PBW THEOREMS AND FROBENIUS STRUCTURES FOR QUANTUM MATRICES

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Abstract. Let $G \in \{\text{Mat}_n(\mathbb{C}), \text{GL}_n(\mathbb{C}), \text{SL}_n(\mathbb{C})\}$, let $\mathcal{O}_q(G)$ be the quantum function algebra – over $\mathbb{Z}[q, q^{-1}]$ – associated to $G$, and let $\mathcal{O}_\ell(G)$ be the specialisation of the latter at a root of unity $\ell$, whose order $\ell$ is odd. There is a quantum Frobenius morphism that embeds $\mathcal{O}(G)$, the function algebra of $G$, in $\mathcal{O}_\ell(G)$ as a central Hopf subalgebra, so that $\mathcal{O}_\ell(G)$ is a module over $\mathcal{O}(G)$. When $G = \text{SL}_n(\mathbb{C})$, it is known by [3], [4] that (the complexification of) such a module is free, with rank $\ell^\dim(G)$. In this note we prove a PBW-like theorem for $\mathcal{O}_q(G)$, and we show that – when $G$ is $\text{Mat}_n$ or $\text{GL}_n$ – it yields explicit bases of $\mathcal{O}_\ell(G)$ over $\mathcal{O}(G)$. As a direct application, we prove that $\mathcal{O}_\ell(\text{GL}_n)$ and $\mathcal{O}_\ell(\text{M}_n)$ are free Frobenius extensions over $\mathcal{O}(\text{GL}_n)$ and $\mathcal{O}(\text{M}_n)$, thus extending some results of [5].

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1. The general setup. Let $G$ be a complex semisimple, connected, simply connected affine algebraic group. One can introduce a quantum function algebra $\mathcal{O}_q(G)$, a Hopf algebra over the ground ring $\mathbb{C}[q, q^{-1}]$, where $q$ is an indeterminate, as in [7]. If $\ell$ is any root of 1, one can specialize $\mathcal{O}_q(G)$ at $q = \ell$, which means taking the Hopf $\mathbb{C}$-algebra $\mathcal{O}_\ell(G) := \mathcal{O}_q(G)/((q - \ell)\mathcal{O}_q(G))$. In particular, for $\ell = 1$ one has $\mathcal{O}_1(G) \cong \mathcal{O}(G)$, the classical (commutative) function algebra over $G$. Moreover, if the order $\ell$ of $\ell$ is odd, then there exists a Hopf algebra monomorphism $\mathfrak{z}: \mathcal{O}(G) \cong \mathcal{O}_1(G) \hookrightarrow \mathcal{O}_\ell(G)$, called quantum Frobenius morphism for $G$, which embeds $\mathcal{O}(G)$ inside $\mathcal{O}_\ell(G)$ as a central Hopf subalgebra. Therefore, $\mathcal{O}_\ell(G)$ is naturally a module over $\mathcal{O}(G)$. It is proved in [4] and in [3] that such a module is free, with rank $\ell^\dim(G)$. In the special case of $G = \text{SL}_2$, a stronger result was given in [8], where an explicit basis was found. We shall give similar results when $G$ is $\text{GL}_n$ or $\text{M}_n := \text{Mat}_n$; namely we provide explicit bases of $\mathcal{O}_\ell(G)$ as a free module over $\mathcal{O}(G)$, where in addition everything is defined replacing $\mathbb{C}$ with $\mathbb{Z}$. The proof is via some (more or less known) PBW theorems for $\mathcal{O}_q(M_n)$ and $\mathcal{O}_q(\text{GL}_n)$ – and $\mathcal{O}_q(\text{SL}_n)$ as well – as modules over $\mathbb{Z}[q, q^{-1}]$.

Let $\text{Mat}_n := \text{Mat}_n(\mathbb{C})$. The algebra $\mathcal{O}(\text{M}_n)$ of regular functions on $\text{M}_n$ is the unital associative commutative $\mathbb{C}$-algebra with generators $\tilde{t}_{i,j} (i, j = 1, \ldots, n)$. The semigroup structure on $\text{M}_n$ yields on $\mathcal{O}(\text{M}_n)$ the natural bialgebra structure given by matrix product – see [6], Ch. 7. We can also consider the semigroup-scheme $(\text{M}_n)_\mathbb{Z}$ associated to $\text{M}_n$, for which a like analysis applies: in particular, its function algebra $\mathcal{O}_\mathbb{Z}(\text{M}_n)$ is a $\mathbb{Z}$-bialgebra, with the same presentation as $\mathcal{O}(\text{M}_n)$ but over the ring $\mathbb{Z}$.
Now we define quantum function algebras. Let $R$ be any commutative ring with unity, and let $q \in R$ be invertible. We define $O^R_q(M_n)$ as the unital associative $R$-algebra with generators $t_{ij}$ $(i, j = 1, \ldots, n)$ and relations

$$
t_{ij}t_{ik} = qt_{ik}t_{ij}, \quad t_{kk}t_{ij} = qt_{kk}t_{ij} \quad \forall \ j < k, i < h, \\
t_{ij}t_{kl} = t_{jk}t_{il}, \quad t_{ii}t_{jk} - t_{ij}t_{ik} = (q - q^{-1}) t_{ij}t_{jk} \quad \forall \ i < j, k < l.
$$

It is known that $O^R_q(M_n)$ is a bialgebra, but we do not need this extra structure in the present work (see [6] for further details – cf. also [1] and [12]).

As to specialisations, set $\mathbb{Z}_q := \mathbb{Z}[q, q^{-1}]$, let $\ell \in \mathbb{N}_+$ be odd, let $\phi_\ell(q)$ be the $\ell$-th cyclotomic polynomial in $q$, and let $\varepsilon := \bar{q} \in \mathbb{Z}_q := \mathbb{Z}_q/(\phi_\ell(q))$, so that $\varepsilon$ is a (formal) primitive $\ell$-th root of 1 in $\mathbb{Z}_\varepsilon$. Then

$$O^\varepsilon_q(M_n) = O^q_q(M_n)/\langle \phi_\ell(q) \rangle O^q_q(M_n) \cong \mathbb{Z}_\varepsilon \otimes \mathbb{Z} O^{q^\varepsilon}_q(M_n).$$

It is also known that there is a bialgebra isomorphism

$$O^{q^\varepsilon}_q(M_n) \cong O^{q^\varepsilon}_q(M_n)/(q - 1)O^{q^\varepsilon}_q(M_n) \hookrightarrow O^\varepsilon_q(M_n), \quad t_{ij} \mod(q - 1)O^{q^\varepsilon}_q(M_n) \mapsto \tau_{ij}
$$

and a bialgebra monomorphism, called quantum Frobenius morphism ($\varepsilon$ and $\ell$ as above),

$$\exists \tau_{\varepsilon}: O^\varepsilon_q(M_n) \cong O^{q^\varepsilon}_q(M_n) \hookrightarrow O^\varepsilon_q(M_n), \quad \tau_{ij} \mapsto t_{ij} \mod(q - 1)^\varepsilon$$

whose image is central in $O^\varepsilon_q(M_n)$. Thus $O^\varepsilon_q(M_n) := \mathbb{Z}_\varepsilon \otimes \mathbb{Z} O^{q^\varepsilon}_q(M_n)$ becomes identified – via $\exists \tau_{\varepsilon}$, which clearly extends to $O^{q^\varepsilon}_q(M_n)$ by scalar extension – with a central subbialgebra of $O^\varepsilon_q(M_n)$, so the latter can be seen as an $O^\varepsilon_q(M_n)$-module. By the result in [4] and [3] mentioned above, we can expect this module to be free, with rank $\ell^n$.

All the previous framework also extends to $GL_n$ and to $SL_n$ instead of $M_n$. Indeed, consider the quantum determinant $D_q := \sum_{\sigma \in S_n} (-q)^{l(\sigma)} t_{1, \sigma(1)} t_{2, \sigma(2)} \cdots t_{n, \sigma(n)} \in O^R_q(M_n)$, where $l(\sigma)$ denotes the length of any permutation $\sigma$ in the symmetric group $S_n$. Then $D_q$ belongs to the centre of $O^R_q(M_n)$, hence one can extend $O^R_q(M_n)$ by a formal inverse to $D_q$, i.e. defining the algebra $O^R_q(GL_n) := O^R_q(M_n)[D_q^{-1}]$. Similarly, we can define also $O^R_q(SL_n) := O^R_q(M_n)/(D_q - 1)$. Now $O^R_q(GL_n)$ and $O^R_q(SL_n)$ are Hopf $R$-algebras, and the maps $O^R_q(M_n) \hookrightarrow O^R_q(GL_n), O^R_q(GL_n) \twoheadrightarrow O^R_q(SL_n), O^R_q(SL_n) \twoheadrightarrow O^R_q(SL_n)$ (the third one being the composition of the first two) given by $t_{ij} \mapsto t_{ij}$ are epimorphisms of $R$-bialgebras, and even of Hopf $R$-algebras in the second case. The specialisations

$$O^\varepsilon_q(GL_n) = O^q_q(GL_n)/\langle \phi_\ell(q) \rangle O^q_q(GL_n) \cong \mathbb{Z}_\varepsilon \otimes \mathbb{Z} O^{q^\varepsilon}_q(GL_n)$$

$$O^\varepsilon_q(SL_n) = O^q_q(SL_n)/\langle \phi_\ell(q) \rangle O^q_q(SL_n) \cong \mathbb{Z}_\varepsilon \otimes \mathbb{Z} O^{q^\varepsilon}_q(SL_n)$$

enjoy the same properties as above, namely there exist isomorphisms $O^\varepsilon_q(GL_n) \cong O^\varepsilon_q(GL_n)$ and $O^\varepsilon_q(SL_n) \cong O^\varepsilon_q(SL_n)$ and there are quantum Frobenius morphisms

$$\exists \tau_{\varepsilon}: O^\varepsilon_q(GL_n) \cong O^\varepsilon_q(GL_n) \hookrightarrow O^\varepsilon_q(GL_n), \quad \exists \tau_{\varepsilon}: O^\varepsilon_q(SL_n) \cong O^\varepsilon_q(SL_n) \hookrightarrow O^\varepsilon_q(SL_n)$$
described by the same formulae as for $M_n$. Moreover, $D_q^{\pm 1}\mod (q - 1) \mapsto D^{\pm 1}$ in the isomorphisms and $D_q^{\pm 1} \cong D_q^{\pm 1}\mod (q - 1) \mapsto D_q^{\pm 1}\mod (q - \varepsilon)$ in the quantum Frobenius morphisms for $GL_n$ (which extend those of $M_n$). In addition, all these isomorphisms and quantum Frobenius morphisms are compatible (in the obvious sense) with the natural maps which link $O_q^Z(M_n)$, $O_q^Z(GL_n)$ and $O_q^Z(SL_n)$, and their specialisations, to each other.

Like for $M_n$, the image of the quantum Frobenius morphisms are central in $O_q^Z(GL_n)$ and in $O_q^Z(SL_n)$. Thus $O_q^Z(GL_n) := \mathbb{Z}_q \otimes \mathbb{Z} O_q^Z(GL_n)$ identifies to a central Hopf subalgebra of $O_q^Z(GL_n)$, and $O_q^Z(SL_n) := \mathbb{Z}_q \otimes \mathbb{Z} O_q^Z(SL_n)$ identifies to a central Hopf subalgebra of $O_q^Z(SL_n)$; so $O_q^Z(GL_n)$ is an $O_q^Z(GL_n)$-module and $O_q^Z(SL_n)$ is an $O_q^Z(SL_n)$-module.

In §2, we shall prove (Theorem 2.1) a PBW-like theorem providing several different bases for $O_q^Z(M_n)$, $O_q^Z(GL_n)$ and $O_q^Z(SL_n)$ as $R$-modules. As an application, we find (Theorem 2.2) explicit bases of $O_q^Z(M_n)$ as an $O_q^Z(M_n)$-module, which then in particular is free of rank $\ell \dim (M_n)$. The same bases are also $O_q^Z(GL_n)$-bases for $O_q^Z(GL_n)$, which then is free of rank $\ell \dim (GL_n)$. Both results can be seen as extensions of some results in [4].

Finally, in §3 we use the above mentioned bases to prove that $O_q^Z(M_n)$ is a free Frobenius extension of its central subalgebra $O_q^Z(M_n)$, and to explicitly compute the associated Nakayama automorphism. The same we do for $O_q^Z(GL_n)$ as well. Everything follows from the ideas and methods in [5], now applied to the explicit bases given by Theorem 2.2.

2. PBW–like theorems.

THEOREM 2.1. (PBW theorem for $O_q^R(M_n)$, $O_q^R(GL_n)$ and $O_q^R(SL_n)$ as $R$-modules) Assume $(q - 1)$ is not invertible in $R_q := (q, q^{-1})$, the subring of $R$ generated by $q$ and $q^{-1}$.

(a) Let any total order be fixed in $\{1, \ldots, n\}^2$. Then the following sets of ordered monomials are $R$-bases of $O_q^R(M_n)$, resp. $O_q^R(GL_n)$, resp. $O_q^R(SL_n)$, as modules over $R$:

$$B_M := \left\{ \prod_{i,j=1}^n t_{i,j}^{N_{i,j}} \left| N_{i,j} \in \mathbb{N} \forall i, j \right. \right\}$$

$$B_{GL}^N := \left\{ \prod_{i,j=1}^n t_{i,j}^{N_{i,j}} D_q^{-N} \left| N, N_{i,j} \in \mathbb{N} \forall i, j; \min (\{N_{i,j}\}_{1 \leq i \leq n} \cup \{N\}) = 0 \right. \right\}$$

$$B_{GL}^Z := \left\{ \prod_{i,j=1}^n t_{i,j}^{N_{i,j}} D_q^{-Z} \left| Z \in \mathbb{Z}, N_{i,j} \in \mathbb{N} \forall i, j; \min (\{N_{i,j}\}_{1 \leq i \leq n} = 0 \right. \right\}$$

$$B_{SL} := \left\{ \prod_{i,j=1}^n t_{i,j}^{N_{i,j}} \left| N_{i,j} \in \mathbb{N} \forall i, j; \min (\{N_{i,j}\}_{1 \leq i \leq n} = 0 \right. \right\}.$$ 

(b) Let $\succeq$ be any total order fixed in $\{1, \ldots, n\}^2$ such that $(i, j) \succeq (h, k) \succeq (l, m)$ whenever $j > n+1-i$, $k = n+1-h$, $m < n+1-l$. Then the following sets of ordered
monomials are $R$-bases of $O_q^R(GL_n)$, resp. $O_q^R(SL_n)$, as modules over $R$:

$$B_{GL}^{\wedge-} := \left\{ \prod_{i,j=1}^n t_{i,j}^{-N} D_q^{-N} \middle| N, N_{i,j} \in \mathbb{N} \forall i, j; \min\{\{N_{i,n+1-i}\}_{1 \leq i \leq n} \cup \{N\}\} = 0 \right\}$$

$$B_{GL}^{\vee-} := \left\{ \prod_{i,j=1}^n t_{i,j}^{-N} D_q^{-Z} \middle| Z \in \mathbb{Z}, N_{i,j} \in \mathbb{N} \forall i, j; \min\{\{N_{i,n+1-i}\}_{1 \leq i \leq n} = 0 \right\}$$

$$B_{SL} := \left\{ \prod_{i,j=1}^n t_{i,j}^{-N} N_{i,j} \in \mathbb{N} \forall i, j; \min\{\{N_{i,n+1-i}\}_{1 \leq i \leq n} = 0 \right\}.$$  

**Proof.** Roughly speaking, our method is a (partial) application of the diamond lemma (see [2]): however, we do not follow it in all details, as we use a specialisation trick as a shortcut.

If we prove our results for the algebras defined over $R_q$ instead of $R$, then the same results will hold as well by scalar extension. Thus we can assume $R = R_q$, and then we note that, by our assumption, the specialised ring $\mathcal{R} := R/(q-1)R \neq \{0\}$ is non-trivial.

**Proof of (a):** (see also [10], Theorem 3.1, and [12], Theorem 3.5.1)

We begin with $O_q^R(M_n)$. It is clearly spanned over $R$ by the set of all (possibly unordered) monomials in the $t_{ij}$'s: so we must only prove that any such monomial belongs to the $R$-span of the ordered monomials. In fact, the latter are linearly independent, since such are their images via specialisation $O_q^R(M_n) \longrightarrow O_q^R(M_n)/(q-1)O_q^R(M_n) \cong \overline{O_q^R(M_n)}$.

Thus, take any (possibly unordered) monomial in the $t_{ij}$'s, say $t := t_{i_1,j_1} t_{i_2,j_2} \cdots t_{i_k,j_k}$, where $k$ is the degree of $t$: we associate to it its weight, defined as

$$w(t) := (k, d_1, d_2, \ldots, d_{n,1}, d_{n,1}, d_{n,2}, \ldots, d_{2,2}, d_{3,1}, \ldots, d_{n-1,1}, d_{n,1}, d_{n,2}, \ldots, d_{n,n})$$

where $d_{i,j} := |\{s \in \{1, \ldots, k\} | (i_s, j_s) = (i, j)\}|$ = number of occurrences of $t_{i,j}$ in $t$. Then $w(t) \in \mathbb{N}^{n^2+1}$, and we consider $\mathbb{N}^{n^2+1}$ as a totally ordered set with respect to the (total) lexicographic order $\leq_{\text{lex}}$. By a quick look at the defining relations of $O_q^R(M_n)$, namely

$$t_{i,j} t_{i,k} = q t_{i,k} t_{i,j}, \quad t_{i,k} t_{h,k} = q t_{h,k} t_{i,k} \quad \forall \quad j < k, i < h,$$

$$t_{i,l} t_{j,k} = t_{j,k} t_{i,l}, \quad t_{i,k} t_{j,l} - t_{j,i} t_{i,k} = (q - q^{-1}) t_{i,l} t_{j,k} \quad \forall \quad i < j, k < l.$$  

one easily sees that the weight defines an algebra filtration on $O_q^R(M_n)$.

Now, using these same relations, one can re-order the $t_{ij}$'s in any monomial according to the fixed total order. During this process, only two non-trivial things may occur, namely:

-1) some powers of $q$ show up as coefficients (when a relation in the first line is employed);

-2) a new summand is added (when the bottom-right relation is used);

If only steps of type 1) occur, then the process eventually stops with an ordered monomial in the $t_{ij}$'s multiplied by a power of $q$. Whenever instead a step of type 2) occurs, the newly added term is just a coefficient $(q - q^{-1})$ times a (possibly unordered) monomial in the $t_{ij}$'s, call it $t'$: however, by construction $w(t') \leq_{\text{lex}} w(t)$. Then, by induction on the weight, we can assume that $t'$ lies in the $R$-span of the ordered
monomials, so we can ignore the new summand. The process stops in finitely many steps, and we are done with \( \Omega_q^R(M_n) \).

Second, we look at \( \Omega_q^R(GL_n) \). Let us consider \( f \in \Omega_q^R(GL_n) \). By definition, there exists \( N \in \mathbb{N} \) such that \( fD_q^N \in \Omega_q^R(M_n) \); therefore, by the result for \( \Omega_q^R(M_n) \) just proved, we can expand \( fD_q^N \) as an \( R \)-linear combination of ordered monomials, call them \( t = \prod_{i,j=1}^n t_{i,j}^{N_{i,j}} \). Thus, \( f \) itself is an \( R \)-linear combination of monomials \( tD_q^{-N} \), so the latter span \( \Omega_q^R(GL_n) \).

Now consider an ordered monomial \( t = \prod_{i,j=1}^n t_{i,j}^{N_{i,j}} \) in which \( N_{i,j} > 0 \) for all \( i \). Then we can re-arrange the \( t_{i,j}'s \) in \( t \) so to single out a factor \( t_{1,1}t_{1,2} \cdots t_{n-1,n-1}t_{n,n} \), up to “paying the cost” (perhaps) of producing some new summands of lower weight: the outcome reads

\[
\ell = q^s t_0 t_{1,1} t_{1,2} \cdots t_{n-1,n-1} t_{n,n} + \text{l.t.'s} \tag{2.1}
\]

for some \( s \in \mathbb{Z} \), with \( t_0 := \prod_{i,j=1}^n t_{i,j}^{-N_{i,j}} \) having lower weight than \( t \), and the expression l.t.’s standing for an \( R \)-linear combination of some monomials \( \ell \) such that \( w(\ell) \preceq_{\text{lex}} w(t) \). Then we re-write the monomial \( t_{1,1} t_{1,2} \cdots t_{n-1,n-1} t_{n,n} \) using the identity

\[
t_{1,1} t_{1,2} \cdots t_{n-1,n-1} t_{n,n} = D_q - \sum_{\sigma \in S_n, \sigma \neq \text{id}} (-q)^{l(\sigma)} t_{1,\sigma(1)} t_{2,\sigma(2)} \cdots t_{n,\sigma(n)} = D_q + \text{l.t.'s} \tag{2.2}
\]

and we replace the right-hand side of (2.2) inside (2.1). We get \( \ell = q^s t_0 D_q + \text{l.t.'s} \) (for \( D_q \) is central!), where now \( t_0 \) and all monomials within l.t.’s have strictly lower weight than \( t \).

If we look now at \( tD_q^z \) (for some \( z \in \mathbb{Z} \)), we can re-write \( t \) as above, thus getting

\[
tD_q^z = q^s t_0 D_q D_q^z + \text{l.t.'s} = q^s t_0 D_q^{z+1} + \text{l.t.'s} \tag{2.3}
\]

where l.t.’s is an \( R \)-linear combination of monomials \( tD_q^{z+1} \) such that \( w(\ell) \preceq_{\text{lex}} w(t) \).

By repeated use of (2.3) as a reduction argument, we can easily show – by induction on the weight – that any monomial of type \( tD_q^{-N} \) \((N \in \mathbb{N})\) can be expanded as an \( R \)-linear combination of elements of \( B_{GL}^c \) or elements of \( B_{GL}^\vee \). Thus, both these sets do span \( \Omega_q^R(GL_n) \).

To finish with, both \( B_{GL}^c \) and \( B_{GL}^\vee \) are \( R \)-linearly independent, as their image through the specialisation epimorphism \( \Omega_q^R(GL_n) \to \Omega^R(GL_n) \cong \Omega^R(GL_n) \) are \( R \)-bases of \( \Omega^R(GL_n) \).

As to \( \Omega_q^R(SL_n) \), we can repeat the argument for \( \Omega_q^R(GL_n) \). First, \( B_{SL} \) is linearly independent, for its image through specialisation \( \Omega_q^R(SL_n) \to \Omega^R(SL_n) \cong \Omega^R(SL_n) \) is an \( R \)-basis of \( \Omega^R(SL_n) \). Second, the epimorphism \( \Omega_q^R(M_n) \to \Omega_q^R(SL_n)(t_{1,j} \mapsto t_{i,j}) \), and the result for \( \Omega_q^R(M_n) \), imply that the \( R \)-span of \( S_{SL} := \{ \prod_{i,j=1}^n t_{i,j}^{N_{i,j}} \mid N_{i,j} \in \mathbb{N} \forall i,j \} \) is \( \Omega_q^R(SL_n) \). Thus one is only left to prove that each monomial \( \ell = \prod_{i,j=1}^n t_{i,j}^{N_{i,j}} \in S_{SL} \) belongs to the \( R \)-span of \( B_{SL} \): as before, this can be done by induction on the weight, using the reduction formula \( \ell = q^s t_0 D_q + \text{l.t.'s} \) (see above), and plugging into the relation \( D_q = 1 \).
Alternatively, we recall there is an isomorphism $O_q^R(SL_n) \otimes_R R[x, x^{-1}] \cong O_q^R(GL_n)$ (of $R$-algebras) given by $t_{ij} \otimes x^2 \mapsto D_q^{-b_{i,j}} \cdot D_q^b$ (cf. [11]). This along with the result about $B_{GL}^\omega$ clearly implies that also $B_{SL}^\omega$ is an $R$-basis for $O_q^R(SL_n)$, as claimed.

**Proof of (b):** First look at $O_q^R(GL_n)$. If $f \in O_q^R(GL_n)$, as in the proof of (a) we expand $f \tilde{D}_q^N$ as an $R$-linear combination of ordered (according to $\preceq$) monomials of type $\tilde{t} = \tilde{t}^- \tilde{t}^+$, with $\tilde{t}^- := \prod_{j=n+1-1} t_{ij}^{N_{ij}}$, $\tilde{t}^+ := \prod_{j=1} t_{ij}^{N_{ij}}$ and $\tilde{t}^+ := \prod_{j<n+1-1} t_{ij}^{N_{ij}}$. So $f$ is an $R$-linear combination of monomials $\tilde{t}^- \tilde{t}^+ D_q^N$, hence the latter span $O_q^R(GL_n)$.

We show that each (ordered) monomial $\tilde{t}^- \tilde{t}^+ D_q^N$ belongs both to the $R$-span of $B_{GL}^\omega$ and of $B_{GL}^\vee$, by induction on the (total) degree of the monomial $\tilde{t}^-$. The basis of induction is deg$(\tilde{t}^-) = 0$, so that $\tilde{t}^- = 1$ and $\tilde{t}^- \tilde{t}^+ D_q^N = \tilde{t}^- \tilde{t}^+ D_q^N \in B_{GL}^\omega \cap B_{GL}^\vee$.

As a matter of notation, let $\mathcal{N}^-$, resp. $\mathcal{H}$, resp. $\mathcal{N}^+$, be the $R$-subalgebra of $O_q^R(M_n)$ generated by the $t_{ij}$'s with $j > n+1-i$, resp. $j = n+1-i$, resp. $j < n+1+i$. Note that $\mathcal{H}$ is Abelian, and $\tilde{t}^- \in \mathcal{H}$, $\tilde{t}^+ \in \mathcal{N}^+$. Now assume that all the exponents $N_{i,n+1-i}$'s in the factor $\tilde{t}^-$ are strictly positive. As $\mathcal{H}$ is Abelian, we can draw out of $\tilde{t}^-$ (even out of $\tilde{t} = \tilde{t}^- \tilde{t}^+$) a factor $t_{n1} t_{n2} \cdots t_{22} t_{11}$. Now recall that $D_q$ can be expanded as $D_q = \sum_{\sigma \in S_n} (-q)^{\ell(\sigma)} t_{n,\sigma(1)} t_{n-1,\sigma(n-1)} \cdots t_{2,\sigma(2)} t_{1,\sigma(1)}$ (see, e.g., [12] or [10]). Then we can re-write the monomial $t_{n1} t_{n2} \cdots t_{22} t_{11}$ as

$$t_{n1} t_{n2} \cdots t_{22} t_{11} = (-q)^{-\ell(\sigma_0)} D_q - \sum_{\sigma \in S_n, \sigma \neq \sigma_0} (-q)^{\ell(\sigma) - \ell(\sigma_0)} t_{n,\sigma(1)} t_{n-1,\sigma(n-1)} \cdots t_{1,\sigma(1)} \quad (2.4)$$

where $\sigma_0 \in S_n$ is the permutation $i \mapsto (n+1-i)$. Note also that we can reorder the factors in the summands of (2.4) so that all factors $t_{ij}$ from $\mathcal{N}^-$ are on the left of those from $\mathcal{N}^+$.

Now we replace the right-hand side of (2.4) in the factor $\tilde{t}^-$ within $\tilde{t} = \tilde{t}^- \tilde{t}^+$, thus

$$\tilde{t}^- \tilde{t}^+ = (-q)^{-\ell(\sigma_0)} \tilde{t}_0^- D_q \tilde{t}_0^+ + l.t.'s = (-q)^{-\ell(\sigma_0)} \tilde{t}_0^- \tilde{t}_0^+ D_q + l.t.'s.$$

Here $\tilde{t}_0^- := \tilde{t}_0^{-} (t_{n1} t_{n2} \cdots t_{22} t_{11})^{-1}$ has lower (total) degree than $\tilde{t}^-$, and the expression $l.t.'s$ stands for an $R$-linear combination of some other monomials $\tilde{t}^- \tilde{t}^+$ (like $\tilde{t}^- \tilde{t}^+$ above) in which again the degree of $\tilde{t}^-$ is lower than the degree of $\tilde{t}^-$. In fact, this holds because when any factor $t_{i,\sigma(i)} \in \mathcal{N}^-$ is pulled from the right to the left of any monomial in $\tilde{t}^- \in \mathcal{H}$ the degree of $\tilde{t}^-$ is not increased. By induction on this degree, we can easily conclude that every ordered monomial $\tilde{t}^- \tilde{t}^+ D_q^N$ (with $z \in \mathbb{Z}$) belongs to both the $R$-span of $B_{GL}^\omega$ and the $R$-span of $B_{GL}^\vee$. That is, both sets span $O_q^R(GL_n)$.

Eventually, both $B_{GL}^\omega$ and $B_{GL}^\vee$ are linearly independent, as their image through the specialisation epimorphism $O_q^R(GL_n) \longrightarrow O_1^R(GL_n) \cong \mathcal{R}^R(GL_n)$ are $\mathcal{R}$-bases of $O_q^R(GL_n)$.

Second, we look at $O_q^R(SL_n)$. As for claim (a), we can repeat again – mutatis mutandis – the argument for $O_q^R(GL_n)$, which does work again – one only has to plug in the additional relation $D_q = 1$ too. Otherwise, as an alternative proof, we can note that the isomorphism $O_q^R(SL_n) \otimes_R R[x, x^{-1}] \cong O_q^R(GL_n)$ together with the result about $B_{GL}^\omega$ easily implies that $B_{SL}^\omega$ too is an $R$-basis for $O_q^R(SL_n)$, q.e.d. □
Remark 2.2. (1) Claim (a) of Theorem 2.1 for $M_n$ only was independently proved in [12] and in [10], but taking a field as ground ring. In [10], claim (b) for $GL_n$ only was proved as well. Similarly, the analogue of claim (b) for $SL_n$ only was proved in [9], §7, but taking as ground ring the field $k(q)$ – for any field $k$ of zero characteristic. Our proof then provides an alternative, unifying approach, which yields stronger results over $R$.

(2) We would better point out a special aspect of the basic assumption of Theorem 2.1 about $q$ and $R$. Namely, if the subring (1) of $R$ generated by 1 has prime characteristic (hence it is a finite field) then the condition on $(q - 1)$ is equivalent to $q$ being trascendental over $R_q$ or $q = 1$. But if instead the characteristic of (1) is zero or positive non-prime, then $(q - 1)$ might be non-invertible in $R_q$ even though $q$ is algebraic (or even integral) over (1).

The end of the story is that Theorem 2.1 holds true in the “standard” case of trascendental values of $q$, but also in more general situations.

(3) The argument used in the proof of Theorem 2.1 to get the result for $O^R_q(SL_n)$ from those for $O^R_q(GL_n)$, via the isomorphism $O^R_q(SL_n) \otimes_R R[x, x^{-1}] \cong O^R_q(GL_n)$, actually works both ways. Therefore, one can also prove the results directly for $O^R_q(SL_n)$ – as we have sketched above – and from them deduce those for $O^R_q(GL_n)$. Even more, as we have proved independently the results for $O^R_q(GL_n)$ – i.e., $B^\vee_{GL}$ and $B^{\vee -}_{GL}$ are $R$-bases – and for $O^R_q(SL_n)$ – i.e., $B_{SL}$ and $B^-_{SL}$ are $R$-bases – we can use them to prove that the algebra morphism $O^R_q(SL_n) \otimes_R R[x, x^{-1}] \longrightarrow O^R_q(GL_n)$ is in fact bijective.

(4) The orders considered in claim (b) of Theorem 2.1 refer to a triangular decomposition of $O^R_q(GL_n)$ and $O^R_q(SL_n)$ which is opposite to the standard one. This opposite decomposition was introduced – and its importance was especially pointed out – in [10].

We are now ready to state and prove the main result of this paper:

Theorem 2.3. (PBW theorem for $O^G_q(G)$ as an $O^G_q(G)$-module, for $G \in \{M_n, GL_n\}$)

Let any total order be fixed in $\{1, \ldots, n\} \times \mathbb{Z}$. Then the set of ordered monomials

$$B^M_{GL} := \left\{ \prod_{i,j=1}^n t_{i,j}^{N_{ij}} \middle| 0 \leq N_{i,j} \leq \ell - 1, \forall i, j \right\}$$

thought of as a subset of $O^Z_q(M_n) \subset O^Z_q(GL_n)$, is a basis of $O^Z_q(M_n)$ as a module over $O^Z_q(M_n)$, and a basis of $O^Z_q(GL_n)$ as a module over $O^Z_q(GL_n)$.

In particular, both modules are free of rank $\ell^{\text{dim}(G)}$, with $G \in \{M_n, GL_n\}$.

Proof. When specialising, Theorem 2.1 (a) implies that $O^Z_q(M_n)$ is a free $\mathbb{Z}$-module with $B_M|_{q=\epsilon} = (\prod_{i,j=1}^n t_{i,j}^{N_{ij}} | N_{i,j} \in \mathbb{N} \forall i, j)$ as basis – where, by abuse of notation, we write again $t_{i,j}$ for $t_{i,j}|_{q=\epsilon}$.

Now, whenever the exponent $N_{i,j}$ is a multiple of $\ell$, the power $t_{i,j}^{N_{ij}}$ belongs to the isomorphic image $\mathfrak{S}_x^G(O^Z_q(M_n))$ of $O^Z_q(M_n)$ inside $O^Z_q(M_n)$, hence it is a scalar for the $O^Z_q(M_n)$-module structure of $O^Z_q(M_n)$.

Therefore, reducing all exponents modulo $\ell$ we find that $B^G_{GL}$ is a spanning set for the $O^Z_q(M_n)$-module $O^Z_q(M_n)$. In addition, $O^Z_q(M_n)$ clearly admits as $\mathbb{Z}$-basis the set $B_M = (\prod_{i,j=1}^n t_{i,j}^{N_{ij}} | N_{i,j} \in \mathbb{N} \forall i, j)$. It follows that $\mathfrak{S}_x^G(B_M) = (\prod_{i,j=1}^n t_{i,j}^{N_{ij}} | N_{i,j} \in \mathbb{N} \forall i, j)$ is a $\mathbb{Z}$-basis of $\mathfrak{S}_x^G(O^Z_q(M_n))$. This last fact easily implies that $B^M_{GL}$ is also $O^Z_q(M_n)$-linearly independent, hence it is a basis of $O^Z_q(M_n)$ over $O^Z_q(M_n)$ as claimed.
As to $\mathcal{O}_e^Z(GL_n)$, from definitions and the analysis in §1 we get (with $D_e := D_q |_e$)

$$\mathcal{O}_e^Z(GL_n) = \mathcal{O}_e^Z(M_n)[D_e^{-1}] = \mathcal{O}_e^Z(M_n)[D_e^{-\ell}]$$

$$= \mathcal{O}_e^Z(M_n)[D^{-1}] \otimes_{\mathcal{O}_e^Z(M_n)} \mathcal{O}_e^Z(M_n) \otimes_{\mathcal{O}_e^Z(M_n)} \mathcal{O}_e^Z(M_n)$$

thus the result for $\mathcal{O}_e^Z(GL_n)$ follows at once from that for $\mathcal{O}_e^Z(M_n)$.

$\square$

3. Frobenius structures.

3.1 Frobenius extensions and Nakayama automorphisms. Following [5], we say that a ring $R$ is a free Frobenius extension over a subring $S$, if $R$ is a free $S$-module of finite rank, and there is an isomorphism $F: R \rightarrow \text{Hom}_S(R, S)$ of $R - S$-bi-modules. Then $F$ provides a non-degenerate associative $S$-bilinear form $\mathbb{B}: R \times R \rightarrow S$, via $\mathbb{B}(r, t) = F(t)(r)$. Conversely, one can characterise Frobenius extensions using such forms. When $S = \mathcal{Z}$ is contained in the centre of $R$, there is a $\mathcal{Z}$-algebra automorphism $\nu: R \rightarrow R$, given by $r F(1) = F(1) \nu(r)$ (for all $r \in R$), and such $\mathbb{B}(x, y) = \mathbb{B}(\nu(y), x)$. This is called the Nakayama automorphism, and it is uniquely determined by the pair $\mathcal{Z} \subseteq R$, up to Int$(R)$.

**Proposition 3.2.** (cf. [5], §2)

Let $R$ be a ring, $\mathcal{Z}$ an affine central subalgebra of $R$. Assume that $R$ is free of finite rank as a $\mathcal{Z}$-module, with a $\mathcal{Z}$-basis $\mathcal{B}$ that satisfies the following condition: there exists a $\mathcal{Z}$-linear functional $\Phi: R \rightarrow \mathcal{Z}$ such that for any non-zero $a = \sum_{b \in \mathcal{B}} z_b b \in R$ there exists $x \in R$ for which $\Phi(xa) = uz_b$ for some unit $u \in \mathcal{Z}$ and some non-zero $z_b \in \mathcal{Z}$.

Then $R$ is a free Frobenius extension of $\mathcal{Z}$. Moreover, for any maximal ideal $\mathfrak{m}$ of $\mathcal{Z}$, the finite dimensional quotient $R/\mathfrak{m}R$ is a finite dimensional Frobenius algebra.

This result is used in [5] to show that many families of algebras – in particular, some related to $\mathcal{O}_e(G)$, where $G$ is a (complex, connected, simply-connected) semisimple affine algebraic group – are indeed free Frobenius extensions. But the authors could not prove the same for $\mathcal{O}_e(G)$, as they did not know an explicit $\mathcal{O}(G)$-basis of $\mathcal{O}_e(G)$. Now, following their strategy and using Theorem 2.3, I shall now prove that $\mathcal{O}_e^Z(G)$ is free Frobenius over $\mathcal{O}_e^Z(G)$ when $G$ is $M_n$ or $GL_n$.

**Theorem 3.3.** Let $G$ be $M_n$ or $GL_n$. Then $\mathcal{O}_e^Z(G)$ is a free Frobenius extension of $\mathcal{O}_e^Z(G)$, with Nakayama automorphism $\nu$ given by $\nu(t_{ij}) = e^{2(i+j-n-1)} t_{ij}$ ($i, j = 1, \ldots, n$).

**Proof.** We prove that there is a suitable $\mathcal{O}_e^Z(G)$-linear functional $\Phi: \mathcal{O}_e^Z(G) \rightarrow \mathcal{O}_e^Z(G)$ as required in Proposition 3.2, so that this result applies to $R := \mathcal{O}_e^Z(G)$ and $\mathcal{Z} := \mathcal{O}_e^Z(G)$.

Define $\Phi$ on the elements of the $\mathcal{O}_e^Z(G)$-basis $\mathcal{B}_{GL}^M$ of $\mathcal{O}_e^Z(G)$ (see Theorem 2.3) by

$$\Phi \left( \prod_{i,j=1}^n t_{ij}^{N_{ij}} \right) := \prod_{i,j=1}^n \delta_{N_{ij}, \ell-1} = \begin{cases} 1, & \text{if } N_{ij} = \ell - 1 \forall i, j \\ 0, & \text{if not} \end{cases}$$

(3.1)

(for all $0 \leq N_{ij} \leq \ell - 1$), and extend to all of $\mathcal{O}_e^Z(G)$ by $\mathcal{O}_e^Z(G)$-linearity. In other words, $\Phi$ is the unique $\mathcal{O}_e^Z(G)$-valued linear functional on $\mathcal{O}_e^Z(G)$ whose value is 1 on
the basis element \( t_{ij}^{\ell-1} := \prod_{i,j=1}^{n} t_{ij}^{\ell-1} \) and is zero on all other elements of the \( \mathcal{O}^{Z_i}(G) \)-basis \( B^{M}_{GL} \).

We claim that \( \Phi \) satisfies the assumptions of Proposition 3.2, so the latter applies and proves our statement. Indeed, let us consider any non-zero \( a = \sum_{t \in B^{M}_{GL}} z_{t} \in \mathcal{O}^{Z_i}(G) \), and let \( t_{0} = \prod_{i,j=1}^{n} t_{ij}^{N_{ij}} \) in \( B^{M}_{GL} \) be such that \( z_{t_{0}} \neq 0 \) and \( w(t_{0}) \) is maximal (w.r.t. \( \leq_{\text{lex}} \)). Then define \( \tilde{t}_{0} := \prod_{i,j=1}^{n} t_{ij}^{N'_{ij}} \in B^{M}_{GL} \) with \( N'_{ij} := \ell - 1 - N_{ij} \) for all \( i,j = 1, \ldots, n \). Quoting from the proof of Theorem 2.1(a), we know that \( \tilde{t}_{0} t_{0} = e^{s_{t_{0}}} t_{0} + l.t.'s \), where \( s \in \mathbb{Z} \) and the expression \( l.t.'s \) now stands for an \( \mathcal{O}^{Z}(G) \)-linear combination of monomials \( t \in B^{M}_{GL} \) such that \( w(t) \leq_{\text{lex}} w(t_{0}) \); in particular, \( \Phi(t) = 0 \) for all these \( \tilde{t} \), hence eventually \( \Phi(t_{0} t_{0}') = e^{s} \Phi(t_{0}^{\ell-1}) = e^{s} \). Similarly, if \( t' \in B^{M}_{GL} \) is such that \( w(t') <_{\text{lex}} w(t) \), then \( t_{0} t' \) is an \( \mathcal{O}^{Z}(G) \)-linear combination of PBW monomials whose weight is at most \( w(t_{0} t') \), hence \( \Phi(t_{0} t') = 0 \). As we chose \( t_{0} \) so that \( w(t_{0}) \) is maximal, we eventually find

\[
\Phi(t_{0} a) = \sum_{t \in B^{M}_{GL}} z_{t} \Phi(t) = z_{t_{0}} \Phi(t_{0}) = e^{s} z_{t_{0}}
\]

where \( e^{s} \) is a unit in \( \mathcal{O}^{Z}(G) \). So \( \Phi \) satisfies the assumptions of Proposition 3.2, as claimed.

As to the Nakayama automorphism \( \nu: \mathcal{O}^{Z}(G) \rightarrow \mathcal{O}^{Z}(G) \), it is characterized (see § 3.1) by the property that \( B(x, y) = B(\nu(y), x) \) for all \( x, y \in \mathcal{R} \). Here \( B \) is a \( \mathbb{Z} \)-bilinear form as in § 3.1, which now is related to \( \Phi \) by the formula \( B(x, y) = \Phi(xy) \) for all \( x, y \in \mathcal{R} \).

As \( \Phi \) is an automorphism, and \( \mathcal{O}^{Z}(G) \) is generated – over \( \mathcal{O}^{Z}(G) \) – by the \( t_{ij} \)'s, the claim about \( \nu \) is proved if we show that

\[
\Phi \left( \prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}} \cdot t_{ij} \right) = \Phi \left( e^{2(i+j-n-1)} t_{ij} \cdot \prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}} \right). \tag{3.2}
\]

Now, our usual argument shows that the expansions of the product of a generator \( t_{ij} \) and a PBW monomial \( \prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}} \) (in either order of the factors) as an \( \mathcal{O}^{Z}(G) \)-linear combination of elements of the \( \mathcal{O}^{Z}(G) \)-basis \( B^{M}_{GL} \) are of the form

\[
\prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}} \cdot t_{ij} = e^{i+j-2n} \prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}+\delta_{r,j}\delta_{s,i}} + l.t.'s
\]

\[
t_{ij} \cdot \prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}} = e^{2(i+j-n)} \prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}+\delta_{r,j}\delta_{s,i}} + l.t.'s.
\]

This along with (3.1) gives

\[
\Phi \left( \prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}} \cdot t_{ij} \right) = e^{i+j-2n} \Phi \left( \prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}+\delta_{r,j}\delta_{s,i}} \right) = e^{i+j-2n} \text{ if } e_{r,s} = \ell - 1 - \delta_{r,j}\delta_{s,i}
\]

\[
\Phi \left( \prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}} \cdot t_{ij} \right) = e^{i+j-2n} \Phi \left( \prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}+\delta_{r,j}\delta_{s,i}} \right) = 0 \text{ if not}
\]
and similarly

$$
\Phi \left( \prod_{r,s=1}^{n} t_{r,s}^{e_{i,j}+\delta_{i,j}} \right) = \varepsilon^{2-i-j} \Phi \left( \prod_{r,s=1}^{n} t_{r,s}^{e_{i,j}} \right) = \varepsilon^{2-i-j} \quad \text{if } e_{r,s} = \ell - 1 - \delta_{r,i} \delta_{s,j},
$$

$$
\Phi \left( \prod_{r,s=1}^{n} t_{r,s}^{e_{i,j}} \right) = 0 \quad \text{if not}.
$$

Direct comparison now shows that (3.2) holds, q.e.d.

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**References**