Canad. Math. Bull. Vol. **37** (1), 1993 pp. 42–45

THE σ-LINKEDNESS OF THE MEASURE ALGEBRA

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ABSTRACT. It is shown that the measure algebra on the space 2^{2^ω} is σ-n-linked for each $n \in \omega$.

1. **Introduction.** The Lebesgue measure, λ on 2^X can be defined by defining the measure of elementary open sets and then using these to approximate other sets. In particular, if $f: X \longrightarrow 2$ is a finite partial function then $[f]$ will denote the closed and open set in the product space 2^X consisting of all $\phi: X \longrightarrow 2$ such that $f \subseteq \phi$. If $|f| = n$ then $\lambda([f]) = 2^{-n}$. Lemma 1.1 is known as the *Lebesgue Density Lemma* and a proof can be found in [4].

LEMMA 1.1. *If $A \subseteq 2^X$ is a set of positive measure and $\rho < 1$ then there is a finite partial function $f: X \longrightarrow 2$ such that $\lambda(A \cap [f]) > \rho\lambda([f])$.*

A Boolean algebra $\mathbb{B}$ is called *σ-k-linked* if $\mathbb{B} = \bigcup_{n\in\omega} \mathbb{B}_n$ such that for each $n \in \omega$ and $\{b_1, b_2, \ldots, b_k\} \subseteq \mathbb{B}_n$ the meet $\wedge_{i=1}^{k} b_i$ is not empty. This paper is concerned with this property for algebras of measurable subsets of 2^κ modulo the ideal of sets X such that $\lambda(X) = 0$—such algebras will be denoted by $\mathbb{M}(\kappa)$. It is a direct consequence of Lemma 1.1 that $\mathbb{M}(\omega)$—or, equivalently, the measure algebra on $\mathbb{R}$—is σ-k-linked for each $k \in \omega$. To prove this, simply choose for each set X of positive measure a basic open set $\mathcal{U}$ such that $\lambda(\mathcal{U} \cap X) > \frac{k}{k+1}\lambda(\mathcal{U})$ and use the countable base to obtain a decomposition into σ-k-linked families. Observe however, that this proof does not generalise to $\mathbb{M}(\kappa)$ for $\kappa \geq \omega_1$ because the space 2^κ does not have a countable base in this case. The use of a countable base is crucial in this proof and the fact that 2^ω has a countable dense subset is not sufficient. Moreover, it is not possible to prove that $\mathbb{M}(\omega)$ is σ-centred—in other words, $\mathbb{M}(\omega)$ is not the union of countably many families which have the property that *all* finite subsets have non-empty intersection. It seems, therefore, that the classical result of Lebesgue is the best possible. It is the purpose of this paper to provide a proof of this result which does not rely on the fact that $2^{\aleph_0}$ has a countable base. Indeed, the argument shows that $\mathbb{M}(\kappa)$ is σ-n-linked for every n if $\kappa \leq 2^{\aleph_0}$. This answers a question of Fremlin who has, independently, proved the result for $\kappa = \omega_1$ [3].

The interest in this question stems from the fact that cellularity properties, such as σ-linkedness, are widely studied by set theorists and topologists. The measure algebra $\mathbb{M}(\kappa)$ was the last of the classical algebras for which it was not known exactly which

Both authors are partially supported by NSERC.
Received by the editors April 1, 1992.
AMS subject classification: 28A60, 04A20.
Key words and phrases: measure algebra, σ-linked.

https://doi.org/10.4153/CMB-1994-007-7 Published online by Cambridge University Press

cellularity properties it satisfied. The fact that $\mathbb{M}(\kappa)$ is σ-n-linked for every n if $\kappa \leq 2^{\aleph_0}$ settles all such questions.

2. **$\mathbb{M}(2^{\aleph_0})$ is σ-k-linked.** The following lemma follows directly from the definition of the product measure.

LEMMA 2.1. *If a_i and b_i are finite partial functions from X to 2 such that*

$$\mathrm{dom}(a_j) \cap \mathrm{dom}(b_i) = \emptyset$$

for $j \in k$ and $i \in m$ and $A = \bigcup_{i<k}[a_i]$ and $B = \bigcup_{i<m}[b_i]$ then $\lambda(A \cap B) = \lambda(A)\lambda(B)$.

DEFINITION 2.1. Given a partition $Z_0 \cup Z_1 \cup Z_2 = X$ of X and integers m and k define $L(Z_0, Z_1, Z_2, m, k)$ to be the set of all measurable subsets $A \subseteq 2^X$ such that there are finite partial functions $f\colon X \longrightarrow 2$ and $t_i\colon X \longrightarrow 2$, for $i \in M$, such that

(1) $\lambda([f]) = 2^{-m}$
(2) $\lambda(A \cap [f]) > (\frac{k}{k+1} + \frac{1}{k+2})2^{-m}$
(3) $t_i \supset f$ for each $i \in M$
(4) $f^{-1}\{i\} \subseteq Z_i$ for $i \in 2$
(5) $\mathrm{dom}(t_i \setminus f) \subseteq Z_2$ for $i \in M$
(6) $\lambda\big(\bigcup_{i\in M}[t_i]\Delta(A \cap [f])\big) < 2^{-km}(\frac{1}{2(k+2)})$

LEMMA 2.2. *If $\{A_i : i \in k\} \subseteq L(Z_0, Z_1, Z_2, m, k)$ then $\lambda(\bigcap_{i\in k} A_i) > 0$.*

PROOF. For each $j \in k$ let f_j and $t_i^j\colon X \longrightarrow 2$, for $i \in M_j$, witness that $A_j \in L(Z_0, Z_1, Z_2, m, k)$—in other words

- $\lambda([f_j]) = 2^{-m}$
- $\lambda(A_j \cap [f_j]) > (\frac{k}{k+1} + \frac{1}{k+2})2^{-m}$
- $\lambda\big((\bigcup_{i\in M_j}[t_i^j])\Delta(A_j \cap [f])\big) < 2^{-km}(\frac{1}{2(k+2)})$

Let $g\colon X \longrightarrow 2$ be any finite partial function such that

- $g^{-1}\{i\} \subseteq Z_i$ for $i \in 2$
- $\lambda([g]) = 2^{-km}$
- $g \supseteq \bigcup_{i\in k} f_i$

and observe that the last condition can be satisfied since Condition 4 of Definition 2.1 implies that $\bigcup_{i\in k} f_i$ is a function. It suffices to show that $\lambda(A_j \cap [g]) > \frac{k}{k+1}\lambda([g])$ for each $j \in k$.

To this end note that $\mathrm{dom}(t_i^j) \cap \mathrm{dom}(g \setminus f_j) = \emptyset$ for each $j \in k$ and $i \in M_j$. Hence it follows from Lemma 2.1 that

$$\begin{aligned}
&\lambda\Big([g \setminus f_j] \cap \Big(\bigcup_{i\in M_j}[t_i^j]\Big)\Big)\\
&\quad = \lambda([g \setminus f_j])\lambda\Big(\bigcup_{i\in M_j}[t_i^j]\Big) = 2^{-(k-1)m}\lambda\Big(\bigcup_{i\in M_j}[t_i^j]\Big)\\
&\quad \geq 2^{-(k-1)m}\lambda\Big(A_j \cap \Big(\bigcup_{i\in M_j}[t_i^j]\Big)\Big) \geq 2^{-(k-1)m}\Big(\lambda(A_j \cap [f_j]) - \lambda\Big((A_j \cap [f_j])\Delta \bigcup_{i\in M_j}[t_i^j]\Big)\Big)\\
&\quad \geq 2^{-(k-1)m}\Big(\frac{k}{k+1} + \frac{1}{k+2}\Big)2^{-m} - 2^{-km}\Big(\frac{1}{2(k+2)}\Big) \geq 2^{-km}\Big(\frac{k}{k+1} + \frac{1}{2(k+2)}\Big)
\end{aligned}$$

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Notice also that, because $t_i^j \supset f_j$ it follows that $[g \setminus f_j] \cap \bigcup_{i\in M_j}[t_i^j] = [g] \cap (\bigcup_{i\in M_j}[t_i^j])$. Hence

$$\lambda([g]\cap A_j) \geq \lambda\Big([g]\cap\Big(\bigcup_{j\in M_i}[t_i^j]\Big)\cap A_j\Big) \geq \lambda\Big([g]\cap\Big(\bigcup_{j\in M_i}[t_i^j]\Big)\Big) - \lambda\Big(\bigcup_{j\in M_i}[t_i^j]\Delta(A_j\cap[g])\Big)$$

and so, since $f_j \subseteq g$, it follows that

$$\lambda([g]\cap A_j) \geq \lambda\Big([g]\cap\Big(\bigcup_{j\in M_i}[t_i^j]\Big)\Big) - \lambda\Big(\Big(\bigcup_{j\in M_i}[t_i^j]\Big)\Delta(A_j\cap[f_j])\Big)$$
$$\geq 2^{-km}\Big(\frac{k}{k+1}+\frac{1}{2(k+2)}\Big) - \frac{2^{-km}}{2(k+2)} = \frac{k}{k+1}\lambda([g])$$

as was required. ∎

THEOREM 2.1. *The measure algebra on* $2^{\mathfrak{c}}$, $\mathbb{M}(2^{\mathfrak{c}})$, *is σ-k-linked for each* $k \in \omega$.

PROOF. Let $\{(Z_0^n, Z_1^n, Z_2^n) : n \in \omega\}$ be a family of partitions of $\mathfrak{c}$ such that for any three disjoint finite sets x_0, x_1 and x_2 there is some $n \in \omega$ such that $x_i \subseteq Z_i^n$ for each $i \in 3$—if $\mathfrak{c}$ is identified with the irrationals then $\{(Z_0^n, Z_1^n, Z_2^n) : n \in \omega\}$ can be thought of as the family of finite unions of intervals with rational end points. It suffices to show that if $A \subseteq 2^{\mathfrak{c}}$ is a set of positive measure then there is some n and m such that $A \in L(Z_0^n, Z_1^n, Z_2^n, m, k)$.

To see this use Lemma 1.1 to find a finite partial function $f : \mathfrak{c} \longrightarrow 2$ such that $|f| = m$ and $\lambda(A \cap [f]) > (\frac{k}{k+1} + \frac{1}{k+2})2^{-m}$. Then approximate $A \cap [f]$ by an open set $\mathcal{W}$ so that $A\cap[f] \subseteq \mathcal{W}$ and $\lambda\big(\mathcal{W}\setminus(A\cap[f])\big) < \frac{2^{-km}}{4(k+2)}$. Now choose finite partial functions $t_i : \mathfrak{c} \longrightarrow 2$ for $i \in M$ such that $[t_i] \subseteq \mathcal{W}$ for each $i \in M$ and such that $\lambda(\mathcal{W} \setminus \bigcup_{i\in M}[t_i]) < \frac{2^{-km}}{4(k+2)}$ it follows that $\lambda\big(A\Delta(\bigcup_{i\in M}[t_i])\big) < \frac{2^{-km}}{2(k+2)}$. It may, without loss of generality, be assumed that $f \subseteq t_i$ for each $i \in M$. Now choose $n \in \omega$ such that $\{\xi \in \mathfrak{c} : f(\xi) = i\} \subseteq Z_i^n$ and $\bigcup_{i\in M} \mathrm{dom}(t_i) \subseteq Z_2^n$. It now follows that $A \in L(Z_0^n, Z_1^n, Z_2^n, m, k)$ as required. ∎

3. **Remarks.** It is worth noting that Theorem 2.1 is the best possible in the sense that $\mathbb{M}(\kappa)$ is easily seen to be not σ-2-linked if $\kappa > 2^{\aleph_0}$. The reason for this is that $\mathbb{M}(\kappa)$ contains the algebra of clopen subsets of 2^κ and it is known (p. 122 [2]) that 2^κ is not separable if $\kappa > 2^{\aleph_0}$. The algebra of clopen subsets of 2^κ is easily seen to be σ-2-linked if and only if 2^κ is separable. This leads to the following corollary.

COROLLARY 3.1. *The measure algebra of a probability space is σ-n-linked for some n if and only if it is σ-n-linked for every n.*

PROOF. This follows directly from Theorem 2.1 and Maharam's Theorem [5] which states that the measure algebra of a homogeneous probability space is isomorphic to $\mathbb{M}(\kappa)$ for some κ and every algebra is obtained from disjoint unions of such spaces. ∎

This corollary is not true for arbitrary Boolean algebras. Examples can be found in [1].

https://doi.org/10.4153/CMB-1994-007-7 Published online by Cambridge University Press

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