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A SUFFICIENT CONDITION FOR SOLVABILITY IN GROUPS ADMITTING ELEMENTARY ABELIAN OPERATOR GROUPS

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1. Introduction. Generalizing a celebrated theorem of Thompson, R. P. Martineau has established [**4**; **5**] that a finite group which admits an elementary abelian group of automorphisms with trivial fixed-point subgroup is necessarily solvable. A critical observation in his approach to this problem is the fact that, corresponding to each prime divisor of its order, such a group contains a unique Sylow subgroup invariant (as a set) under the action. Hence, the theorem we shall derive here represents a modest extension of Martineau's result.

THEOREM. *Let G be a finite group admitting an elementary abelian group of automorphisms A with $(|G|, |A|) = 1$ such that, for each prime p dividing its order, G has a unique A-invariant Sylow p-subgroup. If the fixed-point subgroup $C_G(A)$ has odd order, then G is solvable.*

As with the Thompson-Martineau theorem, the argument splits into three cases according to whether the rank of A is one, two, or at least three. Fortunately, the first case is quickly resolved by a fusion result of Glauberman (which may be regarded as a far-reaching generalization of the normal p-complement theorem used in Thompson's original argument) while the last is a direct consequence of some more recent work of Martineau [**6**]. It is the case that A has rank two which occupies the bulk of this paper. The approach here, as in Martineau's proof, depends heavily on the "maximal subgroups" technique pioneered by Bender, although in the end it is Glauberman's fusion theorem which is used to settle this case also. However, it should be pointed out that the price of greater generality is high, for in addition to requiring this sophisticated fusion result, our conclusion (unlike Martineau's) rests ultimately on the Feit-Thompson theorem on groups of odd order.

In the final section, we indicate how by imposing a slightly more technical hypothesis, the theorem may be extended to encompass an earlier generalization of Martineau's theorem due to J. N. Ward [**10**].

2. Preliminaries. All groups considered here are finite. Any notation not explicitly defined conforms to that of Gorenstein's book [**3**] and we shall assume familiarity with basic results on coprime operators, as found in Sections 5.3 and 6.2 of that reference.

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The first lemma is crucial. It is an easy corollary of a well-known result of Baer and a strong fusion theorem of Glauberman.

LEMMA 2.1. *A group G is 2-closed if and only if $N_G(Z(J(P)))$ is 2-closed for every odd order Sylow subgroup P of G.*

Proof. See the lemma in [**7**].

For the remaining results in this section, we will assume H is a group admitting a group of operators A satisfying all the hypotheses of the theorem (though, as is usually clear from the proofs, most of the lemmas are valid in a more general context).

LEMMA 2.2. *Suppose A is non-cyclic and H is a p-group for some prime p. Then*

(a) $H = \langle C_H(\alpha) : \alpha \in A^\# \rangle$, *and*

(b) *if* $\alpha_0 \in A^\#$, $[H, \alpha_0] \subseteq \langle C_H(\alpha) : \alpha \in A \backslash \langle \alpha_0 \rangle \rangle$.

Proof. For (a), see [**3**, Theorem 6.2.2]. The second statement follows by applying (a) to the group $K = [H, \alpha_0]$ and observing that, since $K = [K, \alpha_0]$, α_0 is fixed-point-free on $K/\Phi(K)$ so $C_K(\alpha_0) \subseteq \Phi(K)$.

LEMMA 2.3. *Suppose H is a p-group for some prime p and $K \subseteq H$ such that $C_H(K) \subseteq K$. If $\alpha \in A$ acts trivially on K, it acts trivially on all of H.*

Proof. Since H is a p-group, we may assume $K \trianglelefteq H$. Then $[H, K] \subseteq K \subseteq C_H(\alpha)$, so $[H, K, \alpha] = [K, \alpha, H] = 1$. By the "3-subgroups" lemma, $[H, \alpha] \subseteq C_H(K) \subseteq K \subseteq C_H(\alpha)$ so $[H, \alpha] = 1$.

LEMMA 2.4. (Glauberman) *If H is solvable, P is a Sylow p-subgroup of H, and $O_{p'}(H) = 1$, then $H = N_H(J(P))C_H(Z(P))$.*

Proof. This is a consequence of [**1**, Corollary 1].

LEMMA 2.5. *Suppose H is solvable, π is a set of primes, and $\alpha_0 \in A$. If K is an A-invariant π'-subgroup of H such that K is normalized by a Hall π-subgroup of $C_H(\alpha_0)$, then $[K, \alpha_0] \subseteq O_{\pi'}(H)$.*

Proof. Let H be a counterexample with $|H| + |A|$ minimal. Considering $H/O_{\pi'}(H)$, we conclude that $O_{\pi'}(H) = 1$ so $F(H)$ is a π-group. In fact, the minimality assumption implies that $H = F(H)K$, that K is a p-group for some prime p, and that $F(H)$ is a faithful irreducible module for $[K]A$ over the field of q elements for some prime $q \neq p$. By the hypothesis, $C_{F(H)}(\alpha_0)$ normalizes K and hence, it centralizes K, so the irreducibility of $F(H)$ yields $C_{F(H)}(\alpha_0) = 1$. If A is cyclic, then we are done either by Thompson's theorem [**3**, Theorem 10.2.1] if $p = 2$ (since $|C_H(A)|$ is odd), or by a result of Shult [**9**, Corollary 3.2] if p is odd. If A is non-cyclic, we use Lemma 2.2 and the minimality assumption to conclude that $K = C_K(\alpha_1)$ for some $\alpha_1 \in A \backslash \langle \alpha_0 \rangle$. Then, if A_0 is a maximal subgroup of A containing α_0 but not α_1, we have $C_H(A_0) = C_H(A)$ and the required result follows since $|H| + |A_0| < |H| + |A|$.

The next lemma strengthens a special case of Lemma 2.5. Here (for the first time) we will need the uniqueness hypothesis on the A-invariant Sylow subgroups.

LEMMA 2.6. *Suppose H is a $\{p, q\}$-group for distinct primes p and q, with P and Q the corresponding A-invariant Sylow subgroups. If A is generated by elements α satisfying $C_Q(\alpha) \subseteq N_H(P)$, then $[P, A] \trianglelefteq H$.*

Proof. Assume H is a counterexample with $|H| + |A|$ minimal. By the preceding lemma, $[P, A] \subseteq O_{q'}(H) = O_p(H)$, so minimality implies $H = O_p(H)Q$ and $P = O_p(H)$.

Suppose A is cyclic. Since Q is the unique A-invariant Sylow q-subgroup of H, $C_G(A) \subseteq N_H(Q)$ so from Lemma 2.5 (applied to Q), we conclude that $[Q, A] \subseteq O_{p'}(H) \subseteq C_H(P)$. On the other hand, $P \trianglelefteq H$ so $C_G(A)$ normalizes $[P, A]$. Since $[P, A] \trianglelefteq P$ and $Q = [Q, A]C_Q(A)$, we are done in this case.

Now assume A is non-cyclic. Lemma 2.2 and the minimality of our alleged counterexample then forces $Q = C_Q(\alpha_1)$ for some $\alpha_1 \in A^{\#}$, whence $[P, \alpha_1] = [H, \alpha_1] \trianglelefteq H$. Since α_1 centralizes $\bar{H} = H/[P, \alpha_1]$, $\bar{A} = A/\langle\alpha_1\rangle$ acts naturally on $\bar{H}$. From $|\bar{H}| + |\bar{A}| < |H| + |A|$, we conclude that $[\bar{P}, \bar{A}] \trianglelefteq \bar{H}$, where $\bar{P} = P/[P, \alpha_1]$, so $[P, A] \trianglelefteq H$ as required.

3. The rank of A. From now on, assume G is a minimal counterexample to the theorem and that A has minimal rank among groups of automorphisms of G which satisfy the hypotheses. Since the hypotheses are inherited by A-invariant subgroups and quotients, it follows that if H is a non-trivial proper A-invariant subgroup of G, then $N_G(H)$ is solvable. This fact will be used implicitly throughout the argument.

LEMMA 3.1. *A has rank two.*

Proof. The uniqueness of the A-invariant Sylow 2-subgroup of G implies that it is normalized by $C_G(A)$. Hence, in the case that A is cyclic, the fact that $C_G(A)$ has odd order may be used together with Lemma 2.5 to conclude that every proper A-invariant subgroup of G is 2-closed. Then by Lemma 2.1, G is 2-closed and hence, by Feit-Thompson, is solvable.

On the other hand, assuming the theorem also to be valid when A has rank two, a contradiction is reached in the higher rank cases by the argument presented in [**6**] (since Lemma 2.4 applies to the A-invariant proper subgroups of G).

4. A-invariant $\{p, q\}$-subgroups. In this section, we fix two distinct primes p and q, with P and Q the corresponding A-invariant Sylow subgroups of G. The aim is to investigate the consequences of assuming that (as must be the case for *some* choice of p and q) G has no A-invariant Hall $\{p, q\}$-subgroups or equivalently, that $PQ \neq QP$.

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Following the notation of [**6**], let X be the (unique) largest A-invariant p-subgroup of G with $XQ = QX$, and Y be the largest A-invariant q-subgroup such that $PY = YP$. Then XQ and PY are the unique maximal A-invariant $\{p, q\}$-subgroups of G containing Q and P respectively.

Observe that since $C_G(A)$ normalizes P and Q, it must also normalize Y and X. Also, $C_P(A) \subseteq X$ and $C_Q(A) \subseteq Y$.

The assumption that A has rank two (Lemma 3.1) will be implicit in the remainder of the argument.

Let $\mathcal{N}^* = \mathcal{N}^*(A; p, q)$ be the set of maximal A-invariant $\{p, q\}$-subgroups of G.

LEMMA 4.1. *Suppose* $H \in \mathcal{N}^*$ *and* K *is an* A*-invariant subgroup of* $F(H)$ *with* $O_p(K) \neq 1 \neq O_q(K)$. *Then* H *is the unique member of* $\mathcal{N}^*$ *containing* K.

Proof. See Lemma 4 of [**5**].

LEMMA 4.2. *If* $H \in \mathcal{N}^*\backslash\{XQ, PY\}$, *then* $O_p(H) \neq 1 \neq O_q(H)$ *and* $X \cap O_p(H) = 1 = Y \cap O_q(H)$.

Proof. Since the Glauberman factorization theorem, Lemma 2.4, applies to A-invariant proper subgroups of G, the proofs of Lemmas 2.3 and 2.5 of [**6**] may be used.

Actually, our attention will be focused not on all of $\mathcal{N}^*$, but on the subset $\mathcal{N}_0^* = \{H \in \mathcal{N}^* : C_P(A)C_Q(A) \subseteq H\}$.

As we have already noted, $\{XQ, PY\} \subseteq \mathcal{N}_0^*$. The next lemma states that only in a very special case may $\mathcal{N}_0^*$ contain any other groups. However, in contrast to [**4**], the exceptional situation does not appear to lead to an immediate contradiction.

LEMMA 4.3. *If* $\mathcal{N}_0^* \neq \{XQ, PY\}$, *then* A *is generated by two elements* α_1 *and* α_2 *such that* $Q \subseteq C_G(\alpha_1)$ *and* $P \subseteq C_G(\alpha_2)$.

Proof. Suppose $H \in \mathcal{N}_0^*\backslash\{XQ, PY\}$ and set $U = O_p(H)$ and $V = O_q(H)$. If $\alpha \in A^\#$, then $C_G(A) \subseteq N_G(Q) \cap C_G(\alpha) \subseteq N_G(C_Q(\alpha))$, so by Lemma 2.5, $[C_Q(\alpha), A] \subseteq O_q(C_G(\alpha))$. Similarly, $[C_P(\alpha), A] \subseteq O_p(C_G(\alpha))$. But by Lemma 4.2, $U \neq 1$ and $C_U(A) \subseteq X \cap U = 1$, so $C_U(\alpha) = [C_U(\alpha), A] \subseteq [C_P(\alpha), A]$. $[C_Q(\alpha), A]$ then centralizes $C_U(\alpha)$, whence it follows from Lemma 4.1 (with $K = Z(F(H))$) that either $C_U(\alpha) = 1$ or $[C_Q(\alpha), A] \subseteq H$. But since $C_Q(A) \subseteq H$ by the definition of $\mathcal{N}_0^*$, the latter case implies $C_Q(\alpha) = [C_Q(\alpha), A]C_Q(A) \subseteq H$. In summary then, for every $\alpha \in A^\#$, either $C_U(\alpha) = 1$ or $C_Q(\alpha) \subseteq H$. Since $Q \nsubseteq H$, we conclude from Lemma 2.2 that $C_U(\alpha_0) = 1$ for some $\alpha_0 \in A^\#$.

Suppose now that $C_U(\alpha) \neq 1$ for every $\alpha \in A\backslash\langle\alpha_0\rangle$, so $[Q, \alpha_0] \subseteq H$ by Lemma 2.2. Since $C_U(\alpha_0) = 1$, Lemma 2.5 (applied to $[Q, \alpha_0]U$) implies $[Q, \alpha_0]$ centralizes U, whence $\langle U, Q\rangle \subseteq N_G([Q, \alpha_0])$. If $[Q, \alpha_0] \neq 1$, this yields $U \subseteq X$, so $U = X \cap U = 1$, a contradiction. Therefore, in this case $Q \subseteq C_G(\alpha_0)$, so we may take $\alpha_1 = \alpha_0$.

On the other hand, suppose there exists some element of A outside $\langle \alpha_0 \rangle$ which is fixed-point-free on U. Then $[H \cap Q, A]$ centralizes U by Lemma 2.5, so for every $\alpha \in A^{\#}$, we have $\langle U, H \cap Q \rangle \subseteq N_G([H \cap Q, \alpha])$. Let $L = N_Q(H \cap Q)$, so $L \neq H \cap Q$ since $Q \nsubseteq H$. If $[H \cap Q, \alpha] \neq 1$, Lemma 4.1 (with $K = F(H)$) implies that $C_L(\alpha) \subseteq N_Q([H \cap Q, \alpha]) \subseteq H \cap Q$, so from Lemma 2.2 we conclude that $H \cap Q \subseteq C_G(\alpha_1)$ for some $\alpha_1 \in A^{\#}$. Now $C_Q(H \cap Q) \subseteq C_Q(V) \subseteq N_Q(V) = H \cap Q$ (by the maximality of H), so it follows from Lemma 2.3 that $Q \subseteq C_G(\alpha_1)$.

Repeating the above argument with p and q interchanged yields the required result. (Of course, α_1 and α_2 generate A since otherwise, $PQ = QP$.)

For the remainder of this section, let $M = N_P(X)$ and $N = N_Q(Y)$. Note that if $PQ \neq QP$, then M and N are strictly larger than X and Y respectively.

LEMMA 4.4. *Suppose $PQ \neq QP$ and $\alpha \in A^{\#}$. If $C_M(\alpha) \nsubseteq X$, then $X \subseteq C_G(\alpha)$ and similarly, if $C_N(\alpha) \nsubseteq Y$, then $Y \subseteq C_G(\alpha)$.*

Proof. The lemma is obviously symmetric in p and q, so we will prove (by contradiction) only the second statement. Assume that both $C_N(\alpha) \nsubseteq Y$ and $[Y, \alpha] \neq 1$.

By Lemma 4.3, we may assume $\mathcal{U}_0^* = \{XQ, PY\}$ (since $C_Q(A) \subseteq Y$). Let $R = X \cap O_p(PY)$ and $L = N_G(R) \cap O_p(PY)$, so $R \trianglelefteq L$. Since $R^Y \subseteq X^Y \cap O_p(PY) \subseteq XQ \cap O_p(PY) = R$, Y normalizes R and hence, L. Therefore, Y acts naturally on the quotient $\bar{L} = L/R$ and the semi-direct product $[\bar{L}]Y$ admits A.

Now since $C_N(\alpha) \nsubseteq Y$ and $\mathcal{U}_0^* = \{XQ, PY\}$, we conclude that $C_P(\alpha) \subseteq X$, so $C_L(\alpha) \subseteq R$. Therefore, α is fixed-point-free on $\bar{L}$ and it follows from Lemma 2.5 that $[Y, \alpha]$ centralizes $\bar{L}$. In other words, $[L, [Y, \alpha]] \subseteq R$. On the other hand, $C_N(\alpha) \subseteq N_Q([Y, \alpha])$ so the hypothesis also implies $N_Q([Y, \alpha]) \nsubseteq Y$. Since $C_G(A) \subseteq N_G([Y, \alpha]) \neq G$ (using $[Y, \alpha] \neq 1$), we obtain $N_P([Y, \alpha]) \subseteq X$ and, in particular, $C_L([Y, \alpha]) \subseteq R$. Applying the usual factorization for coprime action, we find that $L = [L, [Y, \alpha]]C_L([Y, \alpha]) \subseteq R$, so since $L = N_G(R) \cap O_p(PY)$, it must be that $O_p(PY) = R \subseteq X$.

However, $1 \neq Y \neq N_Q(Y) \subseteq N_G(J(Y)) \cap N_G(Z(Y))$ so, since $C_G(A) \subseteq N_G(Y)$, we must have $N_P(J(Y)) \subseteq X$ and $N_P(Z(Y)) \subseteq X$. By Lemma 2.4, $P = O_P(PY)N_P(J(Y))C_P(Z(Y))$, so the preceding paragraph implies $P = X$, contradicting the hypothesis that $PQ \neq QP$.

At this point, it is convenient to partition the set $\pi(G)$ of prime divisors of $|G|$ into three subsets as follows: If $r \in \pi(G)$ with R the A-invariant Sylow r-subgroup of G, let $r \in \pi_i$ (where $i = 0, 1$, or 2) if the subgroup of A generated by all elements $\alpha \in A$ satisfying $C_R(\alpha) = C_R(A)$ has rank i.

LEMMA 4.5. *If $PQ \neq QP$, then one of the following holds:*

(a) *Interchanging p and q if necessary, $p \in \pi_1$ and, if $C_P(\alpha) = C_P(A)$, then $Q \subseteq C_G(\alpha)$ (so $q \in \pi_2$).*

(b) $\{p, q\} \subseteq \pi_2$.

Proof. Since $X \neq N_P(X) = M$ and $Y \neq N_Q(Y) = N$, Lemma 2.2 implies that for some $\alpha_1, \alpha_2 \in A^{\#}$, $C_N(\alpha_1) \not\subseteq Y$ and $C_M(\alpha_2) \not\subseteq X$, whence by the preceding lemma, $Y \subseteq C_G(\alpha_1)$ and $X \subseteq C_G(\alpha_2)$. Also, by Lemma 4.3, we may assume $\mathcal{N}_0{}^* = \{XQ, PY\}$ (else (b) certainly holds) so $C_P(\alpha_1) \subseteq X$, $C_Q(\alpha_2) \subseteq Y$, and $A = \langle \alpha_1, \alpha_2 \rangle$. Therefore, $C_P(\alpha_1) = C_P(A)$ and $C_Q(\alpha_2) = C_Q(A)$, so $\{p, q\} \subseteq \pi_1 \cup \pi_2$.

Now if (b) is false, we may assume without loss of generality that $p \in \pi_1$, so $C_P(\alpha) \neq C_P(A)$ for every $\alpha \in A \backslash \langle \alpha_1 \rangle$. The argument of the preceding paragraph then implies $C_N(\alpha) \subseteq Y$ for every $\alpha \in A \backslash \langle \alpha_1 \rangle$, so by Lemma 2.2, $[N, \alpha_1] \subseteq Y \subseteq C_G(\alpha_1)$. It follows that $N \subseteq C_G(\alpha_1)$ and hence, by Lemma 2.3, $Q \subseteq C_G(\alpha_1)$, so (a) holds.

It is perhaps worth pointing out that Case (a) of Lemma 4.5 is the direct analogue of Lemma 10 of [**4**]. In the situation discussed there, Case (b) is eliminated using a consequence of the Thompson normal p-complement theorem (Lemma 3). Actually (and this provides a slightly different way of finishing Martineau's proof), Case (a) also cannot occur when A is fixed-point-free, for it is an elementary property of coprime action that any two elements of $C_G(\alpha)$ which are conjugate in G are, in fact, conjugate in $C_G(\alpha)$. But if $C_G(A) = 1$, $C_G(\alpha)$ is nilpotent by Thompson's theorem, so if $Q \subseteq C_G(\alpha)$ then Q controls fusion within itself and G has a normal q-complement (even if $q = 2$). Unfortunately, in the situation under consideration here, there appears to be no such quick means of dispatching either of the possibilities defined by Lemma 4.5, and we must derive some simple "global" consequences of the result before we can apply a fusion theorem.

5. The contradiction. In this section, p, q, and r will denote arbitrary primes and P, Q, and R will stand for the corresponding A-invariant Sylow subgroups of G. The sets π_0, π_1, and π_2 are as defined immediately preceding Lemma 4.5.

LEMMA 5.1. $\pi_0 = \emptyset$.

Proof. If $p \in \pi_0$ and $q \in \pi_2$, then $PQ = QP$ by Lemma 4.5, so from Lemma 2.6, $Q \subseteq N_G([P, A])$. On the other hand, Lemma 4.5 implies (by Hall's characterization of solvability) that G contains a solvable A-invariant Hall $(\pi_0 \cup \pi_1)$-subgroup H. It follows that $G = N_G([P, A])H$, so $[P, A]^G = [P, A]^H \subseteq H \neq G$. Since G contains no proper non-trivial A-invariant normal subgroups, we get $P \subseteq C_G(A)$, contradicting $p \in \pi_0$.

LEMMA 5.2. $\pi_1 = \emptyset$.

Proof. Suppose $p \in \pi_1$ and $\alpha_0 \in A^{\#}$ with $C_P(\alpha_0) = C_P(A)$, and let $\lambda = \{q \in \pi(G) : PQ = QP\}$. We claim that if $\{q, r\} \subseteq \lambda$, then $QR = RQ$.

If $\{q, r\} \subseteq \pi_1$, then $QR = RQ$ by Lemma 4.5. If $\{q, r\} \subseteq \pi_2$, then Lemma 2.6 yields $\langle Q, R \rangle \subseteq N_G([P, A])$ and, since $p \in \pi_1$ certainly implies $[P, A] \neq 1$, we have $QR = RQ$ in this case also. Thus, by Lemma 5.1, we may assume $q \in \pi_2$

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and $r \in \pi_1 \backslash \{p\}$. Let $\alpha_1 \in A^\#$ such that $C_R(\alpha_1) = C_R(A)$ and suppose $QR \neq RQ$. By Lemma 4.5, $Q \subseteq C_G(\alpha_1)$ so $[P, \alpha_1] = [QP, \alpha_1] \trianglelefteq QP$. But Lemma 2.5 applied to PR yields $[P, \alpha_1] \subseteq O_p(PR)$ and $[R, \alpha_0] \subseteq O_r(PR)$. Therefore, $\langle [R, \alpha_0], Q \rangle \subseteq N_G([P, \alpha_1])$. Now $[P, \alpha_1] \neq 1$ since $p \in \pi_1$, so if XQ is the maximal A-invariant $\{q, r\}$-subgroup of G containing Q (with X an A-invariant r-group), it follows that $[R, \alpha_0] \subseteq X$. Since $X \neq R$, $X \neq N_R(X) = M$ so, since $M = [M, \alpha_0] C_M(\alpha_0)$, we have $C_M(\alpha_0) \nsubseteq X$. Lemma 4.4 implies $[R, \alpha_0] \subseteq X \subseteq C_G(\alpha_0)$, so $R \subseteq C_G(\alpha_0)$, contradicting $r \in \pi_1$. This proves the claim.

The upshot of the preceding paragraph is that G contains an A-invariant solvable Hall λ-subgroup L. But if $q \notin \lambda$, then $Q \subseteq C_G(\alpha_0)$ by Lemma 4.5. Thus, $G = C_G(\alpha_0)L$, so $[G, \alpha_0] = [L, \alpha_0] \subseteq L \neq G$. Again, since $[G, \alpha_0]$ is a non-trivial A-invariant normal subgroup of G, we have a contradiction.

Proof of theorem. To obtain the final contradiction, we now argue as in the case that A is cyclic. Namely, since as a result of the previous two lemmas, $\pi(G) = \pi_2$, Lemma 2.5 and the fact that $C_G(A)$ has odd order imply that every proper A-invariant subgroup of G is 2-closed, so by Lemma 2.1, G is 2-closed.

6. Some remarks. The conclusion of the theorem may, in fact, be valid without the hypothesis of uniqueness on the A-invariant Sylow subgroups, although this would appear to require some fresh ideas. Certainly, by the recent signalizer functor theorem of Glauberman [**2**], it is only the rank one and two cases that need to be settled. There is perhaps somewhat more hope for a resolution of this question under the hypothesis that $C_G(A)$ is a $\{2, 3\}'$-group since enough is now known about 2-fusion to treat the cyclic case [**8**]. (Note also that in the cyclic case of the present theorem, we needed to assume only that the A-invariant Sylow 2-subgroup was unique.)

Assuming uniqueness of A-invariant Sylow subgroups, the hypothesis that $C_G(A)$ has odd order can easily be replaced by a somewhat weaker hypothesis. Suppose, for example, we assume that the Sylow 2-subgroup of $C_G(A)$ is contained in $O_2(C_G(B))$ for every maximal subgroup B of A. (This is actually equivalent to assuming $C_G(B)$ is 2-closed for every such B.) If p is an odd prime and if S and P are, respectively, the A-invariant Sylow 2 and p-subgroups of G, then since $C_G(A)$ normalizes P, we have $[C_S(A), C_P(B)] = 1$ for every maximal B. It is an easy extension of Lemma 2.2 that P is generated by the $C_P(B)$'s, so P centralizes $C_S(A)$ for every odd p, whence $G = C_G(C_S(A))S$. Then $C_S(A)^G = C_S(A)^S \subseteq S$, so $C_S(A) \subseteq O_2(G)$. By the theorem just proved, $G/O_2(G)$ is solvable, so G is solvable. The result stated with this slightly weaker hypothesis extends [**10**, Theorem 1]. Of course, it does not seem unreasonable to ask whether *any* hypothesis beyond the uniqueness of the A-invariant Sylow subgroups is really necessary.

REFERENCES

1. G. Glauberman, *Failure of factorization in p-solvable groups*, Quart. J. Math. Oxford (2) *24* (1973), 71–77.

2. ——— *On solvable signalizer functors in finite groups*, Proc. London Math. Soc. (3) *33* (1976), 1–27.
3. D. Gorenstein, *Finite groups* (Harper & Row, New York, 1968).
4. R. P. Martineau, *Solubility of groups admitting a fixed-point-free automorphism group of type* (p, p), Math. Z. *124* (1972), 67–72.
5. ——— *Elementary abelian fixed-point-free automorphism groups*, Quart. J. Math. Oxford (2) *23* (1972), 205–212.
6. ——— *Solubility of groups admitting certain fixed-point-free automorphism groups*, Math. Z. *130* (1973), 143–147.
7. M. R. Pettet, *A note on finite groups having a fixed-point-free automorphism*, Proc. Amer. Math. Soc. *52* (1975), 79–80.
8. ——— *A remark on groups with a coprime automorphism fixing a* $\{2, 3\}'$*-subgroup*, to appear, J. London Math. Soc.
9. E. Shult, *On groups admitting fixed point free abelian operator groups*, Illinois J. Math. *9* (1965), 701–720.
10. J. N. Ward, *On groups admitting an elementary abelian automorphism group*, Proceedings of the Second International Conference on the Theory of Groups, Lecture Notes in Mathematics, Vol. 372 (Springer-Verlag, Berlin, 1974).

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