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PERIODIC SOLUTIONS OF NON-LINEAR EVOLUTION EQUATIONS IN BANACH SPACES

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In this paper the theory of Browder [2] and of Lions [3] on periodic solutions of non-linear evolution equations in Banach spaces is put in a more general framework so as to include the Navier-Stokes equations and their variants.

An abstract existence theorem is proved in § 1. Applications are given in § 2. The existence of periodic solutions of the Navier-Stokes equations without any restriction on the dimension of the space domain is established. Application of the abstract theorem to the following problem is given:

$$\begin{cases} D_t u_\epsilon - \Delta u_\epsilon + \sum_{j=1}^{n} u_{j\epsilon} D_j u_\epsilon + \frac{1}{2}(\operatorname{div} u_\epsilon)u_\epsilon + \operatorname{grad} p_\epsilon = f \quad \text{on } G \times [0, T]; \\ \operatorname{div}(u_\epsilon) = -p_\epsilon \cdot \epsilon; \qquad u_\epsilon(x, t) = 0 \quad \text{on } \partial G \times [0, T]; \\ u_\epsilon(x, 0) = u_\epsilon(x, T) \quad \text{on } G. \end{cases}$$

1. Let H be a Hilbert space and $(.\,,.)_H$ the inner product in H. Let V and W be two reflexive separable Banach spaces with $W \subset V \subset H$. W is dense in V and V is dense in H. The natural injection mappings of W into V and of V into H are compact.

Let V^* be the dual of V and $\{.\,,.\}$ the pairing between V and V^*. The pairing between W and its dual W^* is denoted by $(.\,,.)$.

Consider the Banach space $F = L^p(0, T; V)$ of equivalence classes of functions $u(t)$ from $[0, T]$ to V with the norm:

$$||u||_F = \left\{ \int_0^T ||u(t)||_V^p \, dt \right\}^{1/p}, \qquad 2 \leqq p < \infty.$$

$\langle .\,,. \rangle$ is the pairing between F and its dual F^*. Let $Y = L^r(0, T; W)$ with $2 \leqq p < r < \infty$ and let $((.\,,.))$ be the pairing between Y and its dual Y^*.

Thus $\langle u, v \rangle = \int_0^T (u, v)_H \, dt$ if $u \in L^p(0, T; H)$ and v is in F. Similarly for $((.\,,.))$.

Set $X = F \cap L^\infty(0, T; H)$. We shall say that $u_n \to u$ weakly in X if $u_n \to u$ weakly in F and $u_n \to u$ in the weak-star topology of $L^\infty(0, T; H)$.

In this paper we consider non-linear operators A mapping X and Y into Y^* and satisfying the following assumption.

Assumption (I). (i) *A is continuous from line segments in X to the weak* topology of Y^*.*

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(ii) *Let* $u_n \to u$ *weakly in* X, u_n *in* Y, $u_n(0) = u_n(T)$; $u_n' \to u'$ *weakly in* Y^*, $Au_n \to g$ *weakly in* Y^* *with* $g + u'$ *in* F^* *and*

$$\limsup \operatorname{Re} ((u_n' + Au_n, u_n)) \leqq \operatorname{Re} \langle g + u', u\rangle.$$

Then $Au = g$.

(iii) *If* $u_n \to u$ *weakly in* Y *and* $u_n' \to u'$ *weakly in* Y^*, *then* $Au_n \to Au$ *weakly in* Y^* *and* $((Au_n, u_n)) \to ((Au, u))$.

We have shown in [**7**] that all the semi-monotone operators considered by Browder [**2**] and by Lions [**3**] as well as all the weakly continuous operators from F into F^* satisfy Assumption (I).

The main result of this paper is the following theorem.

THEOREM 1. *Let* A *be a non-linear operator mapping* X *and* Y *into* Y^* *and satisfying Assumption* (I). *Suppose further that:*

(i) A *maps bounded sets of* X *and of* Y *into bounded sets of* Y^*;

(ii) $\operatorname{Re}((Au, u)) \geqq c(||u||_F)||u||_F$ *for all* u *in* Y, $c(r)$ *is a positive continuous function with* $c(r) \to +\infty$ *as* $r \to +\infty$;

(iii) $\operatorname{Re}(Au, u) \geqq 0$ *for all* u *in* Y *and for almost all* t *in* $[0, T]$;

(iv) *There exists a positive continuous function* $\varphi(r)$ *such that:*

$$\operatorname{Re}((Au, u)) \leqq \varphi(||u||_F) \quad \text{for all } u \text{ in } Y.$$

Then for each f *in* F^*, *there exists* u *in* X *with* u' *in* Y^* *such that:*

$$u' + Au = f, \qquad u(0) = u(T).$$

Theorem 1 will be derived from the following result.

THEOREM 2. *Let* J *be the duality mapping from* Y *into* Y^* *associated with the gauge function* $\psi(s) = s^{r-1}$. *Suppose that all the hypotheses of Theorem* 1 *are satisfied. Then for each* ϵ, $0 < \epsilon < 1$, *and for each* f *in* F^*, *there exists* u_ϵ *in* Y *with* u_ϵ' *in* Y^* *such that:*

$$u_\epsilon' + \epsilon J u_\epsilon + Au_\epsilon = f, \qquad u_\epsilon(0) = u_\epsilon(T).$$

Moreover, $||u_\epsilon||_F + \epsilon||u_\epsilon||_Y^r + ||u_\epsilon'||_{Y^*} + ||u_\epsilon||_{L^\infty(0,T;H)} \leqq M$. M *is a constant independent of* ϵ.

Proof of Theorem 1 *using Theorem* 2. Since Y is a reflexive Banach space, by taking an equivalent norm if necessary, we may assume that Y^* is strictly convex. It is well known that the duality mapping J from Y into Y^* associated with the gauge function $\psi(s)$ exists. Since Y^* is strictly convex, J is uniquely defined.

From the weak compactness of the unit ball in a reflexive Banach space, we have by taking a subsequence if necessary:

$u_\epsilon \to u$ weakly in F, $u_\epsilon \to u$ in the weak* topology of $L^\infty(0, T; H)$,

$u_\epsilon' \to u'$ weakly in Y^* and $\epsilon^{1/r}u_\epsilon \to 0$ weakly in Y as $\epsilon \to 0$.

Since the natural injection mapping of V into W^* is compact, it follows then that $u_\epsilon(t) - u(t) \to 0$ in W^* *for all* t in $[0, T]$. But $u_\epsilon(0) = u_\epsilon(T)$, thus $u(0) = u(T)$.

By hypothesis, A maps bounded sets of X into bounded sets of Y^*, hence $Au_\epsilon \to g$ weakly in Y^* as $\epsilon \to 0$.

On the other hand, $\epsilon||Ju_\epsilon||_{Y^*} = \epsilon||u_\epsilon||_Y^{r-1} \to 0$ as $\epsilon \to 0$. Therefore $\limsup \operatorname{Re}((u_\epsilon' + Au_\epsilon, u_\epsilon)) \leqq \operatorname{Re}\langle f, u\rangle = \operatorname{Re}\langle g + u', u\rangle$. It follows from Assumption (I) that $Au = g$.

Proof of Theorem 2. It is well known that the duality mapping J is monotone from Y into Y^* and is continuous from the strong topology of Y to the weak topology of Y^*.

For u, v in Y, let $L(u, v) = Au + \epsilon Jv$. Then L maps bounded sets of $Y \times Y$ into bounded sets of Y^*. Moreover, $L(u, \cdot)$ is monotone and is continuous from line segments in Y to the weak topology of Y^*. It is also clear that $L(u, u)$ is coercive.

If $u_n \to u$ weakly in Y and $u_n' \to u'$ weakly in Y^*, then it follows from Assumption (I) that $Au_n \to Au$ weakly in Y^* and $((Au_n, u_n)) \to ((Au, u))$. Therefore $((L(u_n, \varphi), u_n)) \to ((L(u, \varphi), u))$ and $((L(u_n, \varphi), v)) \to ((L(u, \varphi), v))$ for any φ, v in Y.

It follows from [**3**] that there exist u_ϵ in Y, u_ϵ' in Y^* such that

$$u_\epsilon' + \epsilon Ju_\epsilon + Au_\epsilon = f, \qquad u_\epsilon(0) = u_\epsilon(T).$$

We easily obtain $||u_\epsilon||_F + \epsilon||u_\epsilon||_Y^r \leqq M$. M is independent of ϵ.

It remains to show that u_ϵ is uniformly bounded in $L^\infty(0, T; H)$. It is the crucial part of the theorem and indeed of the paper.

First, we show that $||u_\epsilon(0)||_H$ is uniformly bounded. Let $\theta \in C^1(0, T)$ and $\theta(T) = 0$, $\theta(0) = 1$. Set $v_\epsilon = \theta u_\epsilon$. Then $v_\epsilon' + \epsilon\theta Ju_\epsilon + \theta Au_\epsilon = \theta f + \theta' u_\epsilon$. Hence:

$$\tfrac{1}{2}||v_\epsilon(0)||_H^2 = \tfrac{1}{2}||u_\epsilon(0)||_H^2$$
$$\leqq \operatorname{Re}\int_0^T \{-\theta\{f, u_\epsilon\} - \theta'\theta||u_\epsilon||_H^2 + \epsilon\theta^2(Ju_\epsilon, u_\epsilon) + \theta^2(Au_\epsilon, u_\epsilon)\}\, dt.$$

Since by hypothesis $\operatorname{Re}((Au_\epsilon, u_\epsilon)) \leqq \varphi(||u_\epsilon||_F)$ for u_ϵ in Y, we obtain $\operatorname{Re}\int_0^T \theta^2(Au_\epsilon, u_\epsilon)\, dt \leqq K\, \varphi(||u_\epsilon||_F) \leqq C$.

Thus $\tfrac{1}{2}||u_\epsilon(0)||_H^2 \leqq C$. The different constants are all independent of ϵ.

Using a remark as in [**7**], we show that

$$||u_\epsilon(t)||_H^2 \leqq C(||u_\epsilon(0)||_H^2 + 1) \quad \text{for } t \text{ in } [0, T].$$

Indeed,

$$||u_\epsilon(t)||_H^2 = ||u_\epsilon(0)||_H^2 + 2\operatorname{Re}\int_0^t (u_\epsilon', u_\epsilon)\, dt.$$

Thus

$$\|u_\epsilon(t)\|_H^2 = \|u_\epsilon(0)\|_H^2 + 2\,\mathrm{Re}\int_0^t (f - Au_\epsilon - \epsilon Ju_\epsilon, u_\epsilon)\,dt$$

$$\leqq \|u_\epsilon(0)\|_H^2 + 2\,\|f\|_{F^*}\|u_\epsilon\|_F - \mathrm{Re}\int_0^t \epsilon(Ju_\epsilon, u_\epsilon)\,dt,$$

since by hypothesis $\mathrm{Re}(Au_\epsilon, u_\epsilon) \geqq 0$ for almost all t in $[0, T]$. Thus

$$\|u_\epsilon(t)\|_H^2 \leqq \|u_\epsilon(0)\|_H^2 + 2\|f\|_{F^*}\|u_\epsilon\|_F + 2\epsilon\|Ju_\epsilon\|_{Y^*}\|u_\epsilon\|_Y$$

$$\leqq C(\|u_\epsilon(0)\|_H^2 + \|u_\epsilon\|_F + \epsilon\|u_\epsilon\|_{Y}{}^{r}) \leqq M.$$

M is independent of t and of ϵ.

The theorem is proved.

2. We now give some applications of Theorem 1 to the study of periodic solutions of strongly non-linear parabolic equations.

Let G be a bounded open subset of R^n with a smooth boundary ∂G. The points of G will be denoted by $x = (x_1, \ldots, x_n)$. Set $D_j = i^{-1}\,\partial/\partial x_j$, $j = 1, \ldots, n$. For each n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers, we write:

$$D^\alpha = \prod_{j=1}^n D_j^{\alpha_j} \quad \text{with } |\alpha| = \sum_{j=1}^n \alpha_j.$$

The points of E^1 will be denoted by t and differentiation in t by D_t. Let k be a positive integer. By functions we mean k-vector-valued functions $u = (u_1, \ldots, u_k)$ where each u_j is a real-valued function on G or on $G \times [0, T]$.

$W^{k,p}(G)$ is the Banach space

$$W^{k,p}(G) = \{u\colon u \text{ in } L^p(G),\ D^\alpha u \text{ in } L^p(G),\ |\alpha| \leqq k\}$$

with the norm

$$\|u\|_{k,p} = \left\{\sum_{|\alpha|\leqq k} \|D^\alpha u\|_{L^p(G)}^p\right\}^{1/p}, \qquad 1 < p < \infty.$$

(I) *Periodic solutions of strongly non-linear parabolic equations.* The existence of periodic solutions of the strongly non-linear parabolic equations considered by Lions [**3**] may be established by applying Theorem 1 and the remark following Assumption (I).

(II) *Periodic solutions of the Navier-Stokes equations.* Let

$$S = \{\varphi\colon \varphi \text{ in } C_c^\infty(G);\ \mathrm{div}\,\varphi = 0\}.$$

H, V, W are the completion of S in the $L^2(G)$-norm, the $(\|\cdot\|_{1,2})$-norm, and the $(\|\cdot\|_{m,4})$-norm, respectively, where $m = 1 + [n/4]$.

Then $W \subset V \subset H$. W is dense in V and V is dense in H. The natural injection mappings of W into V and of V into H are compact since G is bounded.

Take Y to be the Banach space $Y = L^4(0, T; W)$ and $F = L^2(0, T; V)$. Consider the problem:

$$\begin{cases} D_t u - \Delta u + \sum_{j=1}^{n} u_j D_j u + \operatorname{grad} p = f \quad \text{on } G \times [0, T]; \\ \operatorname{div} u = 0 \text{ on } G \times [0, T] \qquad u(x, t) = 0 \text{ on } \partial G \times [0, T]; \\ \qquad u(x, 0) = u(x, T) \quad \text{on } G. \end{cases}$$

THEOREM 3. *For each f in $L^2(0, T; V^*)$, there exists u in $L^2(0, T; V)$ and in $L^\infty(0, T; H)$ with u' in $L^2(0, T; W^*)$ such that*

$$-((u, \varphi')) + \sum_{j=1}^{n} \int_0^T (u_j D_j u, \varphi)_H \, dt + \sum_{j=1}^{n} \int_0^T (D_j u, D_j \varphi)_H \, dt = \langle f, \varphi \rangle$$

for all φ in Y with φ' in Y^* and $\varphi(0) = \varphi(T)$.

Proof. From the Sobolev embedding theorem we have:

$$W \subset C(\operatorname{cl} G).$$

The natural injection mapping of W into $C(\operatorname{cl} G)$ is continuous.

Let

$$a(u, v) = \sum_{j,k=1}^{n} \int_0^T \{(D_j u_k, D_j v_k)_H + (u_j D_j u_k, v_k)_H\} \, dt,$$

where u is in $X = L^2(0, T; V) \cap L^\infty(0, T; H)$ and v in Y. $a(u, v)$ is well-defined and, moreover, continuous, linear in v on Y. Hence $a(u, v) = ((Au, v))$.

To prove the theorem, we shall apply Theorem 1.

We check that A satisfies all the hypotheses of Assumption (I). Suppose that $u_n \to u$ weakly in X and $u_n' \to u'$ weakly in Y^*. Since the natural injection mapping of V into H is compact, it follows from a result of Aubin [**1**] that $u_n \to u$ in $L^2(0, T; H)$.

An easy argument, using the Lebesgue convergence theorem yields: $Au_n \to Au$ weakly in Y^*.

It remains to verify part (iii) of Assumption (I). Suppose that $u_n \to u$ weakly in Y and $u_n' \to u'$ weakly in Y^*. Since the natural injection mapping of W into V is compact, it follows from [**1**] again that $u_n \to u$ in $L^4(0, T; V)$. Hence

$$\begin{aligned} &||u_n D u_n - u D u||_{L^2(0,T;H)} \\ &\qquad \leqq C||u||_Y \{||u_n - u||_{L^4(0,T;H)} + ||Du_n - Du||_{L^4(0,T;H)}\} \to 0 \quad \text{as } n \to +\infty. \end{aligned}$$

It follows that $Au_n \to Au$ in Y^*.

To apply Theorem 1, it suffices to check part (iv) of the hypotheses of Theorem 1.

For u in Y,

$$a(u, u) = ((Au, u)) = \sum_{j,k=1}^{n} \int_0^T (D_j u_k, D_j u_k)_H \, dt \leqq C||u||_F^2.$$

Applying Theorem 1, we obtain u in X with u' in Y^* such that

$$u' + Au = f, \qquad u(0) = u(T).$$

Since $||Au||_{L^2(0,T;W^*)} \leqq M\{||u||_F + ||u||_{L^\infty(0,T;H)}||u||_F\}$, we have $u' = f - Au$ in $L^2(0, T; W^*)$.

The theorem is proved.

The existence of periodic solutions of the Navier-Stokes equations without any restriction on the dimension of the space domain was shown by Prouse [**5**] when *f is periodic in time* and by Lions [**4**] for f in $L^2(0, T; V^*)$. Lions solved the initial-value problem in a finite-dimensional space, then used a fixed-point theorem (in order to obtain a uniform estimate in $L^\infty(0, T; H)$ of the approximate solutions) to show the existence of periodic solutions in finite-dimensional subspaces. Finally by going to the limit, the approximate solutions are shown to converge weakly in $L^2(0, T; V)$ to a periodic solution of the Navier-Stokes equations.

(III) *Periodic solutions of an equation considered by Temam* [**6**]. Consider the problem:

$$\begin{cases} D_t u_\epsilon - \Delta u_\epsilon + \sum_{j=1}^n u_{j\epsilon} D_j u_\epsilon + \frac{1}{2}(\operatorname{div} u_\epsilon)u_\epsilon + \operatorname{grad} p_\epsilon = f \quad \text{on } G \times [0, T]; \\ \operatorname{div}(u_\epsilon) = -p_\epsilon \cdot \epsilon, \qquad u_\epsilon(x, t) = 0 \quad \text{on } \partial G \times [0, T]; \\ u_\epsilon(x, 0) = u_\epsilon(x, T) \quad \text{on } G. \end{cases}$$

The initial-value problem for the above equation was studied by Temam in [**6**] when $n = 2, 3$.

Let H be the Hilbert space $L^2(G)$ and V, W the completion of $C_c^\infty(G)$ with respect to the $(||\cdot||_{1,2})$-norm and the $(||\cdot||_{m,4})$-norm, respectively, with $m = 1 + [n/4]$.

Take $F = L^2(0, T; V)$ and $Y = L^4(0, T; W)$ with $X = F \cap L^\infty(0, T; H)$. Let

$$a_\epsilon(u, v; w) = \sum_{j=1}^n \int_0^T \int_G D_j u \cdot D_j w \, dxdt + \int_0^T \int_G \epsilon^{-1} \operatorname{div}(u) \operatorname{div}(w) \, dxdt$$
$$+ \frac{1}{2} \sum_{j,k=1}^n \int_0^T \int_G u_j (D_j v_k \cdot w_k - v_k \cdot D_j w_k) \, dxdt, \qquad u, v \text{ in } X \text{ and } w \text{ in } Y.$$

$a_\epsilon(u, v; w)$ is well-defined and $a_\epsilon(u, u; v) = ((A_\epsilon u, v))$ for u in X and v in Y.

THEOREM 4. *For each f in F^* and for ϵ, $0 < \epsilon < 1$, there exists u_ϵ in X with u_ϵ' in $L^2(0, T; W^*)$ such that*

$$u_\epsilon' + A_\epsilon u_\epsilon = f, \qquad u_\epsilon(0) = u_\epsilon(T).$$

Moreover $||u_\epsilon||_F + ||u_\epsilon||_{L^\infty(0,T;H)} + \epsilon^{-\frac{1}{2}}||\operatorname{div}(u_\epsilon)||_{L^2(G\times(0,T))} + ||u_\epsilon'||_{Y^*} \leqq M$. *$M$ is a constant independent of ϵ.*

Proof. An argument as in the proof of Theorem 3 shows that A_ϵ satisfies all the hypotheses of Theorem 1. It follows from Theorem 2 that there exists $u_{\epsilon\eta}$ in Y with $u_{\epsilon\eta}'$ in Y^* such that:

$$u_{\epsilon\eta}' + \eta J u_{\epsilon\eta} + A_\epsilon u_{\epsilon\eta} = f, \qquad u_{\epsilon\eta}(0) = u_{\epsilon\eta}(T), \qquad 0 < \eta < 1.$$

It is easy to see that

$$||u_{\epsilon\eta}||_F + \eta||u_{\epsilon\eta}||_Y{}^4 + \epsilon^{-\frac{1}{2}}||\operatorname{div}(u_{\epsilon\eta})||_{L^2(G\times(0,T))} \leqq M.$$

M is a constant independent of both ϵ and η.

An argument exactly as in the proof of Theorem 2 yields: $||u_{\epsilon\eta}(0)||_H \leqq M$ and $||u_{\epsilon\eta}(t)||_H \leqq C(1 + ||u_{\epsilon\eta}(0)||_H) \leqq M$. M and C are constants independent of ϵ, η, t.

Thus $||u_{\epsilon\eta}||_{L^\infty(0,T;H)} \leqq M$ and hence $||u_{\epsilon\eta}'||_{L^2(0,T;W^*)} \leqq M$.

Let $\eta \to 0$; then from the weak compactness of the unit ball in a reflexive Banach space, we obtain $u_{\epsilon\eta} \to u_\epsilon$ weakly in F. Theorem 1 shows that u_ϵ is a solution of $u_\epsilon' + A_\epsilon u_\epsilon = f$ with $u_\epsilon(0) = u_\epsilon(T)$. All the other assertions of the theorem are trivial to verify.

THEOREM 5. *Let u_ϵ be a solution of $u_\epsilon' + A_\epsilon u_\epsilon = f$, $u_\epsilon(0) = u_\epsilon(T)$ of Theorem 4. Then as $\epsilon \to 0$, $u_\epsilon \to u$ weakly in F and u is a solution of $u' + Au = f$, $u(0) = u(T)$ of Theorem 3.*

Proof. From the weak compactness of the unit ball in a reflexive Banach space, we obtain by taking a subsequence if necessary:

$u_\epsilon \to u$ weakly in F, $u_\epsilon \to u$ in the weak* topology of $L^\infty(0, T; H)$,
$u_\epsilon' \to u'$ weakly in $L^2(0, T; W^*)$ and $\operatorname{div}(u_\epsilon) \to 0$ in $L^2(G \times (0, T))$ as $\epsilon \to 0$.

Since the injection mapping of V into H is compact, it follows from [**1**] that $u_\epsilon \to u$ in $L^2(0, T; H)$ as $\epsilon \to 0$.

From above, we have $\operatorname{div}(u_\epsilon) \to \operatorname{div}(u)$ weakly in $L^2(G \times (0, T))$ and thus $\operatorname{div}(u) = 0$.

On the other hand, as in the proof on Theorem 1, we could show that $u(0) = u(T)$.

It remains to show that $A_\epsilon u_\epsilon \to Au$ weakly in Y^*. The proof is easy and is therefore omitted.

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