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ON THE EISENSTEIN SERIES FOR THE PRINCIPAL CONGRUENCE SUBGROUPS

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Let Γ be a Fuchsian group (of finite type) acting on the upper half plane. To each parabolic cusp κ_i $(i = 1, \cdots, h)$, corresponds a Eisenstein serie

$$E_i(\tau, s) = \sum_{\Gamma_i \backslash \Gamma} y(\sigma_i^{-1}\sigma\tau)^s$$

where Γ_i is the stationary subgroup of Γ with respect to κ_i and σ_i is an element of $SL(2, \boldsymbol{R})$, such that $\sigma_i \infty = \kappa_i$. (Here we denote by $y(\tau)$ the imaginary part of τ.)

Then,

$$\mathrm{E}(\tau, s) = \begin{pmatrix} E_1(\tau, s) \\ \vdots \\ E_h(\tau, s) \end{pmatrix} \text{ satisfies the functional}$$

equation:

$$\mathrm{E}(\tau, s) = \Phi(s)\, \mathrm{E}(\tau, 1 - s). \qquad (*)$$

(For details, see Kubota [1].)

In this paper, we shall give an elementary proof of the functional equation (*) in case $\Gamma = \Gamma_N$ (the principal congruence subgroup of Stufe N). For the explicit form of $\Phi(s)$, see Proposition 1, 2 in §2 (the case $N = p^n$) and Theorem in §3 (general case).

§1

For a positive integer $N > 1$ and a pair of integers $a = \{a_1, a_2\}$ we put

$$\Theta(t\,; a_1, a_2) = \sum_{\{m,n\} \equiv \{a_1, a_2\}\ (N)} e^{-\pi t |m\tau + n|^2 / y}$$

where $\tau = x + iy$, $y > 0$.

LEMMA 1.

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(1) $$\Theta(t\,;a_1,a_2)=\frac{1}{tN^2}\sum_{\{b_1,b_2\}\bmod N} e^{\frac{2\pi i}{N}\begin{vmatrix}a_1,a_2\\ b_1,b_2\end{vmatrix}}\,\Theta\Big(\frac{1}{tN^2}\,;b_1,b_2\Big)\,.$$

Proof is omitted.

To a pair $\{a_1,a_2\}$ such that $(a_1,a_2,N)=1$, there corresponds a Eisenstein series for Γ_N

$$E(\tau,s\,;a_1,a_2)=\sum_{\substack{\{m,n\}\equiv\{a_1,a_2\}\,(N)\\(m,n)=1}}\frac{y^s}{|m\tau+n|^{2s}}\,.$$

Since $\{a_1,a_2\}$ and $\{-a_1,-a_2\}$ give rise to the same Eisenstein series, there are $\frac{1}{2}N^2\prod_{p|N}\Big(1-\frac{1}{p^2}\Big)$ distinct series for $N>2$. (For $N=2$, there are three such series.)

Moreover, we put

$$E^*(\tau,s\,;a_1,a_2)=\sum_{\{m,n\}\equiv\{a_1,a_2\}\,(N)}\frac{y^s}{|m\tau+n|^{2s}}\,.$$

These series converge uniformly on compact sets in the upper half plane, if Re $s>1$.

From the definition, we have

(2) $$\int_0^\infty\Theta(t\,;a_1,a_2)t^{s-1}dt=\pi^{-s}\Gamma(s)E^*(\tau,s\,;a_1,a_2)\,.$$

For a character mod N, such that $\chi(-1)=1$, we put

$$\Theta(t\,;a,\chi)=\sum_{\substack{(u,N)=1\\u\bmod N}}\overline{\chi(u)}\;\Theta(t\,;ua_1,ua_2).$$

From (1), it follows that

(1′) $$\Theta(t\,;a,\chi)=\frac{1}{tN^2}\sum_{b\bmod N}e^{\frac{2\pi i}{N}\begin{vmatrix}a_1,a_2\\ b_1,b_2\end{vmatrix}}\,\Theta\Big(\frac{1}{tN^2}\,;b,\bar{\chi}\Big)\,.$$

$E(\tau,s\,;a,\chi)$ and $E^*(\tau,s\,;a,\chi)$ are defined in the same way.

LEMMA 2.

(3) $$\begin{aligned}E^*(\tau,s\,;a,\chi)&=\prod_{p|N}(1-p^{-2s})\zeta(2s)E(\tau,s\,;a,\chi_0), &&\text{if } \chi=\chi_0\equiv1\\ &=L(2s,\bar{\chi})E(\tau,s\,;a,\chi) &&,\ \text{if } \chi\neq\chi_0\,.\end{aligned}$$

Proof. (1) If $\chi\neq\chi_0$, we have

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$$E(\tau, s; a, \chi) = \sum_{\substack{(u,N)=1 \\ u \bmod N}} \overline{\chi(u)} \left\{ \sum_{\substack{(d,N)=1 \\ d \bmod N}} \left(\sum_{dq \equiv 1\ (N)} q^{-2s} \right) E(\tau, s; dua_1, dua_2) \right.$$

$$= \sum_{\substack{(d,N)=1 \\ d \bmod N}} \chi(d) \sum_{dq \equiv 1\ (N)} q^{-2s} E(\tau, s; a, \chi)$$

$$= \sum_{(q,N)=1} \overline{\chi(q)}\ q^{-2s}\ E(\tau, s; a, \chi).$$

$$= L(2s, \bar{\chi}) E(\tau, s; a, \chi)$$

(2) Let $\chi = \chi_0$. If $N = p_1^{k_1} \cdots p_r^{k_r}$ is a factorization into prime factors, then we have

$$\sum_{(q,N)=1} q^{-2s} = \sum_{i_1, \cdots, i_j} (-1)^j \sum_{p_{i_1} \cdots p_{i_j} \mid q} q^{-2s}$$

$$= \zeta(2s) \sum_{i_1, \cdots, i_j} (-1)^j (p_{i_1} \cdots p_{i_j})^{-2s}.$$

Since, as in (1), we have

$$E^*(\tau, s; a, \chi_0) = \left(\sum_{(q,N)=1} q^{-2s} \right) E(\tau, s; a, \chi_0)$$

we obtain the desired result.

Remark. As is seen from the definition,

$$\sum_{\chi} E(\tau, s; a, \chi) = E(\tau, s; a_1, a_2).$$

Therefore, the functional equation of $E(\tau, s; a_1, a_2)$ can be obtained from that of $E(\tau, s; a, \chi)$.

§2

In this section, we shall prove the functional equation of $E(\tau, s; a, \chi)$ in case $N = p^n$.

Since $E(\tau, s; a, \chi)$ is a χ-homogeneous function, i.e.

$$E(\tau, s; ua, \chi) = \chi(u) E(\tau, s; a, \chi), \quad (ua = \{ua_1, ua_2\}, \ (u, p) = 1)$$

we may restrict ourselves to the case $a \in I$, where

$$I = \{(a_1, a_2); a_1 = 1 \text{ or } a_2 = 1, \ a_1 \equiv 0 \ (p)\}.$$

It is easy to see that, for $a, b \in I$,

$$\langle a, b \rangle = \begin{vmatrix} a_1, a_2 \\ b_1, b_2 \end{vmatrix} \equiv 0 \ (p^k) \text{ if and only if } a \equiv b \ (p^k) \ (1 \leqq k \leqq n).$$

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1) The case $\chi=\chi_0$

(a) Let $n\geqq 2$. Then, for $a\in I$, we have from (1)′

$$\Theta(t;a,\chi_0)-\frac{1}{p}\sum_{a'\in I,\, a'\equiv a(p^{n-1})}\Theta(t;a',\chi_0)$$
$$=\frac{1}{tp^{2n}}\sum_{b}c(b)\Theta\Big(\frac{1}{tp^{2n}};b,\chi_0\Big).$$

If $b\equiv 0\ (p)$, $e^{\frac{2\pi i}{p^n}<a,b>}-\frac{1}{p}\sum_{a'\in I,\, a'\equiv a(p^{n-1})}e^{\frac{2\pi i}{p^n}<a,b>}=0$. Therefore, $c(b)=0$.

For $b\in I$, we have

$$c(b)=\sum_{(t,p)=1,\ t \bmod p^n}\Big\{e^{\frac{2\pi i}{p^n}<a,b>t}-\frac{1}{p}\sum_{\substack{a'\equiv a(p^{n-1})\\ a'\in I}}e^{\frac{2\pi i}{p^n}<a',b>t}\Big\}.$$

If $b\not\equiv a\ (p^{n-1})$, then, as we noted above,

$$\langle a',b\rangle=p^k u \qquad (k<n-1,\ (u,p)=1).$$

Therefore we have

$$\sum_{(t,p)=1,\ t \bmod p^n}e^{\frac{2\pi i}{p^n}<a',b>t}=\sum_{(t,p)=1,\ t \bmod p^n}e^{\frac{2\pi i}{p^r}t}$$
$$=\#\{t;\,t\equiv 1\ (p^r)\}\sum_{(t,p)=1,\ t \bmod p^r}e^{\frac{2\pi i}{p^r}t}=0,$$

because $\sum_{(t,N)=1,\ t \bmod N}e^{\frac{2\pi i}{N}t}=\mu(N)$ (Möbius function) and $r=n-k\geqq 2$.

Hence, $c(b)=0$.

If $b\equiv a\ (p^{n-1})$, we have

$$\sum_{a'\equiv a\ (p^{n-1}),\ a'\in I}e^{\frac{2\pi i}{p^n}<a',b>t}=\sum_{a'\equiv b\ (p^{n-1}),\ a'\in I}e^{\frac{2\pi i}{p^n}<a',b>t}=\sum_{v \bmod p}e^{\frac{2\pi i}{p}v}=0.$$

Therefore

$$c(b)=\sum_{(t,p)=1,\ t \bmod p^n}e^{\frac{2\pi i}{p^n}<a,b>t}=p^n-p^{n-1}\quad\text{if}\quad a=b$$
$$=\#\{t;\,t\equiv 1\ (p)\}\sum_{(t,p)=1,\ t \bmod p}e^{\frac{2\pi i}{p}t}=-p^{n-1}\quad\text{if}\quad a\neq b.$$

Thus we have proved the following formula:

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$$(1'') \qquad \Theta(t;a,\chi_0)-\frac{1}{p}\sum_{a'\in I,\ a'\equiv a\ (p^{n-1})}\Theta(t;a',\chi_0)$$
$$=\frac{1}{tp^n}\left\{\Theta\left(\frac{1}{tp^{2n}};a,\chi_0\right)-\frac{1}{p}\sum_{a'\in I,\ a'\equiv a\ (p^{n-1})}\Theta\left(\frac{1}{tp^{2n}};a',\chi_0\right)\right\}.$$

Now we denote by $E_n(\tau,s;a,\chi)$ the Eisenstein series for Γ_{p^n}. Then, it is easy to see that

$$\sum_{a'\in I,\ a'\equiv a\ (p^{n-1})}E_n^*(\tau,s;a',\chi_0)=E_{n-1}^*(\tau,s;a,\chi_0).$$

We put

$$G(s)=\pi^{-s}\Gamma(s)\zeta(2s)\,(1-p^{-2s})\left\{E_n(\tau,s;a,\chi_0)-\frac{1}{p}E_{n-1}(\tau,s;a,\chi_0)\right\}$$

In view of (2), (3) and (1''), we have

$$G(s)=\int_{1/p^n}^{\infty}\left\{\Theta(t;a,\chi_0)-\frac{1}{p}\sum_{a'\in I,\ a'\equiv a\ (p^{n-1})}\Theta(t;a',\chi_0)\right\}t^{s-1}dt$$
$$+\,p^{n(1-2s)}\int_{1/p^n}^{\infty}\left\{\Theta(t;a,\chi_0)-\frac{1}{p}\sum_{a'}\Theta(t;a,\chi_0)\right\}t^{-s}dt\ .$$

From this immediately follow the analytic continuation of $G(s)$ into the whole s-plane and the functional equation

$$(4) \qquad G(s)=p^{n(1-2s)}G(1-s).$$

(b) In case $n=1$, a similar argument shows that

$$\Theta(t;a,\chi_0)-\frac{1}{p+1}\sum_{a'\in I}\Theta(t;a',\chi_0)=\frac{1}{tp}\left\{\Theta\left(\frac{1}{tp^2};a,\chi_0\right)-\frac{1}{p+1}\sum_{a'\in I}\Theta\left(\frac{1}{tp^2};a',\chi_0\right)\right.$$

Therefore, as in (a), we can prove that

$$G(s)=\pi^{-s}\Gamma(s)\zeta(2s)\,(1-p^{-2s})\left\{E_1(\tau,s;a,\chi_0)-\frac{1}{p+1}E(\tau,s)\right\}$$

is an entire function and satisfies the functional equation

$$(5) \qquad G(s)=p^{1-2s}G(1-s)$$

where $$E(\tau,s)=\sum_{(m,n)=1}\frac{y^s}{|m\tau+n|^{2s}}$$

is the Eisenstein series for the full modular group.

As is well known, $E(\tau,s)$ is meromorphic in the whole s-plane, and satisfies the functional equation

$$E(\tau, s) = \frac{\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)} E(\tau, 1-s). \tag{6}$$

From (4), (5) and (6), we can obtain the following result.

PROPOSITION 1. *Let* $\boldsymbol{E}_n(\tau, s; \chi_0)$ *be the column of the* $p^n + p^{n-1}$ *functions* $E_n(\tau, s; a, \chi_0)$, $(a \in I)$.

Then, $\boldsymbol{E}_n(\tau, s; \chi_0)$ *is a meromorphic function in the whole s-plane and satisfies the following functional equation:*

$$\boldsymbol{E}_n(\tau, s; \chi_0) = \Phi_n(s) \boldsymbol{E}_n(\tau, 1-s; \chi_0)$$

where $\Phi_n(s) = \varphi(s) \langle c^{(n)}(a, b) \rangle$

(*matrix of degree* $p^n + p^{n-1}$)

$$\varphi(s) = \frac{\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)}$$

$$c^{(n)}(a, b) = \frac{p-1}{p^{2s}-1} p^{(n-1)(1-2s)} \qquad \textit{if} \quad a = b$$

$$= p^{-n+r+1} \frac{p^{2s-1}-1}{p^{2s}-1} p^{r(1-2s)} \qquad \textit{if} \quad a \equiv b \ (p^r)$$

$$(0 \leqq r \leqq n-1).$$

Proof. As is easily seen,

$$c^{(n)}(a, b) = \begin{cases} p^{n(1-2s)} \dfrac{1-p^{2s-2}}{1-p^{-2s}} \left(1 - \dfrac{1}{p}\right) + \dfrac{1}{p} c^{(n-1)}(a, b) & (n \geqq 2) \\ p^{1-2s} \dfrac{1-p^{2s-2}}{1-p^{-2s}} \left(1 - \dfrac{1}{p+1}\right) + \dfrac{1}{p+1} & (n = 1) \end{cases}$$

$$\text{if} \quad a = b$$

$$= \begin{cases} -p^{n(1-2s)-1} \dfrac{1-p^{2s-2}}{1-p^{-2s}} + \dfrac{1}{p} c^{(n-1)}(a, b) & (n = 2) \\ -\dfrac{p^{1-2s}}{p+1} \dfrac{1-p^{2s-2}}{1-p^{-2s}} + \dfrac{1}{p+1} & (n = 1) \end{cases}$$

$$\text{if} \quad p^{n-1} \| a - b$$

$$= \frac{1}{p} c^{(n-1)}(a, b) \qquad \text{if} \quad p^k \| a - b \quad (0 \leqq k \leqq n-2).$$

Hence, by induction on n, follows the desired result.

2) The case $\chi \neq \chi_0$

a) Let χ be a primitive character.

For $a=\{a_1, a_2\}$, such that $(a_1, a_2)=p^k u$ $(k \geqq 1,\ (u, p)=1)$, we have

$$\Theta(t; a, \chi) = \sum_{\substack{(u,p)=1 \\ u \bmod p^r}} \chi(u) \Big(\sum_{t \equiv 1\ (p^r)} \chi(t) \Big) \Theta(t; ua_1, ua_2) = 0.$$

$(r = n - k < n)$

Therefore, from (1), it follows that

$$\Theta(t; a, \chi) = \frac{1}{tp^{2n}} \sum_{b \in I} \sum_{\substack{(u,p)=1 \\ u \bmod p^n}} e^{\frac{2\pi i}{p^n} \langle a, b \rangle u} \overline{\chi(u)}\, \Theta\Big(\frac{1}{tp^{2n}}; b, \bar{\chi}\Big).$$

We put $S_\chi = \sum_{u \bmod p^n} e^{\frac{2\pi i}{p^n} u} \chi(u)$ (Gauss sum). Then,

$$\Theta(t; a, \chi) = \frac{1}{tp^{2n}} \sum_{b \in I} S_{\bar{\chi}} \chi(\langle a, b \rangle)\, \Theta\Big(\frac{1}{tp^{2n}}; b, \bar{\chi}\Big). \tag{7}$$

By (2), (3) and (7), we obtain

$$\begin{aligned} \pi^{-s} \Gamma(s) L(2s, \bar{\chi}) E(\tau, s; a, \chi) &= \int_{1/p^n}^{\infty} \Theta(t; a, \chi) t^{s-1} dt \\ &+ \frac{S_{\bar{\chi}}}{p^{2ns}} \sum_{b \in I} \chi(\langle a, b \rangle) \int_{1/p^n}^{\infty} \Theta(t; b, \bar{\chi}) t^{-s} dt. \end{aligned} \tag{8}$$

As is easily seen, we have

$$\sum_{b \in I} \chi(\langle a, b \rangle) \overline{\chi(\langle b, a' \rangle)} = p^n \delta_{a, a'} \qquad (a, a' \in I).$$

Moreover, $\qquad |S_\chi|^2 = p^n$ and $\bar{S}_\chi = S_{\bar{\chi}}$ if $\chi(-1) = 1$.

Therefore, from (8), immediately follows the functional equation

$$G(s, a, \chi) = p^{-2ns} S_{\bar{\chi}} \sum_{b \in I} \chi(\langle a, b \rangle)\, G(1-s, b, \bar{\chi}) \tag{9}$$

where $\qquad G(s, a, \chi) = \pi^{-s} \Gamma(s)\, L(2s, \bar{\chi})\, E(\tau, s; a, \chi).$

By the functional equation of Dirichlet L-function

$$H(s, \chi) = \pi^{-\frac{s}{2}} \Gamma\Big(\frac{s}{2}\Big) L(s, \chi) = S_\chi p^{-ns} H(1-s, \bar{\chi}),$$

(9) can be written as follows.

Let $\boldsymbol{E}(\tau, s; \chi)$ be the column of the $p^n + p^{n-1}$ functions $E(\tau, s; a, \chi)$.

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Then,

$$\boldsymbol{E}(\tau, s; \chi) = \Phi(s,\chi)\boldsymbol{E}(\tau, 1-s, \bar{\chi})$$

where

$$\Phi(s,\chi) = p^{-\frac{n}{2}} \frac{\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)} \frac{L(2s-1,\bar{\chi})}{L(2s,\bar{\chi})} \langle p^{-\frac{n}{2}}\chi(\langle a, b\rangle)\rangle$$

b) We denote by r the integer $\min\{m;\, \chi(v)=1,\;\text{ if }\; v\equiv 1 \bmod p^m\}$. In a) we considered the case $r=n$. Let $r \leqq n-1$.
First we note

$$\sum_{t \bmod p^n} \chi(t)\, e^{\frac{2\pi i}{p^n}ct} = \overline{\chi(c')}S_\chi p^{n-r} \quad \text{if} \quad c = c'p^{n-r},\ (c',p)=1$$
$$= 0 \quad \text{otherwise.}$$

Since $\displaystyle\sum_{a'\equiv a\ (p^{n-1}),\ a'\in I} e^{\frac{2\pi i}{p^n}\langle a', b\rangle u} = e^{\frac{2\pi i}{p^n}\langle a, b\rangle u} \sum_{v \bmod p} e^{\frac{2\pi i}{p}v} = 0$, if $a\equiv b\ (p)$,

$$\Theta(t;a,\chi) - \frac{1}{p}\sum_{\substack{a'\equiv a\ (p^{n-1})\\ a'\in I}} \Theta(t;a',\chi) \tag{10}$$
$$= \frac{1}{tp^{2n}}\, p^{n-r}S_{\bar{\chi}} \sum_{\substack{b\equiv a\ (p^{n-r})\\ b\in I}} \chi\left(\frac{\langle a,b\rangle}{p^{n-r}}\right)\Theta\left(\frac{1}{tp^{2n}};b,\bar{\chi}\right).$$

On the other hand, we have

$$\sum_{\substack{b\equiv a\ (p^{n-r})\\ b\in I}} \chi\left(\frac{\langle a,b\rangle}{p^{n-r}}\right) \sum_{\substack{b'\equiv b\ (p^{n-1})\\ b'\in I}} \Theta(t;b',\chi)$$
$$= \sum_{\substack{b\equiv a\ (p^{n-r})\\ b\in I}} \left\{\sum_{\substack{b'\equiv b\ (p^{n-1})\\ b'\in I}} \chi\left(\frac{\langle a,b'\rangle}{p^{n-r}}\right)\right\}\Theta(t;a,\chi) = 0.$$

Therefore, (10) is unchanged, if $\Theta(t;b,\chi)$ is replaced by

$$\Theta(t;b,\chi) - \frac{1}{p}\sum_{\substack{b'\equiv b\ (p^{n-1})\\ b'\in I}} \Theta(t;b',\chi).$$

We put

$$G(s,a,\chi) = \pi^{-s}\Gamma(s)L(2s,\bar{\chi})\left\{E_n(\tau,s;a,\chi) - \frac{1}{p}E_{n-1}(\tau,s;a,\chi)\right\}.$$

Then, since $\sum_{\substack{a' \equiv a \ (p^{n-1}) \\ a' \in I}} E_n(\tau, s; a, \chi) = E_{n-1}(\tau, s; a, \chi)$

(we note that $r \leqq n-1$), we have

$$G(s, a, \chi) = \int_{1/p^n}^{\infty} \Big\{ \Theta(t; a, \chi) - \frac{1}{p} \sum_{\substack{a' \equiv a \ (p^{n-1}) \\ a' \in I}} \Theta(t; a', \chi) \Big\} t^{s-1} dt$$

$$+ \int_{1/p^n}^{\infty} p^{-2ns} p^{n-r} S_{\bar{\chi}} \sum_{\substack{b \equiv a \ (p^{n-1}) \\ b \in I}} \chi\Big(\frac{\langle a, b \rangle}{p^{n-r}} \Big) \Big\{ \Theta(t; b, \bar{\chi}) - \frac{1}{p} \sum_{\substack{b' \equiv b \ (p^{n-1}) \\ b' \in I}} \Theta(t; b', \bar{\chi}) \Big\} t^{-s} dt$$

$$= p^{-2ns} p^{n-r} S_{\bar{\chi}} \sum_{\substack{b \equiv a \ (p^{n-r}) \\ b \in I}} \chi\Big(\frac{\langle a, b \rangle}{p^{n-r}} \Big) G(1-s, b, \bar{\chi}).$$

(see the remark below)

From this and the result in *a*), we obtain the following

PROPOSITION 2.

$$\boldsymbol{E}_n(\tau, s; \chi) = \Phi_n(s, \chi) \boldsymbol{E}_n(\tau, 1-s; \bar{\chi})$$

where $\Phi_n(s, \chi) = \varphi(s, \chi) \langle c^{(n)}(a, b) \rangle$

$$\varphi(s, \chi) = p^{-r} \frac{\sqrt{\pi}\, \Gamma\big(s - \frac{1}{2}\big)}{\Gamma(s)} \frac{L(2s-1, \bar{\chi})}{L(2s, \bar{\chi})}$$

$$c^{(n)}(a, b) = p^{(1-2s)(n-r-k)-k} \chi\Big(\frac{\langle a, b \rangle}{p^{n-r-k}} \Big)$$

if $p^{n-r-k} \| \langle a, b \rangle \ (0 \leqq k \leqq n-r)$

$= 0$ *otherwise.*

Remark. Let $a \equiv b \ (p^{n-r})$. Then,

$$\sum_{\substack{c \equiv a \ (p^{n-r}) \\ c \in I}} \chi\Big(\frac{\langle a, c \rangle}{p^{n-r}} \Big) \overline{\chi\Big(\frac{\langle b, c \rangle}{p^{n-r}} \Big)} = \begin{cases} p^r - p^{r-1} & \text{if } a = b \\ -p^{r-1} & \text{if } p^{n-1} \| a - b \\ 0 & \text{otherwise.} \end{cases}$$

§3

Let $N = p_1^{n_1} \cdots p_\lambda^{n_\lambda}$ be a factorization into distinct primes.

We put $N = N_i p_i^{n_i} \qquad (1 \leqq i \leqq \lambda).$

Let us choose a set of integers $\{d_1, \cdots, d_\lambda\}$, such that

$$d_i \equiv 0 \ (N_i), \ \equiv 1 \ (p_i^{n_i}) \qquad (1 \leqq i \leqq \lambda).$$

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Then, the mapping

$$\boldsymbol{Z}^{\lambda} \ni \{a_1, \cdots, a_\lambda\} \longrightarrow a = \sum_{i=1}^{\lambda} d_i a_i \in \boldsymbol{Z}$$

induces a ring-isomorphism of $\boldsymbol{Z}/(N)$ onto $\prod_{i=1}^{\lambda} \boldsymbol{Z}/(p_i^{n_i})$.

It is obvious that

(1) $\quad (a^{(1)}, a^{(2)}, N) = 1$ if and only if $(a_i^{(1)}, a_i^{(2)}, p_i) = 1$

(2) $\quad (u, N) = 1$ if and only if $(u_i, p_i) = 1 \qquad (1 \leqq i \leqq \lambda)$

Let $I = I_1 \times \cdots \times I_\lambda$, where $I_i = \{(a_1, a_2) \bmod p_i^{n_i}\,;\, a_1 = 1 \text{ or } a_2 = 1,\ a_1 \equiv 0\ (p_i)\}$. We denote by $\boldsymbol{V}(I)$ the space of functions on I.

Then $$\boldsymbol{V}(I) = \boldsymbol{V}(I_1) \otimes \cdots \otimes \boldsymbol{V}(I_\lambda).$$

For a character χ mod N, there exist characters χ_i mod $p_i^{n_i}$ such that

$$\chi(a) = \prod_{i=1}^{\lambda} \chi_i(a_i).$$

Let
$$\begin{aligned} r_i = r(\chi_i) &\neq 0 \qquad (1 \leqq i \leqq \mu) \\ &= 0 \qquad (\mu + 1 \leqq i \leqq \lambda). \end{aligned}$$

Then there exists a primitive character $\tilde{\chi}$ mod $\tilde{N} = \prod_{i=1}^{\mu} p_i^{r_i}$, such that

$$\chi(a) = \tilde{\chi}(a) \quad \text{if} \quad (a, N) = 1.$$

We put

$$\boldsymbol{T} = \boldsymbol{T}_1 \otimes \cdots \otimes \boldsymbol{T}_\lambda$$

where $\boldsymbol{T}_i$ is a linear transformation in $\boldsymbol{V}_i = \boldsymbol{V}(I_i)$, defined by

$$\begin{aligned} \boldsymbol{T}_i f(a) &= f(a) - \frac{1}{p_i} \sum_{a' \in I_i,\ a' \equiv a\ (p_i^{n_i})} f(a') && \text{if } n_i > 1 \text{ and } n_i > r_i \\ &= f(a) - \frac{1}{p_i + 1} \sum_{a' \in I_i} f(a') && \text{if } n_i = 1 \text{ and } r_i = 0 \\ &= f(a) && \text{if } n_i = r_i . \end{aligned}$$

Moreover, we define an endomorphism of $\boldsymbol{V} = \boldsymbol{V}(I)$ by

$$\boldsymbol{A} f(a) = \sum_{b \in I} A(a, b) f(b)$$

$$A(a, b) = \sum_{\substack{(u, N)=1 \\ u \bmod N}} \overline{\chi(u)}\, e^{\frac{2\pi i}{N}<a, b>u}$$

We have

$$\boldsymbol{A} = c\boldsymbol{A}_1 \otimes \cdots \otimes \boldsymbol{A}_\lambda \quad (c = \bar{S}_{\tilde{\chi}} N \tilde{N}^{-1})$$

where

$$A_i(a, b) = \begin{cases} \chi_i\left(\dfrac{\langle a, b \rangle}{p_i^{n_i - r_i}}\right) & \text{if} \quad a \equiv b \ \ (p_i^{n_i - r_i}) \\ 0 & \text{otherwise} \qquad (1 \leqq i \leqq \mu) \end{cases}$$

$$= \begin{cases} \left(1 - \dfrac{1}{p_i}\right)\overline{\tilde{\chi}(p_i^{n_i})} & \text{if} \quad a = b \\ -\dfrac{1}{p_i}\overline{\tilde{\chi}(p_i^{n_i})} & \text{if} \quad a \neq b, \ \ a \equiv b \ \ (p_i^{n_i - 1}) \qquad (\mu < i \leqq \lambda) \\ 0 & \text{otherwise} \end{cases}$$

For, we have

$$A(a, b) = \prod_{i=1}^{\lambda} \sum_{\substack{(u_i, p_i)=1 \\ u_i \bmod p_i^{n_i}}} \overline{\chi_i(u_i)}\, e^{\frac{2\pi i}{p_i^{n_i}}<a_i, b_i>u_i c_i} \qquad \left(c_i = \frac{d_i}{N_i}\right)$$

$$= \prod_{i=1}^{\mu} p_i^{n_i - r_i} \chi_i(c_i) S_{\bar{\chi}_i} \chi_i\left(\frac{\langle a_i, b_i \rangle}{p_i^{n_i - r_i}}\right)$$

$$\times \prod_{i=\mu+1}^{\lambda} \sum e^{\frac{2\pi i}{p_i^{n_i}}<a_i, b_i>u_i} \quad \text{if} \quad p_i^{n_i - r_i} \| a - b$$

$$= 0 \qquad \text{otherwise}$$

and

$$\prod_{i=1}^{\mu} \chi_i(c_i) S_{\bar{\chi}_i} = \prod_{i=\mu+1}^{\lambda} \overline{\tilde{\chi}(p_i^{n_i})} \bar{S}_{\tilde{\chi}}\,.$$

From the results obtained in §2, we have

$$\begin{aligned} &\boldsymbol{T}_i \boldsymbol{A}_i = \boldsymbol{A}_i \boldsymbol{T}_i, \quad \boldsymbol{A}_i \boldsymbol{A}_i^* \boldsymbol{T}_i = p_i^{r_i} \boldsymbol{T}_i \qquad (1 \leqq i \leqq \lambda) \\ (11) \qquad &\qquad = \tilde{\chi}(p_i^{n_i}) \boldsymbol{T}_i, \quad \text{for} \quad i > \mu \\ &\qquad = \boldsymbol{A}_i \quad , \quad \text{for} \quad i \leqq \mu. \end{aligned}$$

Therefore we have

$$\boldsymbol{TA} = \boldsymbol{AT} \quad \text{and} \quad \boldsymbol{AA}^* \boldsymbol{T} = N^2 \boldsymbol{T}\,.$$

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LEMMA. *For* $f(a,s,\chi)=\pi^{-s}\Gamma(s)E^*(\tau,s;a,\chi)$, *we have*

$$\boldsymbol{T}f(\cdot,s,\chi)=N^{-2s}\boldsymbol{A}\boldsymbol{T}f(\cdot,1-s,\bar{\chi}). \tag{12}$$

Proof. From (1′), we have

$$\boldsymbol{T}\Theta(t;\cdot,\chi)=\frac{1}{tN^2}\sum_{b \bmod N}\Big(\boldsymbol{T}e^{\frac{2\pi i}{N}\langle\cdot,b\rangle}\Big)\Theta\Big(\frac{1}{tN^2};b,\bar{\chi}\Big).$$

If $b_i\equiv 0\ (p_i)$, we have

$$\boldsymbol{T}e^{\frac{2\pi i}{N}\langle\cdot,b\rangle}=0 \qquad (\text{in case } n_i>r_i)$$

or

$$\Theta(t;b,\chi)=0 \qquad (\text{in case } n_i=r_i).$$

Hence,

$$\boldsymbol{T}\Theta(t;\cdot,\chi)=\frac{1}{tN^2}\sum_{b\in I}\boldsymbol{T}\Big(\sum_{(u,N)=1}\overline{\chi(u)}e^{\frac{2\pi i}{N}\langle\cdot,b\rangle u}\Big)\Theta\Big(\frac{1}{tN^2};b,\bar{\chi}\Big)$$
$$=\frac{1}{tN^2}\boldsymbol{T}\boldsymbol{A}\Theta\Big(\frac{1}{tN^2};\cdot,\bar{\chi}\Big).$$

Therefore

$$\boldsymbol{T}f(\cdot,s,\chi)=\int_0^\infty \boldsymbol{T}\Theta(t;\cdot,\chi)t^{s-1}dt$$
$$=\int_{1/N}^\infty \boldsymbol{T}\Theta(t;\cdot,\chi)t^{s-1}dt+N^{-2s}\boldsymbol{A}\int_{1/N}^\infty \boldsymbol{T}\Theta(t;\cdot,\bar{\chi})t^{-s}dt$$
$$=N^{-2s}\boldsymbol{A}\boldsymbol{T}f(\cdot,1-s,\bar{\chi}).$$

Let

$$\boldsymbol{V}_i^{(k)}=\{f\in\boldsymbol{V}_i;\ f(a')=f(a)\ \text{ if }\ a'\equiv a\ (p_i^k)\}\quad (r_i\leqq k\leqq n_i)$$
$$=\{0\}\qquad\qquad (k<r_i).$$

We denote by $\boldsymbol{P}_k$ the projection operator on $\boldsymbol{V}_i^{(k)}\ominus\boldsymbol{V}_i^{(k-1)}$. Then, for $f\in\boldsymbol{V}$, we have

$$\sum_{k_i=r_i}^{n_i}\boldsymbol{P}_{k_1\cdots k_\lambda}f=f \tag{13}$$

where

$$\boldsymbol{P}_{k_1\cdots k_\lambda}=\boldsymbol{P}_{k_1}\otimes\cdots\otimes\boldsymbol{P}_{k_\lambda}$$

LEMMA.

$$\boldsymbol{P}_{k_1\cdots k_\lambda}E_N^*(\tau,s;\cdot,\chi)=N'N^{-1}\prod_{k_i=0}(p_i+1)^{-1}\Big(1-\frac{\chi(p_i)}{p_i^{2s}}\Big)\boldsymbol{T}^{(N')}E_{N'}^*(\tau,s;\cdot,\chi). \tag{14}$$

Proof. First we note that

$$(15)\qquad \boldsymbol{P}_k f(a)=\frac{1}{c_k}\sum_{a'\equiv a\ (p^k)} f(a')-\begin{cases}\dfrac{1}{c_{k-1}}\displaystyle\sum_{a'\equiv a\ (p^{k-1})} f(a') & \text{for}\quad k>r\\ 0 & \text{for}\quad k=r\end{cases}$$

$$=\frac{1}{c_k}\boldsymbol{T}^{(p^k)}\sum_{a'\equiv a\ (p^k)} f(a')$$

($c_k=p^{n-k}$ or p^n+p^{n-1} according as $k>0$ or $k=0$).

We have only to prove the following.

$$1^\circ\qquad \sum_{\substack{a'\equiv a\ (N')\\ a'\in I}} E_N^*(\tau,s;a,\chi)=\left(1-\frac{\chi(p)}{p^{2s}}\right)E_{N'}^*(\tau,s;a,\chi)\quad \text{if}\quad (p,N)=1$$

$$2^\circ\qquad \qquad =E_{N'}^*(\tau,s;a,\chi)\qquad \text{if}\quad p\,|\,N.$$

In case $(p,N)=1$, we have

$$\sum_{a'\equiv a\ (N')} E_N^*(\tau,s;a,\chi)=\sum_{\substack{(u,N')=1\\ u \bmod N'}}\overline{\tilde\chi(u)}\sum_{\substack{\{m,n\}\equiv\{ua^{(1)},ua^{(2)}\}\ (N')\\ \not\equiv 0\quad (p)}}\frac{y^s}{|m\tau+n|^{2s}}$$

$$=E_{N'}^*(\tau,s;a,\chi)-\sum_u\overline{\tilde\chi(u)}\sum_{\substack{\{m,n\}\equiv_p cu\{a^{(1)},a^{(2)}\}(N)\\ (c\equiv p^{-1}\ (N'))}}\frac{y^s}{|m\tau+n|^{2s}}$$

$$=\left(1-\frac{\overline{\tilde\chi(p)}}{p^{2s}}\right)E_{N'}^*(\tau,s;a,\chi).$$

For the proof of 2°, it is sufficient to note that

$\{m,n\}\equiv\{ua^{(1)},\ ua^{(2)}\}\ (p^{k-1})$ if and only if $\{m,n\}\equiv\{uvb^{(1)},\ uvb^{(2)}\}\ (p^k)$ for some $b=\{b^{(1)},\ b^{(2)}\}\equiv a\ (p^{k-1})$ and $v\equiv 1\ (p^{k-1})$.

THEOREM. *$E(\tau,s;a,\chi)$ is a meromorphic function in the whole s-plane and satisfies the following functional equation.*

$$E(\tau,s;\cdot,\chi)=\frac{\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\,\frac{L(2s-1,\tilde\chi)}{L(2s,\tilde\chi)}\,\Phi^{(1)}\otimes\cdots\otimes\Phi^{(\lambda)}E(\tau,1-s;\cdot,\bar\chi)$$

where, for $i\leqq\mu$, $\Phi^{(i)}(a,b)=\chi_i\left(\dfrac{\langle a,b\rangle}{p_i^k}\right)p_i^{(1-2s)k-n+k}$ *if* $p_i^k\| a-b$ $(0\leqq k\leqq n-r)$

$=0$ *otherwise*

for $i>\mu$, $\Phi^{(i)}(a,b)=(\tilde\chi(p_i)p_i^{2s-1})^{1-n}\dfrac{p-1}{\tilde\chi(p_i)p_i^{2s}-1}$, *if* $a=b$

$$= p_i^{k-n+1} \frac{\tilde{\chi}(p_i)p_i^{2s-1}-1}{\tilde{\chi}(p_i)p_i^{2s}-1} (\tilde{\chi}(p_i)p_i^{2s-1})^{-k}$$

$$\textit{if} \quad p_i^k \| a-b \qquad (0 \leqq k \leqq n-1).$$

Proof. We put $\gamma(s,\chi) = \pi^{-s}\Gamma(s)L(2s,\bar{\tilde{\chi}})$.

Since $L(s,\chi) = \prod_{\mu<i\leqq\lambda}\left(1-\frac{\tilde{\chi}(p_i)}{p_i^{2s}}\right)L(s,\tilde{\chi})$, we have, from (12), (13) and (14),

$$\gamma(s,\chi)E(\tau,s;\cdot,\chi)\prod_{\mu<i\leqq\lambda}\left(1-\frac{\overline{\tilde{\chi}(p_i)}}{p_i^{2s}}\right)$$

$$= \sum_{k_i=r_i}^{n_i} N^{-1}N' \prod_{k_i=0}\left(1-\frac{\overline{\tilde{\chi}(p_i)}}{p_i^{2s}}\right)(p_i+1)^{-1}\boldsymbol{T}^{(N')}\{\pi^{-s}\Gamma(s)E_{N'}(\tau,s;\cdot,\chi)\}$$

$$= \sum_{k_i}\prod_{k_i=0}\left(1-\frac{\overline{\tilde{\chi}(p_i)}}{p_i^{2s}}\right)\left(1-\frac{\tilde{\chi}(p_i)}{p_i^{2-2s}}\right)^{-1} N'^{-2s}\boldsymbol{A}^{(N')}\boldsymbol{P}_{k_1\cdots k_\lambda}\{\pi^{s-1}\Gamma(1-s)E_N(\tau,1-s;\cdot,\chi)\}$$

Since
$$\frac{\gamma(1-s,\bar{\chi})}{\gamma(s,\chi)} = \frac{\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\frac{L(2s-1,\bar{\tilde{\chi}})}{L(2s,\bar{\tilde{\chi}})} S_\chi \tilde{N}^{2s-2},$$

we have

$$E(\tau,s;\cdot,\chi) = \frac{\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\frac{L(2s-1,\bar{\tilde{\chi}})}{L(2s,\bar{\tilde{\chi}})}\Phi^{(1)}\otimes\cdots\otimes\Phi^{(\lambda)}\ E(\tau,1-s;\cdot,\bar{\chi})$$

where
$$\Phi^{(i)} = \sum_{k=r_i}^{n_i} p_i^{(2s-2)r_i}p_i^{(1-2s)}\boldsymbol{A}^{(k)}\boldsymbol{P}_k \qquad (1\leqq i\leqq\mu)$$

$$= \sum_{k=1}^{n_i} p_i^{(1-2s)k}\left(\frac{1-\tilde{\chi}(p_i)p_i^{-2-2s}}{1-\bar{\tilde{\chi}}(p_i)p_i^{-2s}}\right)\overline{\tilde{\chi}(p_i)}^k\boldsymbol{P}_k + \boldsymbol{P}_0$$

$$(\mu<i\leqq\lambda).$$

By (11) and (15), $\Phi^{(i)}$ can be written as stated in our theorem. (cf. the proof of prop. 1, 2)

Literature

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