A NEW APPROACH TO JACOBI'S THEOREMS
VIA RAMANUJAN'S CONTINUED FRACTIONS

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In this paper, we show that Jacobi's two-square and two-triangular number theorems are immediate consequence of Ramanujan's continued fraction identities.

1. INTRODUCTION

Let \( r_k(n) \) denote the number of ways the positive integer \( n \) can be represented as a sum of \( k \) squares, with representations arising from different signs and from different orders being regarded as distinct. Geometrically, \( r_k(n) \) counts the number of lattice points on the \( k \)-dimensional sphere \( x_1^2 + \cdots + x_k^2 = n \). Let

\[
\phi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, |q| < 1.
\]

Then the generating function for \( r_k(n) \) is

\[
\phi^k(q) = \sum_{n=0}^{\infty} r_k(n)q^n.
\]

A triangular number is a number of the form \( k(k+1)/2 \) for some non-negative integer \( k \). Let \( t_k(n) \) denote the number of representations of \( n \) as a sum of \( k \) triangular numbers. Geometrically, \( 2^k t_k(n) \) counts the number of lattice points on the \( k \)-dimensional sphere centred at \((-1/2, \ldots, -1/2)\) with radius \( \sqrt{2n + k/4} \). Let

\[
\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}, |q| < 1.
\]

Then the generating function for \( t_k(n) \) is

\[
\psi^k(q) = \sum_{n=0}^{\infty} t_k(n)q^n.
\]
One of the main problems is to find formulae for determining $r_k(n)$, in terms of simple arithmetical functions such as divisor functions. Jacobi’s two-square and two-triangular-number theorems state that

\begin{align}
  r_2(n) &= 4\left[d_1(n) - d_3(n)\right], \\
  t_2(n) &= d_1(4n+1) - d_3(4n+1).
\end{align}

where $d_i(m)$ is the number of divisors of $m$ congruent to $i$ modulo $4$. Many authors gave proofs of (1.1) and (1.2) using $q$-series identities. Hirschhorn [8] and Ewell [6] have used Jacobi’s triple product identity to obtain (1.1). Askey [3], Adiga [1], Bhargava and Adiga [4], Fine [7] and, Cooper and Lam [5] have employed Ramanujan’s $\psi_1$ summation formula to derive (1.1) and (1.2). We show that (1.1) and (1.2) are immediate consequences of two continued fraction identities of Ramanujan.

Ramanujan’s general theta function and its special cases are defined as follows:

\begin{align}
  f(a, b) &:= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, |ab| < 1, \\
  \phi(q) &:= f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q : -q)_\infty}{(q : -q)_\infty}, \\
  \psi(q) &:= f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2 : q^2)_\infty}{(q : q^2)_\infty},
\end{align}

where $(a; q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n)$.

$\phi(q)$ and $\psi(q)$ have been used for a long time in the theory of elliptic functions and modular forms. The main purpose of this note is to establish continued fractions for $\phi^2(q)$ and $\psi^2(q^3)$, and to employ them to derive Jacobi’s two-square and two-triangular number theorems.

2. Main Results

**Theorem 1.** If $|q| < 1$, then

\begin{equation}
  1 + \phi^2(q) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots,
\end{equation}

where

\begin{align*}
  a_1 &= 2, a_2 = -2q, a_3 = (q - q^2)^2, \ldots, \\
  a_n &= (-1)^{n-3} q^{-3} (q + (-1)^{n-2} q^{-1})^2, n \geq 3, \text{ and} \\
  b_1 &= 1, b_2 = 1 + q, b_3 = 1 + q^3, \ldots, b_n = 1 + q^{2n-3}, n \geq 3.
\end{align*}
PROOF: From ([2, p. 14, Entry 11]), we have
\[
\frac{(-a)_\infty (b)_\infty - (a)_\infty (-b)_\infty}{(-a)_\infty (b)_\infty + (a)_\infty (-b)_\infty} = \frac{a - b}{1 - q} + \frac{(a - bq)(aq - b)}{1 - q^2} + \frac{q(a - bq^2)(aq^2 - b)}{1 - q^5} + \cdots .
\]
This continued fraction identity may be rewritten as
\[
1 + \frac{(-a)_\infty (b)_\infty}{(a)_\infty (-b)_\infty} = \frac{2a - b}{1 - 1 - q} + \frac{(a - bq)(aq - b)}{1 - q^3} + \frac{q(a - bq^2)(aq^2 - b)}{1 - q^5} + \cdots .
\]
Changing \(q\) to \(-q\), \(a\) to \(q\) and \(b\) to \(-q\) in the above continued fraction identity and using (1.4), we complete the proof. □

**Theorem 2.** If \(|q| < 1\), then
\begin{equation}
2 + 4 \sum_{n=1}^{\infty} \frac{(-1)^{n(n-1)/2}q^n}{1 + (-q)^n} = \frac{a_1 + a_2 + a_3 + \cdots}{b_1 + b_2 + b_3 + \cdots},
\end{equation}
where \(a_n\) and \(b_n\) are as defined in Theorem 1.

**Proof:** Let \(P_n/Q_n\) denote the \(n\)-th convergent of the continued fraction
\[
a_1/b_1 + a_2/b_2 + a_3/b_3 + \cdots .
\]
Then
\[
Q_1 = 1, \quad Q_2 = (1 - q), \quad Q_3 = (1 + q^3)(1 - q) + (q - q^3)^2 = (1 - q)(1 + q^2)
\]
and
\[
Q_n = b_nQ_{n-1} + a_nQ_{n-2}.
\]
By induction on \(n\), we can show that
\[
Q_n = \prod_{k=1}^{n-1} (1 + (-1)^kq^k), \quad n \geq 2.
\]
We have
\[
P_n/Q_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}a_1a_2\ldots a_k}{Q_1Q_2\ldots Q_kQ_{k-1}}
= \frac{a_1Q_0}{Q_1Q_0} - \frac{a_1a_2}{Q_2Q_1} + \frac{a_1a_2a_3}{Q_3Q_2} - \cdots + \frac{(-1)^{n+1}a_1a_2a_3\ldots a_n}{Q_nQ_{n-1}}
= \frac{2}{1 + 4q - \frac{4q^3}{1 + q^2} - \frac{4q^6}{1 - q^3} + \frac{4q^{10}}{1 + q^4} + \cdots + 4 (-1)^{(n-1)(n-2)/2}q^n(n-1)/2}{1 + (-q)^{n-1}}.
\]
Letting \(n \to \infty\), we obtain the required result. □

**Theorem 3.** Jacobi's two-square theorem.
\[
r_2(n) = 4 [d_1(n) - d_3(n)].
\]

**Proof:** From (2.1) and (2.2), we have
\begin{equation}
\phi^2(q) = 1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^{n(n-1)/2}q^n(n+1)/2}{1 + (-q)^n}
= 1 + 4 \sum_{n=1}^{\infty} \left( q^{4n-3} - \frac{q^{4n-1}}{1 - q^{4n-3}} \right)
\end{equation}
on summing by the column-row method. Now, comparing the coefficients of $q^n$ on both sides of (2.3), we obtain Jacobi’s two-square Theorem.

**Theorem 4.** If $|q| < 1$, then

\begin{equation}
\psi^2(q^2) = \frac{1}{1-q} + \frac{q(1-q)^2}{(1-q)(1+q^2)} + \frac{q(1-q^3)^2}{(1-q)(1+q^4)} + \frac{q(1-q^5)^2}{(1-q)(1+q^6)} + \cdots.
\end{equation}

**Proof:** From ([2, p. 17, Entry 12]), we have

\[\frac{(a^2q^3;q^4)_\infty}{(a^2q;q^4)_\infty} = \frac{1}{1-ab} + \frac{(a-bq)(b-aq)}{(1-ab)(1+q^2)} + \frac{(a-bq^3)(b-aq^3)}{(1-ab)(1+q^4)} + \cdots.\]

Putting $a = b = \sqrt{q}$ in the above identity and using (1.5), we obtain (2.4).

**Theorem 5.** If $|q| < 1$, then

\begin{equation}
\sum_{n=0}^{\infty} \frac{(-1)^n q^n}{1-q^{2n+1}} = \frac{1}{1-q} + \frac{q(1-q)^2}{(1-q)(1+q^2)} + \frac{q(1-q^3)^2}{(1-q)(1+q^4)} + \cdots.
\end{equation}

**Proof:** Let $P_n/Q_n$ denote the $n$-th convergent of the continued fraction (2.5). Then

\[Q_1 = 1 - q, Q_2 = (1-q)^2(1+q^2) + q(1-q)^2 = (1-q)^2(1+q+q^2) = (1-q)(1-q^3).\]

By induction on $n$, one can show that

\[Q_n = (1-q)(1-q^3)(1-q^5)\ldots(1-q^{2n-1}).\]

Then we have

\[
\frac{P_n}{Q_n} = \sum_{k=1}^{n} \frac{(-1)^{k+1}a_1a_2\ldots a_k}{Q_kQ_{k-1}} = \frac{a_1}{Q_1Q_0} - \frac{a_1a_2}{Q_2Q_1} + \frac{a_1a_2a_3}{Q_3Q_2} - \cdots + \frac{(-1)^{n+1}a_1a_2a_3\ldots a_n}{Q_nQ_{n-1}} = \frac{1}{1-q} - \frac{q}{1-q^3} + \frac{q^2}{1-q^5} - \cdots + \frac{(-1)^{n-1}q^{n-1}}{1-q^{2n-1}}.
\]

Letting $n \to \infty$, we complete the proof of Theorem 5.

**Theorem 6.** Jacobi’s two-triangular number Theorem.

\[t_2(n) = d_1(4n+1) - d_3(4n+1).\]

**Proof:** From (2.4) and (2.5), we have

\begin{equation}
\psi^2(q^2) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{1-q^{2n+1}}.
\end{equation}
Changing $q$ to $-q$ in (2.6) and adding the resulting identity to (2.6), we obtain
\[
2\psi^2(q^2) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{1 - q^{2n+1}} + \sum_{n=0}^{\infty} \frac{q^n}{1 + q^{2n+1}}
\]
\[
= 2 \sum_{n=0}^{\infty} \frac{q^{2n}}{1 - q^{8n+2}} - 2 \sum_{n=0}^{\infty} \frac{q^{6n+4}}{1 - q^{8n+6}}.
\]
(2.7)

Changing $q$ to $q^{1/2}$ in (2.7), we deduce
\[
\psi^2(q) = \sum_{n=0}^{\infty} \frac{q^n}{1 - q^{4n+1}} - \sum_{n=0}^{\infty} \frac{q^{3n+2}}{1 - q^{4n+3}}
\]
\[
= \sum_{n,m=0}^{\infty} q^{4mn+m+n} - \sum_{n,m=0}^{\infty} q^{4mn+3m+3n+2}
\]
\[
= \sum_{n,m=0}^{\infty} q^{((4m+1)(4n+1)-1)/4} - \sum_{n,m=0}^{\infty} q^{((4m+3)(4n+3)-1)/4}.
\]
(2.8)

By comparing the coefficients of $q^n$ on both sides of (2.8), we obtain the Jacobi’s two-triangular number Theorem. 

REFERENCES


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