REVERSES OF THE CAUCHY–BUNYAKOVSKY–SCHWARZ INEQUALITY FOR n-TUPLES OF COMPLEX NUMBERS

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Some new reverses of the Cauchy–Bunyakovsky–Schwarz inequality for n-tuples of real and complex numbers related to Cassels and Shisha–Mond results are given.

1. INTRODUCTION

Let \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) be two positive n-tuples with the property that there exists the positive numbers \( m_i, M_i \) \((i = 1, 2)\) such that

\[
0 < m_i \leq a_i \leq M_i < \infty \quad \text{and} \quad 0 < m_2 \leq b_i \leq M_2 < \infty,
\]

for each \( i \in \{1, \ldots, n\} \).

The following reverses of the Cauchy–Bunyakovsky–Schwarz inequality are well known in the literature:

1. PÓLYA–SZEGÖ’S INEQUALITY [8]

\[
\frac{\sum_{k=1}^{n} a_k^2}{(\sum_{k=1}^{n} a_kb_k)^2} \leq \frac{1}{4} \left( \frac{M_1 M_2}{m_1 m_2} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2; \tag{1.2}
\]

2. SHISHA–MOND’S INEQUALITY [9]

\[
\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 - \sum_{k=1}^{n} a_kb_k \leq \left( \sqrt{\frac{M_1}{m_2}} - \sqrt{\frac{m_1}{M_2}} \right)^2; \tag{1.3}
\]

3. OZEKI’S INEQUALITY [7]

\[
\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 - \left( \sum_{k=1}^{n} a_kb_k \right)^2 \leq \frac{1}{4} n^2 (M_1 M_2 - m_1 m_2)^2; \tag{1.4}
\]

4. DIAZ–METCALF’S INEQUALITY [1]

\[
\sum_{k=1}^{n} b_k^2 + \frac{m_2 M_2}{m_1 M_1} \sum_{k=1}^{n} a_k^2 \leq \left( \frac{M_2}{m_1} + \frac{m_2}{M_1} \right) \sum_{k=1}^{n} a_kb_k. \tag{1.5}
\]
If the weight $w = (w_1, \ldots, w_n)$ is a positive $n$-tuple, then we have the following inequalities, which are also well known.

5. **Cassels' Inequality** [10]

If the positive $n$-tuples $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ satisfy the condition

$$0 < m \leq \frac{a_k}{b_k} \leq M < \infty \quad \text{for each} \quad k \in \{1, \ldots, n\},$$

where $m, M$ are given, then

$$\sum_{k=1}^{n} \frac{w_k a_k^2}{(\sum_{k=1}^{n} w_k a_k b_k)^2} \leq \frac{M + m}{4mM}.$$

6. **Greub–Reinboldt’s Inequality** [4]

If $a$ and $b$ satisfy the condition (1.1), then

$$\sum_{k=1}^{n} \frac{w_k a_k^2}{(\sum_{k=1}^{n} w_k a_k b_k)^2} \leq \frac{(M_1 M_2 + m_1 m_2)^2}{4m_1 m_2 M_1 M_2}.$$

7. **Generalised Diaz–Metcalf Inequality** [1] (see also [6, p. 123])

If $u, v \in [0, 1]$ and $v \leq u, u + v = 1$ and (1.6) holds, then one has the inequality

$$u \sum_{k=1}^{n} w_k b_k^2 + vm \sum_{k=1}^{n} w_k a_k^2 \leq (vm + u M) \sum_{k=1}^{n} w_k a_k b_k.$$

8. **Klamkin–Mclenaghan’s Inequality** [5]

If $a$ and $b$ satisfy (1.6), then we have the inequality

$$\sum_{k=1}^{n} \left( \sum_{k=1}^{n} w_k a_k^2 \right)^2 \leq \left( \sum_{k=1}^{n} w_k a_k b_k \right)^2 \leq \left( \sqrt{M} - \sqrt{m} \right)^2 \sum_{k=1}^{n} w_k a_k b_k \sum_{k=1}^{n} w_k a_k^2.$$

The main aim of this paper is to point out some new reverse inequalities of the classical Cauchy–Bunyakovsky–Schwarz result for both real and complex $n$-tuples.

2. **Some Reverses of the Cauchy–Bunyakovsky–Schwarz Inequality**

The following result holds.

**Theorem 1.** Let $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $p = (p_1, \ldots, p_n) \in \mathbb{R}^n_+$ with $\sum_{i=1}^{n} p_i = 1$. If $b_i \neq 0, \ i \in \{1, \ldots, n\}$ and there exists the constant $\alpha \in \mathbb{K}$ and $r > 0$ such that for any $k \in \{1, \ldots, n\}$

$$\frac{a_k}{b_k} \in \overline{D}(\alpha, r) := \{ z \in \mathbb{K} \mid |z - \alpha| \leq r \},$$
then we have the inequality

\[ \sum_{k=1}^{n} p_k |a_k|^2 + (|\alpha|^2 - r^2) \sum_{k=1}^{n} p_k |b_k|^2 \leq 2 \text{Re} \left[ \overline{\alpha} \left( \sum_{k=1}^{n} p_k a_k b_k \right) \right] \]

\[ \leq 2 |\alpha| \cdot \sum_{k=1}^{n} p_k |a_k b_k| \]

The constant \( c = 2 \) is best possible in the sense that it cannot be replaced by a smaller constant.

**Proof:** From (2.1) we have \(|a_k - \alpha \overline{b}_k|^2 \leq r |b_k|^2\) for each \( k \in \{1, \ldots, n\} \), which is clearly equivalent to

\[ |a_k|^2 + (|\alpha|^2 - r^2) |b_k|^2 \leq 2 \text{Re} \left[ \overline{\alpha}(a_k b_k) \right] \]

for each \( k \in \{1, \ldots, n\} \).

Multiplying (2.3) with \( p_k \geq 0 \) and summing over \( k \) from 1 to \( n \), we deduce the first inequality in (1.2). The second inequality is obvious.

To prove the sharpness of the constant 2, assume that under the hypothesis of the theorem there exists a constant \( c > 0 \) such that

\[ \sum_{k=1}^{n} p_k |a_k|^2 + (|\alpha|^2 - r^2) \sum_{k=1}^{n} p_k |b_k|^2 \leq c \text{Re} \left[ \overline{\alpha} \left( \sum_{k=1}^{n} p_k a_k b_k \right) \right] , \]

provided \( a_k / b_k \in \overline{D}(\alpha, r)\), \( k \in \{1, \ldots, n\} \).

Assume that \( n = 2, p_1 = p_2 = 1/2, b_1 = b_2 = 1, \alpha = r > 0 \) and \( a_2 = 2r, a_1 = 0 \).

Then \( |a_2/b_2 - \alpha| = r, |a_1/b_1 - \alpha| = r \) showing that the condition (2.1) holds. For these choices, the inequality (2.4) becomes \( 2r^2 \leq cr^2 \), giving \( c \geq 2 \).

The case where the disk \( \overline{D}(\alpha, r) \) does not contain the origin, that is, \(|\alpha| > r\), provides the following interesting reverse of the Cauchy–Bunyakovsky–Schwarz inequality.

**Theorem 2.** Let \( a, b, \ p \) as in Theorem 1 and assume that \(|\alpha| > r > 0\). Then we have the inequality

\[ \sum_{k=1}^{n} p_k |a_k|^2 + \sum_{k=1}^{n} p_k |b_k|^2 \leq \frac{1}{|\alpha|^2 - r^2} \left\{ \text{Re} \left[ \overline{\alpha} \left( \sum_{k=1}^{n} p_k a_k b_k \right) \right] \right\}^2 \]

\[ \leq |\alpha|^2 \sum_{k=1}^{n} p_k |a_k b_k|^2 . \]

The constant \( c = 1 \) in the first and second inequality is best possible in the sense that it cannot be replaced by a smaller constant.

**Proof:** Since \(|\alpha| > r\), we may divide (2.2) by \( \sqrt{|\alpha|^2 - r^2} > 0 \) to obtain

\[ \frac{1}{\sqrt{|\alpha|^2 - r^2}} \sum_{k=1}^{n} p_k |a_k|^2 + \sqrt{|\alpha|^2 - r^2} \sum_{k=1}^{n} p_k |b_k|^2 \leq \frac{2}{\sqrt{|\alpha|^2 - r^2}} \text{Re} \left[ \overline{\alpha} \left( \sum_{k=1}^{n} p_k a_k b_k \right) \right] . \]
On the other hand, by the use of the following elementary inequality

\[ \frac{1}{\beta} p + \beta q \geq 2\sqrt{pq} \text{ for } \beta > 0 \text{ and } p, q \geq 0, \]

we may state that

\[ 2 \left( \sum_{k=1}^{n} p_k |a_k|^2 \right)^{1/2} \cdot \left( \sum_{k=1}^{n} p_k |b_k|^2 \right)^{1/2} \leq \frac{1}{\sqrt{|\alpha|^2 - r^2}} \sum_{k=1}^{n} p_k |a_k|^2 + \sqrt{|\alpha|^2 - r^2} \sum_{k=1}^{n} p_k |b_k|^2. \]

Utilising (2.6) and (2.8), we deduce

\[ \left( \sum_{k=1}^{n} p_k |a_k|^2 \right)^{1/2} \cdot \left( \sum_{k=1}^{n} p_k |b_k|^2 \right)^{1/2} \leq \frac{1}{\sqrt{|\alpha|^2 - r^2}} \text{Re} \left[ \frac{c}{\alpha} \left( \sum_{k=1}^{n} p_k a_k b_k \right) \right], \]

which is clearly equivalent to the first inequality in (2.6).

The second inequality is obvious.

To prove the sharpness of the constant, assume that (2.5) holds with a constant \( c > 0 \), that is,

\[ \sum_{k=1}^{n} p_k |a_k|^2 \sum_{k=1}^{n} p_k |b_k|^2 \leq \frac{c}{|\alpha|^2 - r^2} \left\{ \text{Re} \left[ \frac{c}{\alpha} \left( \sum_{k=1}^{n} p_k a_k b_k \right) \right] \right\}^2 \]

provided \( a_k / b_k \in \bar{D}(\alpha, r) \) and \( |a| > r \).

For \( n = 2, b_2 = b_1 = 1, p_1 = p_2 = 1/2, a_2, a_1 \in \mathbb{R}, \alpha, r > 0 \) and \( \alpha > r \), we get from (2.9) that

\[ \frac{a_1^2 + a_2^2}{2} \leq \frac{c\alpha^2}{\alpha^2 - r^2} \left( \frac{a_1 + a_2}{2} \right)^2. \]

If we choose \( a_2 = \alpha + r, a_1 = \alpha - r \), then \( |a_i - \alpha| \leq r, i = 1, 2 \) and by (2.10) we deduce

\[ \alpha^2 + r^2 \leq \frac{c\alpha^4}{\alpha^2 - r^2}, \]

which is clearly equivalent to

\[ (c - 1)\alpha^4 + r^4 \geq 0 \text{ for } \alpha > r > 0. \]

If in this inequality we choose \( \alpha = 1, r = \varepsilon \in (0, 1) \) and let \( \varepsilon \to 0^+ \), then we deduce \( c \geq 1 \).

The following corollary is a natural consequence of the above theorem.
COROLLARY 1. Under the assumptions of Theorem 2, we have the following additive reverse of the Cauchy–Bunyakovsk–Schwarz inequality

\[
0 \leq \sum_{k=1}^{n} p_k |a_k|^2 \sum_{k=1}^{n} p_k |b_k|^2 - \left| \sum_{k=1}^{n} p_k a_k b_k \right|^2 \\
\leq \frac{r^2}{|\alpha|^2 - r^2} \left| \sum_{k=1}^{n} p_k a_k b_k \right|^2.
\]

The constant \( c = 1 \) is best possible in the sense mentioned above.

REMARK 1. If in Theorem 1, we assume that \( |\alpha| = r \), then we obtain the inequality:

\[
\sum_{k=1}^{n} p_k |a_k|^2 \leq 2 \text{Re} \left( \sum_{k=1}^{n} p_k a_k b_k \right) \\
\leq 2 |\alpha| \left| \sum_{k=1}^{n} p_k a_k b_k \right|.
\]

The constant 2 is sharp in both inequalities.

We also remark that, if \( r > |\alpha| \), then (2.2) may be written as

\[
\sum_{k=1}^{n} p_k |a_k|^2 \leq (r^2 - |\alpha|^2) \sum_{k=1}^{n} p_k |b_k|^2 + 2 \text{Re} \left( \sum_{k=1}^{n} p_k a_k b_k \right) \\
\leq (r^2 - |\alpha|^2) \sum_{k=1}^{n} p_k |b_k|^2 + 2 |\alpha| \left| \sum_{k=1}^{n} p_k a_k b_k \right|.
\]

The following reverse of the Cauchy–Bunyakovsk–Schwarz inequality also holds.

THEOREM 3. Let \( a, b, p \) be as in Theorem 1 and assume that \( \alpha \in \mathbb{K}, \alpha \neq 0 \) and \( r > 0 \). Then we have the inequalities

\[
0 \leq \left( \sum_{k=1}^{n} p_k |a_k|^2 \right)^{1/2} \cdot \left( \sum_{k=1}^{n} p_k |b_k|^2 \right)^{1/2} - \left| \sum_{k=1}^{n} p_k a_k b_k \right| \\
\leq \left( \sum_{k=1}^{n} p_k |a_k|^2 \right)^{1/2} \cdot \left( \sum_{k=1}^{n} p_k |b_k|^2 \right)^{1/2} - \text{Re} \left( \frac{\alpha}{|\alpha|} \sum_{k=1}^{n} p_k a_k b_k \right) \\
\leq \frac{1}{2} \cdot \frac{r^2}{|\alpha|} \sum_{k=1}^{n} p_k |b_k|^2.
\]

The constant \( 1/2 \) is best possible in the sense mentioned above.

PROOF: From Theorem 1, we have

\[
\sum_{k=1}^{n} p_k |a_k|^2 + |\alpha|^2 \sum_{k=1}^{n} p_k |b_k|^2 \leq 2 \text{Re} \left( \sum_{k=1}^{n} p_k a_k b_k \right) + r^2 \sum_{k=1}^{n} p_k |b_k|^2.
\]
Since \( \alpha \neq 0 \), we can divide (2.15) by \(|\alpha|\), getting

\[
(2.16) \quad \frac{1}{|\alpha|} \sum_{k=1}^{n} p_k |a_k|^2 + |\alpha| \sum_{k=1}^{n} p_k |b_k|^2 \leq 2 \text{Re} \left[ \frac{\bar{\alpha}}{|\alpha|} \left( \sum_{k=1}^{n} p_k a_k b_k \right) \right] + \frac{r^2}{|\alpha|} \sum_{k=1}^{n} p_k |b_k|^2.
\]

Utilising the inequality (2.7), we may state that

\[
(2.17) \quad 2 \left( \sum_{k=1}^{n} p_k |a_k|^2 \right)^{1/2} \cdot \left( \sum_{k=1}^{n} p_k |b_k|^2 \right)^{1/2} \leq \frac{1}{|\alpha|} \sum_{k=1}^{n} p_k |a_k|^2 + |\alpha| \sum_{k=1}^{n} p_k |b_k|^2.
\]

Making use of (2.16) and (2.17), we deduce the second inequality in (2.14).

The first inequality in (2.14) is obvious.

To prove the sharpness of the constant \(1/2\), assume that there exists a \( c > 0 \) such that

\[
(2.18) \quad \left( \sum_{k=1}^{n} p_k |a_k|^2 \right)^{1/2} \cdot \left( \sum_{k=1}^{n} p_k |b_k|^2 \right)^{1/2} - \text{Re} \left[ \frac{\bar{\alpha}}{|\alpha|} \left( \sum_{k=1}^{n} p_k a_k b_k \right) \right] \leq c \cdot \frac{r^2}{|\alpha|} \sum_{k=1}^{n} p_k |b_k|^2,
\]

provided \(|a_k/b_k - \alpha| \leq r, \alpha \neq 0, r > 0\).

If we choose \( n = 2, \alpha > 0, b_1 = b_2 = 1, a_1 = \alpha + r, a_2 = \alpha - r, \) then from (2.18) we deduce

\[
(2.19) \quad \sqrt{r^2 + \alpha^2} - \alpha \leq c \frac{r^2}{\alpha}.
\]

If we multiply (2.19) with \( \sqrt{r^2 + \alpha^2} + \alpha > 0 \) and then divide it by \( r > 0 \), we deduce

\[
(2.20) \quad 1 \leq \frac{\sqrt{r^2 + \alpha^2} + \alpha}{\alpha} \cdot c
\]

for any \( r > 0, \alpha > 0 \).

If in (2.20) we let \( r \to 0^+ \), then we get \( c \geq 1/2 \), and the sharpness of the constant is proved.

\[\square\]

3. A CASSELS Type Inequality for Complex Numbers

The following result holds.

**Theorem 4.** Let \( a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{K}^n \), where \( \mathbb{K} = \mathbb{R}, \mathbb{C} \) and \( p = (p_1, \ldots, p_n) \in \mathbb{R}_+^n \) with \( \sum_{i=1}^{n} p_i = 1 \). If \( b_i \neq 0, i \in \{1, \ldots, n\} \) and there exist the constants \( \gamma, \Gamma \in \mathbb{K} \) with \( \text{Re}(\Gamma \gamma) > 0 \) and \( \Gamma \neq \gamma \), so that either

\[
(3.1) \quad \left| \frac{a_k}{b_k} - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \quad \text{for each } k \in \{1, \ldots, n\},
\]

available at https://www.cambridge.org/core/terms. https://doi.org/10.1017/S000497270003625X
or, equivalently,

\[ \text{Re} \left[ \left( \Gamma - \frac{a_k}{b_k} \right) \left( \bar{a}_k - \overline{\gamma} \right) \right] \geq 0 \quad \text{for each} \quad k \in \{1, \ldots, n\} \]

holds, then we have the inequalities

\[ \sum_{k=1}^{n} p_k |a_k|^2 \sum_{k=1}^{n} p_k |b_k|^2 \leq \frac{1}{2 \text{Re}(\Gamma \overline{\gamma})} \left\{ \text{Re} \left[ (\overline{\gamma} + \Gamma) \sum_{k=1}^{n} p_k a_k b_k \right] \right\}^2 \leq \left\{ \frac{|\Gamma + \gamma|^2}{4 \text{Re}(\Gamma \overline{\gamma})} \right\} \sum_{k=1}^{n} p_k a_k b_k \right|^2 . \]

The constants 1/2 and 1/4 are best possible in (3.3).

**PROOF:** The fact that the relations (3.1) and (3.2) are equivalent follows by the simple fact that for \( z, u, U \in \mathbb{C} \), the following inequalities are equivalent

\[ |z - u + U| \leq \frac{1}{2} |U - u| \]

and

\[ \text{Re} [(u - z)(\overline{z} - \overline{u})] \geq 0. \]

Define \( \alpha = (\gamma + \Gamma)/2 \) and \( \tau = |\Gamma - \gamma|/2. \) Then

\[ |\alpha|^2 - \tau^2 = \frac{|\Gamma + \gamma|^2}{4} - \frac{|\Gamma - \gamma|^2}{4} = \text{Re}(\Gamma \overline{\gamma}) > 0. \]

Consequently, we may apply Theorem 2, and the inequalities (3.3) are proved.

The sharpness of the constants may be proven in a similar way to that in the proof of Theorem 2, and we omit the details. \( \square \)

The following additive version also holds.

**COROLLARY 2.** With the assumptions in Theorem 4, we have

\[ \sum_{k=1}^{n} p_k |a_k|^2 \sum_{k=1}^{n} p_k |b_k|^2 - \left| \sum_{k=1}^{n} p_k a_k b_k \right|^2 \leq \left\{ \frac{|\Gamma - \gamma|^2}{4 \text{Re}(\Gamma \overline{\gamma})} \right\} \sum_{k=1}^{n} p_k a_k b_k \right|^2 . \]

The constant 1/4 is also best possible.

**REMARK 2.** With the above assumptions and if \( \text{Re}(\Gamma \overline{\gamma}) = 0 \), then by the use of Remark 1, we may deduce the inequality

\[ \sum_{k=1}^{n} p_k |a_k|^2 \leq \text{Re} \left[ (\overline{\gamma} + \Gamma) \sum_{k=1}^{n} p_k a_k b_k \right] \leq |\Gamma + \gamma| \left| \sum_{k=1}^{n} p_k a_k b_k \right|. \]

If \( \text{Re}(\Gamma \overline{\gamma}) < 0 \), then, by Remark 1, we also have

\[ \sum_{k=1}^{n} p_k |a_k|^2 \leq - \text{Re}(\Gamma \overline{\gamma}) \sum_{k=1}^{n} p_k |b_k|^2 + \text{Re} \left[ (\overline{\Gamma} + \gamma) \sum_{k=1}^{n} p_k a_k b_k \right] \]

\[ \leq - \text{Re}(\Gamma \overline{\gamma}) \sum_{k=1}^{n} p_k |b_k|^2 + |\Gamma + \gamma| \left| \sum_{k=1}^{n} p_k a_k b_k \right|. \]
Remark 3. If $a_k, b_k > 0$ and there exist the constants $m, M > 0$ ($M > m$) with

\begin{equation}
3.7 \quad m \leq \frac{a_k}{b_k} \leq M \text{ for each } k \in \{1, \ldots, n\},
\end{equation}

then, obviously (3.1) holds with $\gamma = m$, $\Gamma = M$, also $\Gamma \gamma = Mm > 0$ and by (3.3) we deduce

\begin{equation}
3.8 \quad \sum_{k=1}^{n} p_k a_k^2 \sum_{k=1}^{n} p_k b_k^2 \leq \frac{(M + m)^2}{4mM} \left( \sum_{k=1}^{n} p_k a_k b_k \right)^2,
\end{equation}

that is, Cassels’ inequality.

4. A SHISHA–MOND TYPE INEQUALITY FOR COMPLEX NUMBERS

The following result holds.

**Theorem 5.** Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n) \in \mathbb{K}^n$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $p = (p_1, \ldots, p_n) \in \mathbb{R}_+^n$ with $\sum_{i=1}^{n} p_i = 1$. If $b_i \neq 0$, $i \in \{1, \ldots, n\}$ and there exist the constants $\gamma, \Gamma \in \mathbb{K}$ such that $\Gamma \neq \gamma, -\gamma$ and either

\begin{equation}
4.1 \quad \left| \frac{a_k}{b_k} - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\gamma - \Gamma| \text{ for each } k \in \{1, \ldots, n\},
\end{equation}

or, equivalently,

\begin{equation}
4.2 \quad \text{Re} \left[ \left( \Gamma - \frac{a_k}{b_k} \right) \left( \overline{\frac{a_k}{b_k}} - \gamma \right) \right] \geq 0 \text{ for each } k \in \{1, \ldots, n\},
\end{equation}

holds, then we have the inequalities

\begin{equation}
4.3 \quad 0 \leq \left( \sum_{k=1}^{n} p_k |a_k|^2 \right)^{1/2} \cdot \left( \sum_{k=1}^{n} p_k |b_k|^2 \right)^{1/2} - \left| \sum_{k=1}^{n} p_k a_k b_k \right| \\
\leq \left( \sum_{k=1}^{n} p_k |a_k|^2 \right)^{1/2} \cdot \left( \sum_{k=1}^{n} p_k |b_k|^2 \right)^{1/2} - \text{Re} \left[ \frac{\Gamma + \overline{\gamma}}{|\Gamma + \gamma|} \sum_{k=1}^{n} p_k a_k b_k \right] \\
\leq \frac{1}{4} \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \sum_{k=1}^{n} p_k |b_k|^2.
\end{equation}

The constant $1/4$ is best possible in the sense that it cannot be replaced by a smaller constant.

**Proof:** Follows by Theorem 3 on choosing $\alpha = (\gamma + \Gamma)/2 \neq 0$ and $r = |\Gamma - \gamma|/2 > 0$. The proof for the best constant follows in a similar way to that in the proof of Theorem 3 and we omit the details.
REMARK 4. If $a_k, b_k > 0$ and there exists the constants $m, M > 0$ $(M > m)$ with
\begin{equation}
\frac{a_k}{b_k} \leq M \quad \text{for each} \quad k \in \{1, \ldots, n\},
\end{equation}
then we have the inequality
\begin{equation}
0 \leq \left( \sum_{k=1}^{n} p_k a_k^2 \right)^{1/2} \cdot \left( \sum_{k=1}^{n} p_k b_k^2 \right)^{1/2} - \sum_{k=1}^{n} p_k a_k b_k \\
\leq \frac{1}{4} \cdot \frac{(M - m)^2}{(M + m)} \sum_{k=1}^{n} p_k b_k^2.
\end{equation}
The constant $1/4$ is best possible. For $p_k = 1/n$, $k \in \{1, \ldots, n\}$, we recapture the result from [3, Theorem 5.21] that has been obtained from a reverse inequality due to Shisha and Mond [8].

5. FURTHER REVERSES OF THE CAUCHY–BUNYAKOVSKY–SCHWARZ INEQUALITY

The following result holds.

**Theorem 6.** Let $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ and $r > 0$ such that for $p_i \geq 0$ with $\sum_{i=1}^{n} p_i = 1$
\begin{equation}
\sum_{i=1}^{n} p_i |b_i|^2 - \bar{a_i} |a_i|^2 \leq r^2 < \sum_{i=1}^{n} p_i |a_i|^2.
\end{equation}
Then we have the inequality
\begin{equation}
0 \leq \sum_{i=1}^{n} p_i |a_i|^2 \sum_{i=1}^{n} p_i |b_i|^2 - \left| \sum_{i=1}^{n} p_i a_i b_i \right|^2 \\
\leq \sum_{i=1}^{n} p_i |a_i|^2 \sum_{i=1}^{n} p_i |b_i|^2 - \left[ \text{Re} \left( \sum_{i=1}^{n} p_i a_i b_i \right) \right]^2 \\
\leq r^2 \sum_{i=1}^{n} p_i |b_i|^2.
\end{equation}
The constant $c = 1$ in front of $r^2$ is best possible in the sense that it cannot be replaced by a smaller constant.

**Proof:** From the first condition in (5.1), we have
\begin{equation}
\sum_{i=1}^{n} p_i \left[ |b_i|^2 - 2 \text{Re}(b_i a_i) + |a_i|^2 \right] \leq r^2,
\end{equation}
giving
\begin{equation}
\sum_{i=1}^{n} p_i |b_i|^2 + \sum_{i=1}^{n} p_i |a_i|^2 - r^2 \leq 2 \text{Re} \left( \sum_{i=1}^{n} p_i a_i b_i \right).
\end{equation}
Since, by the second condition in (5.1) we have
\[ \sum_{i=1}^{n} p_i |a_i|^2 - r^2 > 0, \]
we may divide (5.3) by \( \sqrt{\sum_{i=1}^{n} p_i |a_i|^2 - r^2} > 0 \), getting
\[ \frac{\sum_{i=1}^{n} p_i |b_i|^2}{\sqrt{\sum_{i=1}^{n} p_i |a_i|^2 - r^2}} + \sqrt{\sum_{i=1}^{n} p_i |a_i|^2 - r^2} \leq \frac{2 \Re(\sum_{i=1}^{n} p_i a_i b_i)}{\sqrt{\sum_{i=1}^{n} p_i |a_i|^2 - r^2}}. \]

Utilising the elementary inequality
\[ \frac{p}{\alpha} + q\alpha \geq 2\sqrt{pq} \quad \text{for} \quad p, q \geq 0 \quad \text{and} \quad \alpha > 0, \]
we may write that
\[ 2 \sqrt{\sum_{i=1}^{n} p_i |b_i|^2} \leq \frac{\sum_{i=1}^{n} p_i |b_i|^2}{\sqrt{\sum_{i=1}^{n} p_i |a_i|^2 - r^2}} + \sqrt{\sum_{i=1}^{n} p_i |a_i|^2 - r^2}. \]

Combining (5.5) with (5.6) we deduce
\[ \sqrt{\sum_{i=1}^{n} p_i |b_i|^2} \leq \frac{\Re(\sum_{i=1}^{n} p_i a_i b_i)}{\sqrt{\sum_{i=1}^{n} p_i |a_i|^2 - r^2}}. \]

Taking the square in (5.7), we obtain
\[ \sum_{i=1}^{n} p_i |b_i|^2 \left( \sum_{i=1}^{n} p_i |a_i|^2 - r^2 \right) \leq \left[ \Re\left( \sum_{i=1}^{n} p_i a_i b_i \right) \right]^2, \]
giving the third inequality in (5.2).

The other inequalities are obvious.

To prove the sharpness of the constant, assume, under the hypothesis of the theorem, that there exists a constant \( c > 0 \) such that
\[ \sum_{i=1}^{n} p_i |a_i|^2 \sum_{i=1}^{n} p_i |b_i|^2 - \left[ \Re\left( \sum_{i=1}^{n} p_i a_i b_i \right) \right]^2 \leq cr^2 \sum_{i=1}^{n} p_i |b_i|^2, \]
provided
\[ \sum_{i=1}^{n} p_i |b_i - \overline{a}_i|^2 \leq r^2 < \sum_{i=1}^{n} p_i |a_i|^2. \]

Let \( r = \sqrt{\varepsilon} \), \( \varepsilon \in (0, 1) \), \( a_i, e_i \in \mathbb{C} \), \( i \in \{1, \ldots, n\} \) with \( \sum_{i=1}^{n} p_i |a_i|^2 = \sum_{i=1}^{n} p_i |e_i|^2 = 1 \) and \( \sum_{i=1}^{n} p_i a_i e_i = 0. \) Put \( b_i = \overline{a}_i + \sqrt{\varepsilon} e_i. \) Then, obviously
\[ \sum_{i=1}^{n} p_i |b_i - \overline{a}_i|^2 = r^2, \quad \sum_{i=1}^{n} p_i |a_i|^2 = 1 > r \]
and
\[ \sum_{i=1}^{n} p_i |b_i|^2 = \sum_{i=1}^{n} p_i |a_i|^2 + \varepsilon \sum_{i=1}^{n} p_i |e_i|^2 = 1 + \varepsilon, \]
\[ \Re \left( \sum_{i=1}^{n} p_i a_i b_i \right) = \sum_{i=1}^{n} p_i |a_i|^2 = 1 \]
and thus
\[ \sum_{i=1}^{n} p_i |a_i|^2 \sum_{i=1}^{n} p_i |b_i|^2 - \left[ \Re \left( \sum_{i=1}^{n} p_i a_i b_i \right) \right]^2 = \varepsilon. \]
Using (5.8), we may write
\[ \varepsilon \leq c\varepsilon (1 + \varepsilon) \quad \text{for} \quad \varepsilon \in (0, 1), \]
giving \( 1 \leq c(1 + \varepsilon) \) for \( \varepsilon \in (0, 1) \). Making \( \varepsilon \to 0^+ \), we deduce \( c \geq 1 \).

The following result also holds.

**Theorem 7.** Let \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{K}^n, p = (p_1, \ldots, p_n) \in \mathbb{R}_+^n \) with \( \sum_{i=1}^{n} p_i = 1 \) and \( \gamma, \Gamma \in \mathbb{K} \) such that \( \Re(\gamma \Gamma) > 0 \) and either
\[ \gamma \geq \frac{p_1 \Re(\Gamma)}{2}, \]
\[ \gamma \leq \frac{p_1 \Re(\Gamma)}{2}. \]
Then we have the inequalities
\[ \sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i |y_i|^2 \leq \frac{1}{4} \cdot \frac{\Re((\Gamma + \gamma) \sum_{i=1}^{n} p_i x_i y_i)}{\Re(\gamma \Gamma)} \]
\[ \leq \frac{1}{4} \cdot \frac{\Re(\gamma \Gamma) \sum_{i=1}^{n} p_i x_i y_i}{\Re(\gamma \Gamma)} \]
The constant \( 1/4 \) is best possible in both inequalities.

**Proof:** Define \( b_i = x_i \) and \( a_i = (\Gamma + \gamma)/2 \cdot y_i \) and \( r = |\Gamma - \gamma|/2 \left( \sum_{i=1}^{n} p_i |y_i|^2 \right)^{1/2} \).
Then, by (5.10)
\[ \sum_{i=1}^{n} p_i |b_i - a_i|^2 = \sum_{i=1}^{n} p_i \left| x_i - \frac{\gamma + \Gamma}{2} \cdot y_i \right|^2 \]
\[ \leq \frac{1}{4} |\Gamma - \gamma|^2 \sum_{i=1}^{n} p_i |y_i|^2 = r^2, \]
showing that the first condition in (5.1) is satisfied.

We also have
\[ \sum_{i=1}^{n} p_i |a_i|^2 - r^2 = \sum_{i=1}^{n} p_i \left| \frac{\Gamma + \gamma}{2} \right|^2 |y_i|^2 - \frac{1}{4} |\Gamma - \gamma|^2 \sum_{i=1}^{n} p_i |y_i|^2 \]
\[ = \text{Re}(\Gamma \gamma) \sum_{i=1}^{n} p_i |y_i|^2 > 0 \]
since \( \text{Re}(\gamma \Gamma) > 0 \), and thus the condition in (5.1) is also satisfied.

Using the second inequality in (5.2), one may write
\[ \sum_{i=1}^{n} p_i \left| \frac{\Gamma + \gamma}{2} \right|^2 |y_i|^2 \sum_{i=1}^{n} p_i |x_i|^2 - \left[ \text{Re} \sum_{i=1}^{n} p_i \left( \frac{\Gamma + \gamma}{2} \right) y_i x_i \right]^2 \]
\[ \leq \frac{1}{4} |\Gamma - \gamma|^2 \sum_{i=1}^{n} p_i |y_i|^2 \sum_{i=1}^{n} p_i |x_i|^2, \]
giving
\[ \frac{|\Gamma + \gamma|^2 - |\Gamma - \gamma|^2}{4} \sum_{i=1}^{n} p_i |y_i|^2 \sum_{i=1}^{n} p_i |x_i|^2 \leq \frac{1}{4} \text{Re} \left[ (\Gamma + \gamma) \sum_{i=1}^{n} p_i x_i y_i \right]^2, \]
which is clearly equivalent to the first inequality in (5.11).

The second inequality in (5.11) is obvious.

To prove the sharpness of the constant \( 1/4 \), assume that the first inequality in (5.11) holds with a constant \( C > 0 \), that is,
\[ \sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i |y_i|^2 \leq C \cdot \left( \frac{\text{Re}((\Gamma + \gamma) \sum_{i=1}^{n} p_i x_i y_i)}{\text{Re}(\Gamma \gamma)} \right)^2, \]
provided \( \text{Re}(\gamma \Gamma) > 0 \) and either (5.9) or (5.10) holds.

Assume that \( \Gamma, \gamma > 0 \) and let \( x_i = \gamma \gamma_i \). Then (5.9) holds true and by (5.12) we deduce
\[ \gamma^2 \left( \sum_{i=1}^{n} p_i |y_i|^2 \right)^2 \leq C \left( \frac{(\Gamma + \gamma)^2}{\Gamma \gamma} \right) \left( \sum_{i=1}^{n} p_i |y_i|^2 \right)^2, \]
giving
\[ \Gamma \gamma \leq C (\Gamma + \gamma)^2 \text{ for any } \Gamma, \gamma > 0. \]
Let \( \varepsilon \in (0, 1) \) and choose in (5.13) \( \Gamma = 1 + \varepsilon, \gamma = 1 - \varepsilon > 0 \) to get \( 1 - \varepsilon^2 \leq 4C \) for any \( \varepsilon \in (0, 1) \). Letting \( \varepsilon \to 0+ \), we deduce \( C \geq 1/4 \) and the sharpness of the constant is proved.

Finally, we note that the conditions (5.9) and (5.10) are equivalent since in an inner product space \( (H, \langle \cdot, \cdot \rangle) \) for any vectors \( x, z, Z \in H \) one has \( \text{Re}(Z - x, x - z) \geq 0 \) if and only if \( \|x - (z + Z)/2\| \leq \|Z - z\|/2 \) [1]. We omit the details.
6. More Reverses of the Cauchy–Bunyakovsky–Schwarz Inequality

The following result holds.

**Theorem 8.** Let \( \mathbf{a} = (a_1, \ldots, a_n) \), \( \mathbf{b} = (b_1, \ldots, b_n) \) \( \in \mathbb{K}^n \) and \( \mathbf{p} = (p_1, \ldots, p_n) \) \( \in \mathbb{R}_+^n \) with \( \sum_{i=1}^n p_i = 1 \). If \( r > 0 \) and the following condition is satisfied

\[
\sum_{i=1}^n p_i |b_i - \bar{a}_i|^2 \leq r^2,
\]

then we have the inequalities

\[
0 \leq \left( \sum_{i=1}^n p_i |b_i|^2 \sum_{i=1}^n p_i |a_i|^2 \right)^{1/2} - \sum_{i=1}^n p_i a_i b_i 
\leq \left( \sum_{i=1}^n p_i |b_i|^2 \sum_{i=1}^n p_i |a_i|^2 \right)^{1/2} - \sum_{i=1}^n p_i \text{Re}(a_i b_i) 
\leq \left( \sum_{i=1}^n p_i |b_i|^2 \sum_{i=1}^n p_i |a_i|^2 \right)^{1/2} - \sum_{i=1}^n p_i \text{Re}(a_i b_i) 
\leq \frac{1}{2} r^2.
\]

The constant \( 1/2 \) is best possible in (6.2) in the sense that it cannot be replaced by a smaller constant.

**Proof:** The condition (6.1) is clearly equivalent to

\[
\sum_{i=1}^n p_i |b_i|^2 + \sum_{i=1}^n p_i |a_i|^2 \leq 2 \sum_{i=1}^n p_i \text{Re}(b_i a_i) + r^2.
\]

Using the elementary inequality

\[
2 \left( \sum_{i=1}^n p_i |b_i|^2 \sum_{i=1}^n p_i |a_i|^2 \right)^{1/2} \leq \sum_{i=1}^n p_i |b_i|^2 + \sum_{i=1}^n p_i |a_i|^2
\]

and (6.3), we deduce

\[
2 \left( \sum_{i=1}^n p_i |b_i|^2 \sum_{i=1}^n p_i |a_i|^2 \right)^{1/2} \leq 2 \sum_{i=1}^n p_i \text{Re}(b_i a_i) + r^2,
\]
giving the last inequality in (6.2). The other inequalities are obvious.

To prove the sharpness of the constant \( 1/2 \), assume that

\[
0 \leq \left( \sum_{i=1}^n p_i |b_i|^2 \sum_{i=1}^n p_i |a_i|^2 \right)^{1/2} - \sum_{i=1}^n p_i \text{Re}(b_i a_i) \leq cr^2
\]
for any \( a, b \in \mathbb{K}^n \) and \( r > 0 \) satisfying (6.1).

Assume that \( a, e \in H, e = (e_1, \ldots, e_n) \) with \( \sum \limits_{i=1}^{n} p_i |a_i|^2 = \sum \limits_{i=1}^{n} p_i |e_i|^2 = 1 \) and \( \sum \limits_{i=1}^{n} p_i a_i e_i = 0 \). If \( r = \sqrt{\varepsilon} \), \( \varepsilon > 0 \), and if we define \( b = \bar{a} + \sqrt{\varepsilon} e \) where \( \bar{a} = (\bar{a}_1, \ldots, \bar{a}_n) \in \mathbb{K}^n \), then \( \sum \limits_{i=1}^{n} p_i |b_i - \bar{a}_i|^2 = \varepsilon = r^2 \), showing that the condition (6.1) is satisfied.

On the other hand,

\[
\left( \sum \limits_{i=1}^{n} p_i |b_i|^2 \sum \limits_{i=1}^{n} p_i |a_i|^2 \right)^{1/2} - \sum \limits_{i=1}^{n} p_i \text{Re}(b_i a_i)
\]

\[
= \left( \sum \limits_{i=1}^{n} p_i |\bar{a}_i + \sqrt{\varepsilon} e_i|^2 \right)^{1/2} - \sum \limits_{i=1}^{n} p_i \text{Re}[(\bar{a}_i + \sqrt{\varepsilon} e_i)a_i]
\]

\[
= \left( \sum \limits_{i=1}^{n} p_i |a_i|^2 + \varepsilon \sum \limits_{i=1}^{n} |e_i|^2 \right)^{1/2} - \sum \limits_{i=1}^{n} p_i |a_i|^2
\]

\[
= \sqrt{1 + \varepsilon} - 1.
\]

Utilising (6.6), we conclude that

(6.7)

\[ \sqrt{1 + \varepsilon} - 1 \leq c \varepsilon \] for any \( \varepsilon > 0 \).

Multiplying (6.7) by \( \sqrt{1 + \varepsilon} + 1 > 0 \) and thus dividing by \( \varepsilon > 0 \), we get

(6.8)

\[ (\sqrt{1 + \varepsilon} - 1) c \geq 1 \] for any \( \varepsilon > 0 \).

Letting \( \varepsilon \to 0+ \) in (6.8), we deduce \( c \geq 1/2 \), and the theorem is proved.

Finally, the following result also holds.

**Theorem 9.** Let \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{K}^n, p = (p_1, \ldots, p_n) \in \mathbb{R}_+^n \) with \( \sum \limits_{i=1}^{n} p_i = 1 \), and \( \gamma, \Gamma \in \mathbb{K} \) with \( \Gamma \neq \gamma, -\gamma \), so that either

(6.9)

\[ \sum \limits_{i=1}^{n} p_i \text{Re}[(\Gamma y_i - x_i)(\bar{x}_i - \bar{\gamma} y_i)] \geq 0, \]

or, equivalently,

(6.10)

\[ \sum \limits_{i=1}^{n} p_i |\bar{x}_i - \bar{\gamma} + \frac{\Gamma}{2} | y_i |^2 \leq \frac{1}{4} |\Gamma - \gamma|^2 \sum \limits_{i=1}^{n} p_i |y_i|^2 \]

holds. Then we have the inequalities

(6.11)

\[ 0 \leq \left( \sum \limits_{i=1}^{n} p_i |x_i|^2 \sum \limits_{i=1}^{n} p_i |y_i|^2 \right)^{1/2} - \sum \limits_{i=1}^{n} p_i x_i y_i \]

\[
\leq \left( \sum \limits_{i=1}^{n} p_i |x_i|^2 \sum \limits_{i=1}^{n} p_i |y_i|^2 \right)^{1/2} - \sum \limits_{i=1}^{n} p_i \text{Re} \left[ \frac{\bar{x}_i + \bar{\gamma}}{|\bar{x}_i + \bar{\gamma}|} x_i y_i \right].
\]
Reverses of the Cauchy–Bunyakovsky–Schwarz inequality

The constant \(1/4\) in the last inequality is best possible.

**Proof:** Consider \(b_i = x_i, a_i = (\Gamma + \gamma)/2 \cdot y_i, i \in \{1, \ldots, n\}\) and

\[
r := \frac{1}{2} |\Gamma - \gamma| \left( \sum_{i=1}^{n} p_i |y_i|^2 \right)^{1/2}.
\]

Then, by (6.10), we have

\[
\sum_{i=1}^{n} p_i |b_i - a_i|^2 = \sum_{i=1}^{n} p_i \left| x_i - \frac{\gamma + \Gamma}{2} \cdot y_i \right|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2 \sum_{i=1}^{n} p_i |y_i|^2 = r^2
\]

showing that (6.1) is valid.

By the use of the last inequality in (6.2), we have

\[
0 \leq \left( \sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i \left| \frac{\Gamma + \gamma}{2} \right| |y_i|^2 \right)^{1/2} - \sum_{i=1}^{n} p_i \text{Re}\left[ \frac{\Gamma + \gamma}{2} x_i y_i \right]
\]

\[
\leq \frac{1}{8} |\Gamma - \gamma|^2 \sum_{i=1}^{n} p_i |y_i|^2.
\]

Dividing by \(|\Gamma + \gamma|/2 > 0\), we deduce

\[
0 \leq \left( \sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i |y_i|^2 \right)^{1/2} - \sum_{i=1}^{n} p_i \text{Re}\left[ \frac{\Gamma + \gamma}{|\Gamma + \gamma|} x_i y_i \right]
\]

\[
\leq \frac{1}{4} \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \sum_{i=1}^{n} p_i |y_i|^2,
\]

which is the last inequality in (6.11).

The other inequalities are obvious.

To prove the sharpness of the constant \(1/4\), assume that there exists a constant \(c > 0\), such that

\[
(6.12) \quad \left( \sum_{i=1}^{n} p_i |x_i|^2 \sum_{i=1}^{n} p_i |y_i|^2 \right)^{1/2} - \sum_{i=1}^{n} p_i \text{Re}\left[ \frac{\Gamma + \gamma}{|\Gamma + \gamma|} x_i y_i \right] \leq c \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \sum_{i=1}^{n} p_i |y_i|^2,
\]

provided either (6.9) or (6.10) holds.

Let \(n = 2, y = (1, 1), x = (x_1, x_2) \in \mathbb{R}^2, p = (1/2, 1/2)\) and \(\Gamma, \gamma > 0\) with \(\Gamma > \gamma\). Then by (6.12) we deduce

\[
(6.13) \quad \sqrt{2} \sqrt{x_1^2 + x_2^2} - (x_1 + x_2) \leq 2 c \frac{(\Gamma - \gamma)^2}{\Gamma + \gamma}.
\]
If $x_1 = \Gamma$, $x_2 = \gamma$, then $(\Gamma - x_1)(x_1 - \gamma) + (\Gamma - x_2)(x_2 - \gamma) = 0$, showing that the condition (6.9) is valid for $n = 2$ and $p, x, y$ as above. Replacing $x_1$ and $x_2$ in (6.13), we deduce

\[(6.14) \quad \sqrt{2}\sqrt{\Gamma^2 + \gamma^2} - (\Gamma + \gamma) \leq \frac{2c(\Gamma - \gamma)^2}{\Gamma + \gamma}.
\]

If in (6.14) we choose $\Gamma = 1 + \varepsilon$, $\gamma = 1 - \varepsilon$ with $\varepsilon \in (0, 1)$, we deduce

\[(6.15) \quad \sqrt{1 + \varepsilon^2} - 1 \leq 2c\varepsilon^2.
\]

Finally, multiplying (6.15) with $\sqrt{1 + \varepsilon^2} + 1 > 0$ and then dividing by $\varepsilon^2$, we deduce

\[(6.16) \quad 1 \leq 2c(\sqrt{1 + \varepsilon^2} + 1) \text{ for any } \varepsilon > 0.
\]

Letting $\varepsilon \to 0^+$ in (6.16), we get $c \geq 1/4$, and the sharpness of the constant is proved. □

**Remark 5.** The integral version may be stated in a canonical way. The corresponding inequalities for integrals will be considered in another work devoted to positive linear functionals with complex values that is in preparation.

**References**


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