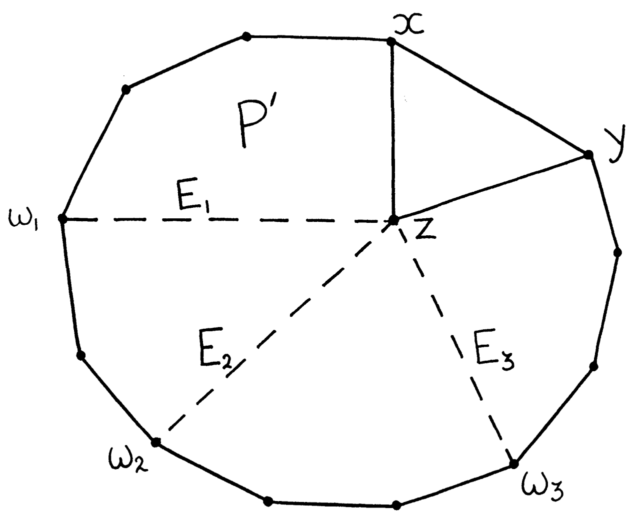

FIGURE 4

The triangles of T other than xyz determine a simplicial dissection S' of P'. It is not necessarily a triangulation, but any internal edge of S' which has both ends in the boundary of P' must have one end at z and the other at a point of L other than x and y. If there are c such edges, and $c > 0$, we enumerate them as $E_1, E_2, \ldots, E_c$, with ends $w_1, w_2, \ldots, w_c$ respectively on L, arranging that the points x, y, w_i occur in the order $x, w_1, w_2, \ldots, w_c, y$, on L.

If $c > 0$ the edges $E_1, \ldots, E_c$ partition P' into $c + 1$ closed regions $P_0, P_1, \ldots, P_c$ bounded by simple closed curves $zxw_1, zw_1w_2, \ldots, zw_cy$ respectively, and the triangulation T induces triangulations $T_0, T_1, \ldots, T_c$ respectively of these regions.

If $c = 0$ we write $P_0 = P'$ and $T_0 = S'$. In this case T_0 has at least four external edges. But if $c > 0$ the triangulations T_q are not restricted in this way.

It is clear that we can obtain the triangulations of P with k fixed external edges, one of which is A, and given values of r and c by starting with the diagram of Figure 4 and taking all sets of triangulations of the regions P_i which agree with T on L, in which xz, $zw_1, \ldots, zw_c$ and zy are edges, and which give rise to the correct value of r.

Let n and m be defined for T by (1.3) and (1.4). Let T_q have k_q external and r_q internal edges, and led n_q and m_q be defined correspondingly. Then

$$\sum_{q=0}^{c} r_q = r - c - 2, \tag{3.1}$$

$$\sum_{q=0}^{c} k_q = k + 2c + 1, \tag{3.2}$$

$$\sum_{q=0}^{c} m_q = m - c + 1, \tag{3.3}$$

$$\sum_{q=0}^{c} n_q = n - 1. \tag{3.4}$$

We deduce from the foregoing considerations that if $r > 0$ then

$$\psi_{n,m} = \sum_{c=0}^{\infty} \sum \prod_{q=0}^{c} \psi_{n_q,m_q} \tag{3.5}$$

where the second summation is over all ordered sets of non-negative integers $n_0, \ldots, n_c, m_0, \ldots, m_c$) which satisfy (3.3) and (3.4). But the sum

$$\sum \prod_{q=0}^{c} \psi_{n_q,\, m_q}$$

occurring in (3.5) is the coefficient of $x^{n-1}y^{m-c+1}$ in $\{\psi(x, y)\}^{c+1}$, that is, the coefficient of $x^n y^m$ in

$$xy^{-2}\{y\psi(x, y)\}^{c+1}.$$

We must not deduce that this expression, summed over c, is identical with $\psi(x, y)$. For one thing $\psi(x, y)$ has a constant term $\psi_{0,0} = 1$, and (3.5) does not apply to the case $r = 0$, that is, $n = m = 0$. For another, the construction associated with Figure 4 requires $k_0 \geqslant 4$, that is, $m_0 \geqslant 1$, if $c = 0$. Taking these limitations into account we have

$$\psi + xy^{-1}g = 1 + xy^{-2} \sum_{c=0}^{\infty} (y\psi)^{c+1}, \tag{3.6}$$

where $\psi = \psi(x, y)$ and $g = g(x)$.

Multiplying both sides of (3.6) by $y(1 - y\psi)$ and rearranging we have

$$(y\psi + xg - y)(1 - y\psi) = x\psi, \tag{3.7}$$

which may be rewritten as

$$y^2\psi^2 + (x + xgy - y - y^2)\psi + y - xg = 0. \tag{3.8}$$

Our problem now is to solve (3.8) for ψ as a power series in x and y and for g as a power series in x (with no negative indices). It is of course required, by (2.1) and (2.2), that $g(x) = \psi(x, 0)$. However, this is not an independent condition; it is obtained from (3.8) when we substitute $y = 0$.

We proceed to show that (3.8) has a unique solution for ψ. Putting $x = 0$ in any such solution we obtain a power series ψ_0 in y, and ψ_0 must satisfy

$$y^2\psi_0^2 - y(y + 1)\psi_0 + y = 0,$$

that is,

$$y(y\psi_0 - 1)(\psi_0 - 1) = 0.$$

But the only series in non-negative powers of y which satisfies this equation identically is the series with the single term 1.

Now write

$$\psi = \sum_{n=0}^{\infty} \psi_n x^n,$$

where the ψ_i are power series in y. We have shown that $\psi_0 = 1$. Equating coefficients of x^n in (3.8), with $n > 0$, we obtain an equation giving $(y^2 - y)\psi_n$ in terms of $\psi_0, \psi_1, \ldots, \psi_{n-1}$. Hence, by induction ψ is uniquely determined by (3.8). (We use the fact that $g(x) = \psi(x, 0) = \sum(\psi_n)_{y=0}x^n$.)

4. Solution of equation (3.8). The solution of (3.8) can be derived from that of the quadratic equation

$$Y^2 + Y(1 - t + \theta t) + \theta t = 0, \tag{4.1}$$

where θ and t are independent variables. There are two solutions Y_1 and Y_2 of (4.1) for Y as a power series in t and θ, corresponding to the constant terms 0 and -1 respectively. They can be obtained by applying the binomial theorem to the expression

$$\tfrac{1}{2}\{-(1 - t + \theta t) \pm \{(1 - t + \theta t)^2 - 4\theta t\}^{\frac{1}{2}}\}. \tag{4.2}$$

This gives Y_1 or Y_2 according as we take the positive or the negative sign before the square root. The series are convergent for sufficiently small values of θ and t. If the positive sign is taken (4.2) vanishes whenever one of θ or t is zero, whatever the value of the other may be. Moreover, the coefficient of θt is then found to be -1. It follows from these observations that

$$\frac{Y_1}{\theta t} \text{ and } \frac{\theta t}{Y_1}$$

are power series in θ and t (without negative indices), and with the constant term -1. Hence we may deduce that there is a power series Φ in θ and t (without negative indices) defined by the equation

$$(4.3)\qquad (1-\theta)^3 t\Phi = \frac{\theta t}{Y_1(1+Y_1)^2} + 1$$
$$= 1 + \theta t\left\{\frac{1}{Y_1} - \frac{1}{1+Y_1} - \frac{1}{(1+Y_1)^2}\right\}.$$

Write

$$U_i = \frac{\theta t}{Y_i(1+Y_i)^2}$$

($i = 1, 2$). Then

$$(4.4)\qquad U_1U_2 = \frac{\theta^2 t^2}{Y_1Y_2(1+Y_1+Y_2+Y_1Y_2)^2}$$
$$= \frac{\theta^2 t^2}{\theta t\cdot(1-(1-t+\theta t)+\theta t)^2} = \theta t^{-1},$$

$$(4.5)\qquad U_1 + U_2 = \frac{\theta t(Y_1 + 2Y_1^2 + Y_1^3 + Y_2 + 2Y_2^2 + Y_2^3)}{Y_1Y_2(1+Y_1+Y_2+Y_1Y_2)}$$
$$= t^{-2}\{(Y_1+Y_2)(Y_1+Y_2+1)^2 - Y_1Y_2(3Y_1+3Y_2+4)\}$$
$$= t^{-2}\{(-1+(1-\theta)t)(1-\theta)^2t^2 - \theta t(1+3(1-\theta)t\}$$
$$= -\theta t^{-1} - (1+\theta-2\theta^2) + t(1-\theta)^3.$$

Hence

$$(U_1+1)(U_2+1) = -\theta(1-2\theta) + t(1-\theta)^3,$$
$$(U_1+1)+(U_2+1) = -\theta t^{-1} + (1-\theta+2\theta^2) + t(1-\theta)^3.$$

It follows that Φ is a solution of the quadratic equation

$$(4.6)\qquad (1-\theta)^6t^2\Phi^2 + (\theta t^{-1} - (1-\theta+2\theta^2) - t(1-\theta)^3)(1-\theta)^3t\Phi$$
$$- \theta(1-2\theta) + t(1-\theta)^3 = 0.$$

This equation becomes identical with (3.8) if we put $\psi = \Phi$ and

$$(4.7)\qquad y = (1-\theta)^3 t,$$

$$(4.8)\qquad x = \theta(1-\theta)^3,$$

$$(4.9)\qquad xg = \theta(1-2\theta).$$

Let us agree that θ is that solution (given by Lagrange's Theorem) of (4.8) as a power series in x which has a zero constant term. Then (4.9) determines g as an analytic function of x (**3**, p. 132). We have in fact, by Lagrange's Theorem,

$$g = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}\left[\frac{d^{n-1}}{d\theta^{n-1}}\frac{1-4\theta}{(1-\theta)^{3n}}\right]_{\theta=0}$$
$$= 1 + \sum_{n=1}^{\infty} \frac{x^n}{(n+1)!}\frac{d^n}{d\theta^n}\left[\frac{4}{(1-\theta)^{3n+2}} - \frac{3}{(1-\theta)^{3n+3}}\right]_{\theta=0}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{x^n}{(n+1)!} \{4(3n+2)(3n+3)\ldots(4n+1)$$
$$- 3(3n+3)(3n+4)\ldots(4n+2)\},$$

$$g = 1 + x + 2\sum_{n=2}^{\infty} \frac{x^n}{(n+1)!}(3n+3)(3n+4)\ldots(4n+1). \tag{4.10}$$

In view of the discussion at the end of § 3 we conclude that ψ is the function Φ defined by (4.3), and that g is indeed given by (4.10). This establishes (1.5).

5. The coefficients in ψ. We can write (4.1) as

$$Y = (t-1) - \theta t\left(1 + \frac{1}{Y}\right). \tag{5.1}$$

We treat t as a fixed non-zero number and use Lagrange's Theorem to expand the function

$$U = \frac{\theta t}{Y(1+Y)^2}$$

in powers of θ. We find

$$U = \theta t\left\{\frac{1}{a(1+a)^2} + \sum_{j=1}^{\infty} \frac{(-\theta t)^j}{j!}\frac{d^{j-1}}{da^{j-1}}\left\{\left(\frac{1+a}{a}\right)^j \frac{d}{da}\left(\frac{1}{a(1+a)^2}\right)\right\}\right\}, \tag{5.2}$$

where a is to be set equal to $t-1$ when the differentiations have been performed. Of course U corresponds to only one of the roots Y of (5.1). We may now write

$$\begin{aligned} U &= \frac{-\theta}{t(1-t)} \\ &\quad -\theta t\sum_{j=1}^{\infty}\frac{(\theta t)^j}{j!}\frac{d^{j-1}}{dt^{j-1}}\left\{\left(\frac{t}{1-t}\right)^j \frac{1}{t^3}\left(\frac{-3}{1-t} + \frac{1}{(1-t)^2}\right)\right\} \\ &= -\frac{\theta}{t} - \frac{\theta}{1-t} \\ &\quad -\theta t\sum_{j=1}^{\infty}\frac{\theta^j t^j}{j!}\frac{d^{j-1}}{dt^{\,j-1}}\left\{t^{j-3}\left(\frac{-3}{(1-t)^{j+1}} + \frac{1}{(1-t)^{j+2}}\right)\right\}. \end{aligned} \tag{5.3}$$

Write

$$\begin{aligned} A &= t^{3j+3}\frac{d}{dt}\left(\frac{t^{1-3j}}{1-t}\right) \\ &= \frac{t^3}{(1-t)^2}(1 - 3j + 3jt) \\ &= 3j(1-t)^2 - (9j+1)(1-t) \\ &\quad + (9j+3) - \frac{3(j+1)}{1-t} + \frac{1}{(1-t)^2}. \end{aligned} \tag{5.4}$$

Then

$$\frac{dA}{dt} = -6j(1-t) + (9j+1) - \frac{3\,.\,1\,.\,(j+1)}{(1-t)^2} + \frac{2}{(1-t)^3}, \tag{5.5}$$

$$\frac{d^2A}{dt^2} = 6j - \frac{3\,.\,2\,.\,1\,.\,(j+1)}{(1-t)^3} + \frac{3\,.\,2}{(1-t)^4}, \tag{5.6}$$

and if $j > 2$ we have

$$\frac{d^jA}{dt^j} = (j+1)!\left\{\frac{-3}{(1-t)^{j+1}} + \frac{1}{(1-t)^{j+2}}\right\}. \tag{5.7}$$

Applying these formulae to (5.3) we obtain

$$\begin{aligned}
U = -\theta t \sum_{j=1}^{\infty} \Bigg[& \frac{\theta^j t^j}{j!(j+1)!} \\
& \times \frac{d^{j-1}}{dt^{j-1}}\left\{t^{j-3}\frac{d^j}{dt^j}\left\{t^{3j+3}\frac{d}{dt}\left(\frac{t^{1-3j}}{1-t}\right)\right\}\right\}\Bigg] - \frac{\theta}{t} - \frac{\theta}{1-t} \\
& + \theta t\left\{\frac{\theta t}{1!2!}t^{-2}(4+6t) + \frac{\theta^2t^2}{2!3!}\frac{d}{dt}12t^{-1}\right\} \\
= -\theta t \sum_{j=1}^{\infty} & \frac{\theta^j t^j}{j!(j+1)!}\sum_{m=1}^{\infty}\frac{(m+2)!(m-1)!(m-3j)}{(m-j+2)!(m-j)!}t^{m-j} \\
& - \frac{\theta}{t} - \frac{\theta}{1-t} + \theta^2(2+3t) - \theta^3 t,
\end{aligned}$$

by expanding $t^{1-3j}(1-t)^{-1}$ as a power series in t and differentiating term by term. We must suppose $|t| < 1$.

Now U is one of the functions U_1 and U_2 of § 4. Let U' denote the other one. By (4.5) U' contains no term in t^{-1} and can be written as a power series in θ and t without negative indices. But U, which contains the term $-\theta/t$ cannot be so written. Hence U' is the function U_1 used in (4.3). Since ψ has been identified with the function Φ of that equation we have, by (4.5) and (4.7)

$$\begin{aligned}
y\psi = \theta t \sum_{m=1}^{\infty} t^m \sum_{j=1}^{\infty} & \frac{(m+2)!(m-1)!(m-3j)}{j!(j+1)!(m-j+2)!(m-j)!}\theta^j \\
& + \frac{\theta}{1-t} - \theta^2(2+3t) + \theta^3 t - (1+\theta-2\theta^2) + t(1-\theta)^3 + 1,
\end{aligned}$$

$$\begin{aligned}
y\psi = t(1-2\theta)& \\
& + \theta t \sum_{m=1}^{\infty} t^m\left\{1 + \sum_{j=1}^{m}\frac{(m+2)!(m-1)!(m-3j)}{j!(j+1)!(m-j+2)!(m-j)!}\theta^j\right\}.
\end{aligned}$$

Hence, by (4.7), we have

$$(5.8)\quad \psi = \frac{1-2\theta}{(1-\theta)^3} + \sum_{m=1}^{\infty} \frac{y^m}{(1-\theta)^{3m+3}} \sum_{j=0}^{m} \frac{(m+2)!(m-1)!(m-3j)}{j!(j+1)!(m-j+2)!(m-j)!}\theta^{j+1}.$$

In order to express ψ directly in terms of x and y we obtain the expansion of $\theta^p(1-\theta)^{-q}$ in powers of x, where p and q are both positive integers.

$$\begin{aligned}
\frac{\theta^p}{(1-\theta)^q} &= \sum_{s=1}^{\infty} \frac{x^s}{s!}\left[\frac{d^{s-1}}{d\theta^{s-1}}\left\{\frac{1}{(1-\theta)^{3s}}\frac{d}{d\theta}\frac{\theta^p}{(1-\theta)^q}\right\}\right]_{\theta=0} \\
&= \sum_{s=1}^{\infty} \frac{x^s}{s!}\left[\frac{d^{s-1}}{d\theta^{s-1}}\left\{\theta^{p-1}\left\{\frac{q}{(1-\theta)^{3s+q+1}} + \frac{p-q}{(1-\theta)^{3s+q}}\right\}\right\}\right]_{\theta=0} \\
&= \sum_{s=1}^{\infty} \frac{x^s}{s!}\left\{\frac{q}{(3s+q)!}\left[\frac{d^{s-1}}{d\theta^{s-1}}\left\{\theta^{p-1}\frac{d^{3s+q}}{d\theta^{3s+q}}\left(\frac{1}{1-\theta}\right)\right\}\right]_{\theta=0}\right. \\
&\qquad \left. + \frac{p-q}{(3s+q-1)!}\left[\frac{d^{s-1}}{d\theta^{s-1}}\left\{\theta^{p-1}\frac{d^{3s+q-1}}{d\theta^{3s+q-1}}\left(\frac{1}{(1-\theta)}\right)\right\}\right]_{\theta=0}\right\} \\
&= \sum_{s=p}^{\infty} \frac{x^s}{s!}\left\{\frac{q(4s+q-p)!(s-1)!}{(3s+q)!(s-p)!} + \frac{(p-q)(4s+q-p-1)!(s-1)!}{(3s+q-1)!(s-p)!}\right\} \\
&= \sum_{s=p}^{\infty} \frac{x^s(4s+q-p-1)!}{s(s-p)!(3s+q)!}\{q(4s+q-p)+(p-q)(3s+q)\},
\end{aligned}$$

$$(5.9)\qquad \frac{\theta^p}{(1-\theta)^q} = (3p+q)\sum_{s=p}^{\infty} \frac{(4s+q-p-1)!\,x^s}{(s-p)!(3s+q)!}.$$

The above argument is valid also for $p > 0$ and $q = 0$. Hence (5.9) gives θ, θ^2, θ^3, etc. in terms of x.

Combining (5.8) and (5.9) we find that, for $m > 0$,

$$\psi_{n,m} = \sum_{j=0}^{m}\left\{\frac{(m+2)!(m-1)!(m-3j)}{j!(j+1)!(m-j+2)!(m-j)!} \times \frac{(4n+3m-j+1)!(3m+3j+6)}{(n-j-1)!(3n+3m+3)!}\right.,$$

where only values of $j \leqslant n-1$ are to be considered. Hence for $m > 0$,

$$(5.10)\quad \psi_{n,m} = \frac{3(m+2)!(m-1)!}{(3n+3m+3)!} \times \sum_{j=0}^{m} \frac{(4n+3m-j+1)!(m+j+2)(m-3j)}{j!(j+1)!(m-j)!(m-j+2)!(n-j-1)!}.$$

For the case $m = 0$ we have

$$(5.11)\qquad \psi_{n,0} = \frac{2(4n+1)!}{(3n+2)!(n+1)!},$$

by (4.10). As special cases of formula (5.10) we have

$$\psi_{n,1} = \frac{72(4n+3)!}{(3n+6)!(n-1)!}, \tag{5.12}$$

$$\psi_{n,2} = \frac{60(13n+9)(4n+5)!}{(3n+9)!(n-1)!}, \tag{5.13}$$

$$\psi_{n,3} = \frac{120(67n^2+138n+68)(4n+7)!}{(3n+12)!(n-1)!}. \tag{5.14}$$

6. An acknowledgement. The foregoing solution of (3.8) was arrived at by the following indirect method. Expansion of ψ as a power series in x, y, and the unknown function g led to the formula

$$-x^{m+1}\psi_m = \sum_{\substack{0\leqslant l\leqslant m+1,\\ 0\leqslant k\leqslant \frac{1}{2}l}} (-1)^l \frac{(2m-l-k+2)!x^k(xg+1)^{l-2k}}{(m-l+1)!(m-l+2)!k!(l-2k)!}. \tag{6.1}$$

where ψ_m is now the sum of the terms of ψ involving y^m, By expanding $(xg+1)^{l-2k}$ and summing over l this was replaced by

$$\begin{aligned} &-x^{m+1}\psi_m \\ &= \sum_{\substack{0\leqslant j<\infty\\ 0\leqslant k<\infty}} \frac{x^{j+k}(-g)^j(m-k+1)!(m-k)!}{j!k!(j+k-1)!(m-2k-j+2)!(m-2k-j+1)!}. \end{aligned} \tag{6.2}$$

Had the coefficients in g been known this would have given the coefficient of x^n on the left-hand side as a polynomial in m, m being sufficiently large. It was argued that this polynomial would vanish identically. Hence it was permissible to put $m = 0$ in the expression on the left and then equate the expression to zero.

$$\sum_{j,k} \frac{(-xg)^j x^k (2k+j-2)!(2k+j-3)!}{j!k!(k-2)!(k-1)!(k+j-1)!} = 0. \tag{6.3}$$

This equation was used to compute the first few terms of g as follows

$$\begin{aligned} g = {} & 1 + x + 3x^2 + 13x^3 + 68x^4 + 399x^5 + 2530x^6 \\ & + 16{,}965x^7 + 118{,}668x^8 + 857{,}956x^9 + 6{,}369{,}838x^{10} + \dots . \end{aligned} \tag{6.4}$$

(This agrees with (5.11).) The inverse function λ of xg was then computed and its coefficients were observed to have only small prime factors. A study of the factorizations led quickly to the recognition of λ as

$$\frac{1}{32}(1 + 20x - 8x^2 - (1-8x)^{3/2}).$$

The relation between g and x was then put in the parametric form of (4.8) and (4.9). Next the coefficients of $y, y^2, \dots, y^8$ in ψ were calculated in terms of θ by equating coefficients of y^m in (3.8). Again the numerical coefficients were found to have simple factors, and the general form (5.8) was recognized.

At this stage the solution of (3.8) was known, but a proof was required. The author is indebted to G. F. D. Duff for showing him how (5.8) could be put in a form recognizable as a Lagrangian expansion and so making possible the construction of the theory of §§ 4 and 5.

7. Simple triangulations. We now discuss the determination of the function h from the identity (2.6). Writing $x = z^2$ and $\xi = zg(z^2)$ we have

$$\xi h(\xi^2) = 2\xi - z. \tag{7.1}$$

Hence if $H(x)$ denotes the inverse function of $xg(x^2)$ we have

$$h(\xi^2) = 2 - \xi^{-1}H(\xi), \tag{7.2}$$

and our problem is reduced to the determination of H. By computing the first few terms we find, writing x for ξ^2,

$$h(x) = 1 + x + x^3 + 3x^4 + 12x^5 + 52x^6 + 241x^7 + 1173x^8 + \dots. \tag{7.3}$$

To obtain a general formula we proceed as follows. We define θ as in § 4, with $x = z^2$. We then have

$$\xi H(\xi) = xg(x) = \theta(1 - 2\theta), \tag{7.4}$$

$$\xi^2 = \frac{(xg(x))^2}{x} = \frac{\theta(1-2\theta)^2}{(1-\theta)^3}, \tag{7.5}$$

$$\theta = \frac{\xi^2(1-\theta)^3}{(1-2\theta)^2}. \tag{7.6}$$

Applying Lagrange's Theorem we find

$$\xi H(\xi) = \sum_{n=0}^{\infty} \frac{\xi^{2n+2}}{(n+1)!}\left[\frac{d^n}{d\theta^n}\left\{\frac{(1-\theta)^{3n+3}}{(1-2\theta)^{2n+2}}(1-4\theta)\right\}\right]_{\theta=0},$$

$$\xi^{-1}H(\xi) = \sum_{n=0}^{\infty} \frac{\xi^{2n}}{(n+1)!}\left[\frac{d^n}{d\theta^n}\left\{\frac{2^{-3n-3}(1+1-2\theta)^{3n+3}(2(1-2\theta)-1)}{(1-2\theta)^{2n+2}}\right\}\right]_{\theta=0}$$

$$= \sum_{n=0}^{\infty} \frac{\xi^{2n}}{2^{3n+3}(n+1)!}\left[\frac{d^n}{d\theta^n}\left\{\sum_{j=0}^{3n+3}\left\{\binom{3n+3}{j}(1-2\theta)^j\right\}\right.\right.$$
$$\left.\left.\times\left\{\frac{2}{(1-2\theta)^{2n+1}} - \frac{1}{(1-2\theta)^{2n+2}}\right\}\right\}\right]_{\theta=0}$$

$$= \sum_{n=0}^{\infty} \frac{\xi^{2n}}{2^{3n+3}(n+1)!}\left[\frac{d^n}{d\theta^n}\left\{2\sum_{j=0}^{3n+3}\binom{3n+3}{j}(1-2\theta)^{j-2n-1}\right.\right.$$
$$\left.\left. - \sum_{j=0}^{3n+3}\binom{3n+3}{j}(1-2\theta)^{j-2n-2}\right\}\right]_{\theta=0}$$

$$= \sum_{n=0}^{\infty} \frac{\xi^{2n}}{2^{3n+3}(n+1)!}\left[\frac{d^n}{d\theta^n}\left\{2(1-2\theta)^{n+2} - (1-2\theta)^{-2n-2}\right.\right.$$
$$\left.\left. + \sum_{j=1}^{3n+3}\left\{2\binom{3n+3}{j-1} - \binom{3n+3}{j}\right\}(1-2\theta)^{j-2n-2}\right\}\right]_{\theta=0}.$$

But

$$2\binom{3n+3}{j-1} - \binom{3n+3}{j} = \frac{(3n+3)!}{j!(3n-j+4)!}\{2j - (3n-j+4)\}$$
$$= \frac{(3n+3)!(3j-3n-4)}{j!(3n-j+4)!}.$$

Hence

$$\xi^{-1}H(\xi) = \sum_{n=0}^{\infty} \frac{\xi^{2n}}{2^{3n+3}(n+1)!}\left\{(-2)^n(n+2)! - 2^n\frac{(3n+1)!}{(2n+1)!}\right.$$
$$\left. + (-2)^n(3n+3)! \sum_{j=1}^{3n+3} \frac{(3j-3n-4)(j-2n-2)(j-2n-3)\dots(j-3n-1)}{j!(3n-j+4)!}\right\}$$
$$= \sum_{n=0}^{\infty} \frac{\xi^{2n}}{2^{2n+3}(n+1)!}\left\{(-1)^n(n+2)!\right.$$
$$+ (3n+3)! \sum_{j=0}^{2n+1} \frac{(3n+1-j)!(3j-3n-4)}{j!(3n-j+4)!(2n+1-j)!}$$
$$\left. + (-1)^n(6n+5)(n+1)! + \tfrac{1}{2}(-1)^n(3n+3)(6n+2)n!\right\}$$

Hence for $n > 0$, we have, by (7.2),

$$(7.7) \qquad \phi_{n,0} = \frac{1}{2^{2n+3}}\left\{- (-1)^n(16n+10) + \frac{(3n+3)!}{(n+1)!} \sum_{q=0}^{n} \frac{M(n,q)}{(n-q)!(n+q+1)!}\right\}$$

where $j = n - q$, $n + q + 1)$ and

$$(7.8 \quad M(n,q) = \frac{3q+4}{(2n+q+4)(2n+q+3)(2n+q+2)} - \frac{3q-1}{(2n-q+3)(2n-q+2)(2n-q+1)}.$$

We can transform (7.8) into

$$(7.9) \qquad M(n,q) = D^{-1}(40n^3 - 12n^2(6q^2+6q-11) - 10n(15q^2+15q-14) - 6(q^4+2q^3+13q^2+12q-8))$$

where D is the product of the denominators in (7.8).

8. Asymptotic formulae. We investigate the behaviour of $\psi_{n,0}$ as $n \to \infty$ by applying Stirling's Theorem to (5.11). We find

$$\psi_{n,0} \sim \frac{2(4n+1)^{4n+1}e^{-4n-1}\sqrt{2\pi(4n+1)}}{(3n+2)^{3n+2}e^{-3n-2}\sqrt{2\pi(3n+2)}(n+1)^{n+1}e^{-n-1}\sqrt{2\pi(n+1)}}$$

$$= \frac{2e^2(4n)^{4n+1}\left(1+\frac{1}{4n}\right)^{4n+1}}{(3n)^{3n+2}\left(1+\frac{2}{3n}\right)^{3n+2} n^{n+1}\left(1+\frac{1}{n}\right)^{n+1}} \sqrt{\frac{4n+1}{2\pi(n+1)(3n+2)}}$$

$$\sim \frac{2}{n^2}\frac{256^n \,.\, 4}{27^n \,.\, 9}\sqrt{\frac{4}{6\pi n}},$$

$$\psi_{n,0} \sim \frac{1}{16}\sqrt{\frac{3}{2\pi}}\, n^{-5/2}\left(\frac{256}{27}\right)^{n+1}. \tag{8.1}$$

The behaviour of $\psi_{n,m}$ as $n \to \infty$ for other fixed values of m can be obtained similarly from (5.10) or the special equations (5.12)-(5.14). Perhaps the behaviour of $\phi_{n,0}$ is more interesting, and we can determine this as follows. We note that

$$\sum_{q=0}^{n} \binom{2n+1}{n-q}(x^{n-q} + x^{n+q+1}) = (1+x)^{2n+1},$$

$$\sum_{q=0}^{n} \binom{2n+1}{n-q}\{(n-q)(n-q-1)x^{n-q-2}$$

$$+ (n+q+1)(n+q)x^{n+q-1}\} = (2n+1)(2n)(1+x)^{2n-1}.$$

Putting $x = 1$ in these identities we obtain

$$\sum_{q=0}^{n} \binom{2n+1}{n-q} = 2^{2n}, \tag{8.2}$$

$$\sum_{q=0}^{n} \binom{2n+1}{n-q}(q^2+q) = \tfrac{1}{2}n.\, 2^{2n}. \tag{8.3}$$

We have also

$$\binom{2n+1}{n-q} = \binom{2n+1}{n}\prod_{s=0}^{q-1}\frac{(n-s)}{(n+2+s)}. \tag{8.4}$$

Write $J = [\frac{1}{2}n^{2/3}]$. Then by (8.2) and (8.4)

$$\binom{2n+1}{n-2J} < 2^{2n}\left\{\prod_{s=J+1}^{2J+1}\frac{(n-s)}{(n+2+s)}\right\}\frac{(n+2J+2)(n+2J+3)}{(n-2J)(n-2J-1)} \tag{8.5}$$

$$< 2^{2n}\left\{1 - \frac{(J+1)}{n}\right\}^{J+1} \times 2$$

$$< 2\,.\,2^{2n}\,.\,\exp\left\{\frac{-(J+1)^2}{n}\right\} < 2\,.\,2^{2n}\,.\,\exp\left\{-\frac{1}{4}n^{1/3}\right\},$$

for sufficiently large n, for if $|x| > 0$ we have $1 - x < e^{-x}$.

The function D of (7.9) satisfies $D^{-1} < n^{-6}$ since $0 \leqslant q \leqslant n$. The coefficients in the polynomial in n and q which is multiplied into D^{-1} have absolute values whose sum is less than 1000. Hence

(8.6) $$|M(n, q)| < 1000n^{-2}.$$

Now

$$\binom{2n+1}{n-q}$$

is a decreasing function of q for $0 \leqslant q \leqslant n$ (by 8.4)). So by (8.5) and (8.6) we have

(8.7) $$\sum_{q=2J+1}^{n} \binom{2n+1}{n-q} < 2n \,.\, 2^{2n} \,.\, \exp\left\{-\frac{1}{4} n^{1/3}\right\},$$

(8.8) $$\sum_{q=2J+1}^{n} \binom{2n+1}{n-q} (q^2 + q) < 2n(n^2+n) \,.\, 2^{2n} \,.\, \exp\left\{-\frac{1}{4} n^{1/3}\right\},$$

(8.9) $$\left|\sum_{q=2J+1}^{n} \binom{2n+1}{n-q} M(n, q)\right| < 2000n^{-1} \,.\, 2^{2n} \,.\, \exp\left\{-\frac{1}{4} n^{1/3}\right\}.$$

Let us now restrict ourselves to values of $q \leqslant 2J$. For these we have

(8.10) $$(2n + 2J + 4)^{-6} < D^{-1} < (2n - 2J + 1)^{-6},$$

(8.11) $$M(n, q) = D^{-1}(40n^3 - 72n^2(q^2 + q)) + \delta,$$

where

(8.12) $$|\delta| < n^{-6} \cdot 1000n^{8/3} = 1000n^{-10/3}.$$

Let ϵ denote an arbitrary positive real number. Write

$$X = \sum_{q=0}^{2J} \binom{2n+1}{n-q} M(n, q).$$

Then, by (8.10), (8.11), and (8.12),

$$\begin{aligned} X &< \{40n^3(2n - 2J + 1)^{-6} + 1000n^{-10/3}\} \sum_{q=0}^{2J} \binom{2n+1}{n-q} \\ &\qquad - \{72n^2(2n + 2J + 4)^{-6}\} \sum_{q=0}^{2J} \binom{2n+1}{n-q} (q^2 + q) \\ &< \{40n^3(2n - 2J + 1)^{-6} + 1000n^{-10/3}\}\{2^{2n}\} \\ &\qquad - \{72n^2(2n + 2J + 4)^{-6}\}\left\{\tfrac{1}{2}n \,.\, 2^{2n} - 2n(n^2 + n) \,.\, 2^{2n} \,.\, \exp\left\{-\frac{1}{4} n^{1/3}\right\}\right\} \end{aligned}$$

by (8.2, (8.3), and (8.8),

$$< n^3 \,.\, 2^{2n}\{40(2n)^{-6}(1 - \epsilon)^{-6} + \epsilon n^{-6} - 36(2n)^{-6}(1 + \epsilon)^{-6}(1 - \epsilon)\}$$

for sufficiently large n. Similarly

$$\begin{aligned} X &> \{40n^3(2n + 2J + 4)^{-6} - 1000n^{-10/3}\}\left\{2^{2n} - 2n \,.\, 2^{2n} \,.\, \exp\left\{-\frac{1}{4} n^{1/3}\right\}\right\} \\ &\qquad\qquad - \{72n^2(2n - 2J + 1)^{-6}\}\{\tfrac{1}{2}n \,.\, 2^{2n}\} \\ &> n^3 \,.\, 2^{2n}\{(40(2n)^{-6}(1 + \epsilon)^{-6} - \epsilon n^{-6})(1 - \epsilon) - 36(2n)^{-6}(1 - \epsilon)^{-6}\} \end{aligned}$$

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for sufficiently large n. Since ϵ may be arbitrarily small we deduce that

$$X \sim \frac{1}{16} n^{-3} 2^{2n}.$$

Hence, by (7.7) and (8.9) we have

$$\phi_{n,0} \sim \frac{1}{2^{2n+3}} \left\{ -(-1)^n(16n+10) + \frac{(3n+3)!}{(n+1)!(2n+1)!} \cdot \frac{2^{2n}}{16n^3} \right\},$$

$$\sim \frac{1}{2^{2n+3}} \left\{ -(-1)^n(16n+10) + \frac{2^{2n}}{16} \sqrt{\frac{3}{\pi}}\, n^{-5/2} \left(\frac{27}{4}\right)^{n+1} \right\},$$

by Stirling's Theorem. It follows that

$$\phi_{n,0} \sim \frac{1}{128} \sqrt{\frac{3}{\pi}}\, n^{-5/2} \left(\frac{27}{4}\right)^{n+1}. \tag{8.13}$$

9. Simplicial dissections of the 2-sphere. In this final section we discuss the behaviour as $n \to \infty$ of the function $Q(n)$ defined as the number of combinatorially distinct simplicial dissections of the 2-sphere with $2n$ triangles.

Isomorphism for such dissections can be defined as in § 1 for triangulations. But as there is now no boundary only conditions (ii) and (iii) are imposed. When we say two dissections are combinatorially distinct we mean they are not isomorphic.

An *automorphism* of a dissection S is an isomorphic mapping of S onto itself. Every dissection S has a "trivial" or "identical" automorphism which maps each vertex onto itself. We call S *symmetrical* if it has another automorphism and *unsymmetrical* otherwise.

We note that $n > 1$ since no two triangles of a simplicial dissection have more than one edge in common. We may write α_0, α_1, and α_2 for the numbers of vertices, edges, and triangles respectively. Then, using the Euler polyhedron formula we have

$$\alpha_0 = n + 2, \quad \alpha_1 = 3n, \quad \alpha_2 = 2n. \tag{9.1}$$

Let t be a triangle of S. Stereographic projection from a point X in the interior of t onto a plane yields a triangulation T in the plane with just three external edges. It is defined by the images of the triangles of S other than t and its external edges are the images of the edges of t. If S is unsymmetrical the six rotations and reflections of the pattern of T define six different triangulations. For if two of them are equivalent we can evidently construct a non-trivial automorphism of S. Similarly, two different triangles of S never give rise to two equivalent triangulations. But if S is symmetrical some of its triangles may yield fewer than six different triangulations, and not all of its triangles will yield different sets of triangulations.

Let $Q_0(n)$ be the number of combinatorially distinct unsymmetrical dissections, and $Q_1(n)$ the number of combinatorially distinct symmetrical ones. Then by the above considerations we have

$$12nQ_0(n) \leqslant \psi_{n-1,0} \leqslant 12n(Q_0(n) + Q_1(n)). \tag{9.2}$$

I presume, though I recall no proof, that $Q_1(n)/Q_0(n) \to 0$ as $n \to \infty$. If this is true it follows from (9.2) and (8.1) that

$$Q(n) \sim \frac{1}{64\sqrt{6\pi}} n^{-7/2} \left(\frac{256}{27}\right)^n. \tag{9.3}$$

A similar argument applies to the number $R(n)$ of "simple" simplicial dissections of the sphere in which no three edges form a simple closed curve unless they bound a triangle of the dissection. (8.13) is used instead of (8.1). The argument indicates that

$$R(n) \sim \frac{1}{512\sqrt{3\pi}} n^{-7/2} \left(\frac{27}{4}\right)^n. \tag{9.4}$$

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