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L^q-SPECTRUM OF SELF-SIMILAR MEASURES WITH OVERLAPS IN THE ABSENCE OF SECOND-ORDER IDENTITIES

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Abstract

For the class of self-similar measures in $\mathbb{R}^d$ with overlaps that are *essentially of finite type*, we set up a framework for deriving a closed formula for the L^q-spectrum of the measure for $q \geq 0$. This framework allows us to include iterated function systems that have different contraction ratios and those in higher dimension. For self-similar measures with overlaps, closed formulas for the L^q-spectrum have only been obtained earlier for measures satisfying Strichartz's second-order identities. We illustrate how to use our results to prove the differentiability of the L^q-spectrum, obtain the multifractal dimension spectrum, and compute the Hausdorff dimension of the measure.

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1. Introduction

Let μ be a bounded positive Borel measure on $\mathbb{R}^d$ whose support $\mathrm{supp}(\mu)$ is compact. For $q \in \mathbb{R}$, the L^q*-spectrum* $\tau(q)$ of μ is defined as

$$\tau(q) := \varliminf_{\delta \to 0^+} \frac{\ln \sup \sum_i \mu(B_\delta(x_i))^q}{\ln \delta},$$

where $B_\delta(x_i)$ is a disjoint family of δ-balls with centers $x_i \in \mathrm{supp}(\mu)$ and the supremum is taken over all such families. The function $\tau(q)$ arises in the theory of multifractal

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decomposition of measures. A major goal of the theory is to compute the following *dimension spectrum*:

$$f(\alpha) := \dim_{\mathrm{H}}\Big\{x \in \mathrm{supp}(\mu) : \lim_{\delta\to 0^+} \frac{\ln \mu(B_\delta(x))}{\ln \delta} = \alpha\Big\},$$

where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension. The multifractal formalism, a heuristic principle first proposed by physicists (see [7, 8] and the references therein), asserts that the dimension spectrum is equal to the Legendre transform of $\tau(q)$, that is,

$$f(\alpha) = \tau^*(\alpha) := \inf\{q\alpha - \tau(q) : \ q \in \mathbb{R}\}.$$

We are mainly interested in self-similar measures. For such measures, the multifractal formalism has been verified rigorously for those satisfying the separated open set condition [1, 3]. For self-similar measures defined by iterated function systems satisfying the weak separation condition, Lau and the first author [13] proved that if $\tau(q)$ is differentiable at $q \geq 0$, then the multifractal formalism at the corresponding point holds. Feng and Lau [5] removed the differentiability condition; they also studied the validity of the multiformal formalism in the region $q < 0$.

The L^q-spectrum also encodes other important information of the measure. For example, $\tau(0)$ is the negative of the box dimension of the corresponding self-similar set; if τ is differentiable at $q = 1$, then $\tau'(1)$ is equal to the Hausdorff dimension of μ (see [9, 13, 19, 23] and the references therein); for $q > 1$, $\tau(q)/(q-1)$ is the L^q-dimension of μ (see [24]).

The computation of L^q-spectrum thus plays a key role in the theory of multifractal measures. For self-similar and graph-directed self-similar measures satisfying the open set condition, $\tau(q)$ is computed by Cawley and Mauldin [1] and Edgar and Mauldin [3]. For self-similar measures with overlaps, the computation is much more difficult. Lau and the first author obtained $\tau(q)$, $q \geq 0$, for the infinite Bernoulli convolution associated with the golden ratio [12] and a class of convolutions of Cantor measures [14]. Feng [4] computed $\tau(q)$ for infinite Bernoulli convolutions associated with a class of Pisot numbers. The graph of $\tau(q)$ for $q < 0$ has been studied in [4, 6, 18].

The computation of $\tau(q)$ in [12] and [14] makes use of Strichartz's second-order self-similar identities. Unfortunately, very few self-similar measures satisfy these identities. Thus, closed formulas for $\tau(q)$ have been obtained for only a few classes of measures that are defined by iterated function systems on $\mathbb{R}$ with the same contraction ratio. The main objective of this paper is to derive a closed formula for $\tau(q)$, $q \geq 0$, for self-similar measures that are so-called *essentially of finite type* (EFT), a condition introduced in [21]. We recall the definition of EFT in Definition 2.16. It is worth mentioning that recently Deng and the first author [2] used an infinite matrix method to obtain the differentiability of the L^q-spectrum for a class of iterated function systems (IFSs) that includes some of those studies in this paper; however, the method does not yield a closed formula for $\tau(q)$.

Throughout this paper an IFS refers to a finite family of contractions defined on a compact subset X of $\mathbb{R}^d$. The derivation of $\tau(q)$ in this paper is based on the following equivalent definition, which holds for $q \geq 0$:

$$\begin{aligned}\tau(q) &= \inf\Big\{\alpha \geq 0 : \overline{\lim_{\delta\to 0^+}} \frac{1}{\delta^{d+\alpha}} \int_X \mu(B_\delta(x))^q\,dx > 0\Big\} \\ &= \sup\Big\{\alpha \geq 0 : \overline{\lim_{\delta\to 0^+}} \frac{1}{\delta^{d+\alpha}} \int_X \mu(B_\delta(x))^q\,dx < \infty\Big\},\end{aligned} \tag{1.1}$$

(see [11, 12] and [13, Proposition 3.1]).

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open subset and μ be a positive finite Borel measure with $\mathrm{supp}(\mu) \subseteq \overline{\Omega}$ and $\mu(\Omega) > 0$. We say that two subsets U and V of Ω with positive μ-measure are *μ-equivalent* if $\mu|_V = w\mu|_U \circ \sigma^{-1}$ for some $w > 0$ and some similitude $\sigma : U \to V$, where $\mu|_F$ denotes the restriction of the measure μ to $F \subseteq \mathbb{R}^d$. A *μ-partition* $\mathbf{P}$ of U is a finite family of measure disjoint sub-cells of U such that $\mu(U) = \sum_{V\in\mathbf{P}} \mu(V)$. A sequence of μ-partitions $\{\mathbf{P}_k\}_{k\geq 1}$ is *refining* if each member of $\mathbf{P}_{k+1}$ is a subset of some member of $\mathbf{P}_k$.

Our main assumption is the EFT condition introduced in [21], which, loosely speaking, holds if there exists some bounded open subset $\Omega \subseteq \mathbb{R}^d$ with $\mathrm{supp}(\mu) \subseteq \overline{\Omega}$ and $\mu(\Omega) > 0$, together with a finite family $\mathbf{B} := \{B_{1,\ell}\}_{\ell\in\Gamma}$ of cells in Ω such that for each $\ell \in \Gamma$, there is a family of refining μ-partition $\{\mathbf{P}_{k,\ell}\}_{k\geq 1}$ satisfying the following conditions: (1) there exists some cell $B \in \mathbf{P}_{2,\ell}$ that is not in $P_{1,\ell}$ such that B has the same measure type with some cell in $\mathbf{B}$; (2) $\mathbf{P}_{k+1,\ell}$ contains all cells in $\mathbf{P}_{k,\ell}$ that have the same measure type with some cells in $\mathbf{B}$ for $k \geq 2$; (3) the sum of the μ-measures of those cells $\mathbf{B} \in \mathbf{P}_{k,\ell}$ that are not μ-equivalent to any cell in $\mathbf{B}$ tends to 0 as $k\to\infty$. In this case, we call Ω an *EFT-set*, $\mathbf{B}$ a *basic family of cells* in Ω, and $(\mathbf{B}, \mathbf{P}) := (\{B_{1,\ell}\}, \{\mathbf{P}_{k,\ell}\}_{k\geq 1})_{\ell\in\Gamma}$ a *basic pair* with respect to Ω. We say that $(\mathbf{B}, \mathbf{P})$ is *weakly regular* if for any $\ell \in \Gamma$, there exists some similitude σ_ℓ such that $\sigma_\ell(\Omega) \subseteq B_{1,\ell}$.

Let μ be a self-similar measure defined by a finite type IFS $\{S_i\}_{i\in\Lambda}$ on $\mathbb{R}^d$ (see [10, 15, 22]) with Ω being a finite type condition set. Assume that μ satisfies EFT with $\Omega \subseteq \mathbb{R}^d$ being an EFT-set and $(\mathbf{B}, \mathbf{P}) := (\{B_{1,\ell}\}, \{\mathbf{P}_{k,\ell}\}_{k\geq 1})_{\ell\in\Gamma}$ being a weakly regular basic pair with respect to Ω. Fix $q \geq 0$, define

$$\varphi_\ell(\delta) := \int_{B_{1,\ell}} \mu(B_\delta(x))^q\,dx, \quad \Phi_\ell^{(\alpha)}(\delta) := \frac{1}{\delta^{d+\alpha}} \varphi_\ell(\delta) \quad \text{for } \ell \in \Gamma. \tag{1.2}$$

Then we can derive renewal equations for $\Phi_\ell^{(\alpha)}(\delta)$, and express them in vector form as:

$$\mathbf{f} = \mathbf{f} * \mathbf{M}_\alpha + \mathbf{z},$$

where $\alpha \in \mathbb{R}$, and

$$\begin{aligned} \mathbf{f} = \mathbf{f}^{(\alpha)}(x) &= [f_\ell^{(\alpha)}(x)]_{\ell\in\Gamma}, \quad x \in \mathbb{R}; \\ f_\ell^{(\alpha)}(x) &:= \Phi_\ell^{(\alpha)}(e^{-x}) \quad \text{for } \ell \in \Gamma; \\ \mathbf{M}_\alpha = [\mu_{m\ell}^{(\alpha)}]_{\ell,m\in\Gamma} &\quad \text{is a finite matrix of Borel measures on } \mathbb{R}; \\ \mathbf{z} = \mathbf{z}^{(\alpha)}(x) = [z_\ell^{(\alpha)}(x)]_{\ell\in\Gamma} &\quad \text{is a vector of error functions.} \end{aligned} \tag{1.3}$$

Let

$$\mathbf{M}_\alpha(\infty) := [\mu_{m\ell}^{(\alpha)}(\mathbb{R})]_{\ell,m\in\Gamma}. \tag{1.4}$$

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For each $\ell \in \Gamma$ and $\alpha \in \mathbb{R}$, define

$$F_\ell(\alpha) := \sum_{m\in\Gamma} \mu_{m\ell}^{(\alpha)}(\mathbb{R}), \quad D_\ell := \{\alpha \in \mathbb{R} : F_\ell(\alpha) < \infty\}. \tag{1.5}$$

If the error functions decay exponentially to 0 as $x \to \infty$, then the L^q-spectrum of μ is given by the unique α such that the spectral radius of $\mathbf{M}_\alpha(\infty)$ is equal to 1. The following is our main result.

THEOREM 1.1. *Let μ be a self-similar measure defined by a finite type IFS $\{S_i\}_{i\in\Lambda}$ on $\mathbb{R}^d$. Assume that μ satisfies EFT with Ω being an EFT-set and $(\mathbf{B},\mathbf{P}) := (\{B_{1,\ell}\}, \{\mathbf{P}_{k,\ell}\}_{k\geq 1})_{\ell\in\Gamma}$ being a weakly regular basic pair with respect to Ω. Let $\mathbf{M}_\alpha(\infty)$ and $F_\ell(\alpha)$ be defined as in (1.4) and (1.5).*

(1) *There exists a unique $\alpha \in \mathbb{R}$ such that the spectral radius of $\mathbf{M}_\alpha(\infty)$ is equal to 1.*
(2) *If we assume, in addition, that for the unique α in (a), there exists $\epsilon > 0$ such that for all $\ell \in \Gamma$, $z_\ell^{(\alpha)}(x) = o(e^{-\epsilon x})$ as $x \to \infty$, then $\tau(q) = \alpha$ for $q \geq 0$.*

In Section 4, we illustrate Theorem 1.1 by the following family of IFSs on $\mathbb{R}$:

$$S_1(x) = \rho x, \quad S_2(x) = rx + \rho(1-r), \quad S_3(x) = rx + 1 - r, \tag{1.6}$$

where the contraction ratios $\rho, r \in (0,1)$ satisfy

$$\rho + 2r - \rho r \leq 1, \tag{1.7}$$

that is, $S_2(1) \leq S_3(0)$ (see Figure 1). This family of IFSs is first studied by Lau and Wang [17], and is used to illustrate the (general) finite type condition in [10, 15]. For a probability vector $(p_i)_{i=1}^3$, we define

$$w_1(k) := p_1 \sum_{j=0}^{k} p_2^{k-j} p_3^j \quad \text{for } k \geq 0. \tag{1.8}$$

THEOREM 1.2. *Let μ be a self-similar measure defined by an IFS in (1.6) together with a probability vector $(p_i)_{i=1}^3$, and $w_1(k)$ be defined as in (1.8). Then for $q \geq 0$, there exists a unique real number $\alpha := \alpha(q)$ satisfying*

$$\rho^{-\alpha}(1 - p_2^q r^{-\alpha})(1 - p_3^q r^{-\alpha}) \sum_{k=0}^{\infty} w_1(k)^q (r^{-\alpha})^k + r^{-\alpha}(p_2^q + p_3^q) = 1. \tag{1.9}$$

Hence $\tau(q) = \alpha$. Moreover, τ is differentiable on $(0,\infty)$ and

$$\begin{aligned}\dim_{\mathrm{H}}(\mu) = \tau'(1) = \Bigg(\bigg(\sum_{i=2}^{3} p_i \ln p_i - p_2 p_3 \sum_{i=2}^{3} \ln p_i\bigg) \sum_{k=0}^{\infty} w_1(k) \\ - \bigg(\prod_{i=2}^{3}(1-p_i)\bigg) \sum_{k=0}^{\infty} w_1(k) \ln w_1(k) - \sum_{i=2}^{3} p_i \ln p_i\Bigg)\end{aligned}$$

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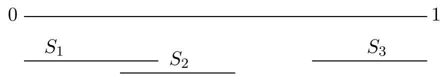

FIGURE 1. The first iteration of $\{S_i\}_{i=1}^3$ defined in (1.6). The figure is drawn with $\rho = 1/3$ and $r = 2/7$.

$$\times\bigg((p_2 + p_3 - 2p_2p_3)\sum_{k=0}^{\infty} w_1(k)\ln r - \bigg(\prod_{i=2}^{3}(1-p_i)\bigg)\sum_{k=0}^{\infty} w_1(k)\ln(\rho r^k) - \sum_{i=2}^{3} p_i \ln r\bigg)^{-1}.$$

REMARK 1.3. Substituting $q = 0$ in (1.9) gives $\rho^{-\tau(0)} + 2r^{-\tau(0)} - (\rho r)^{-\tau(0)} = 1$. Hence $-\tau(0)$ equals the Hausdorff and box dimensions of the corresponding self-similar set (see [10, 15, 17]).

In Section 5, we illustrate Theorem 1.1 by the following family of IFSs on $\mathbb{R}^2$:

$$\begin{aligned} S_1(\mathbf{x}) = \rho\mathbf{x}, \quad S_2(\mathbf{x}) = r\mathbf{x} + (\rho - \rho r, 0), \\ S_3(\mathbf{x}) = r\mathbf{x} + (1 - r, 0), \quad S_4(\mathbf{x}) = r\mathbf{x} + (0, 1 - r), \end{aligned} \tag{1.10}$$

where the contraction ratios $\rho, r \in (0, 1)$ satisfy

$$\rho + 2r - \rho r \le 1, \tag{1.11}$$

that is, $S_2(1, 0) \le S_3(0, 0)$ (see Figure 2(a)). For any probability vector $(p_i)_{i=1}^4$, define

$$w_2(k) := p_1 \sum_{j=0}^{k} p_2^{k-j} p_3^{j} \quad \text{for } k \ge 0. \tag{1.12}$$

THEOREM 1.4. *Let μ be a self-similar measure defined by an IFS in (1.10) together with a probability vector $(p_i)_{i=1}^4$, and $w_2(k)$ be defined as in (1.12). Then for $q \ge 0$, there exists a unique real number $\alpha := \alpha(q)$ satisfying*

$$\rho^{-\alpha}(1 - p_2^q r^{-\alpha})(1 - p_3^q r^{-\alpha}) \sum_{k=0}^{\infty} w_2(k)^q (r^{-\alpha})^k + r^{-\alpha} \sum_{i=2}^{4} p_i^q = 1. \tag{1.13}$$

Hence $\tau(q) = \alpha$. Moreover, τ is differentiable on $(0, \infty)$ and

$$\dim_{\mathrm{H}}(\mu) = \tau'(1) = \bigg(\bigg(\sum_{i=2}^{3} p_i \ln p_i - p_2p_3 \sum_{i=2}^{3} \ln p_i\bigg)\sum_{k=0}^{\infty} w_2(k) - \bigg(\prod_{i=2}^{3}(1-p_i)\bigg)\sum_{k=0}^{\infty} w_2(k)\ln w_2(k) - \sum_{i=2}^{4} p_i \ln p_i\bigg)$$

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$$\times\Big((p_2+p_3-2p_2p_3)\sum_{k=0}^{\infty} w_2(k)\ln r$$
$$-\Big(\prod_{i=2}^{3}(1-p_i)\Big)\sum_{k=0}^{\infty} w_2(k)\ln(\rho r^k)-\sum_{i=2}^{4} p_i\ln r\Big)^{-1}.$$

Remark 1.5. Substituting $q=0$ into (1.13), $\rho^{-\tau(0)}+3r^{-\tau(0)}-(\rho r)^{-\tau(0)}=1$. Again, $-\tau(0)$ equals the Hausdorff and box dimensions of the corresponding self-similar set (see [15, Example 5.2]).

We use the vector-valued renewal theorem of Lau *et al.* [16] to derive the stated formulas for $\tau(q)$; the classical renewal theorem used in [12] and [14] is not sufficient, as a finite number of renewal equations arise in our derivations. New techniques are also used in estimating the error terms and in proving the differentiability of $\tau(q)$.

This paper is organized as follows. In Section 2, we briefly recall the definition of EFT. In Section 3 we derive renewal equations and prove Theorem 1.1. Section 4 illustrates Theorem 1.1 by the class of one-dimensional IFSs (1.6) and proves Theorem 1.2. Section 5 studies IFSs in higher dimension and proves Theorem 1.4. Finally we state some comments and open questions in Section 6.

2. Self-similar measures and measures that are essentially of finite type

In this section, we recall the definition of EFT and then prove that it is satisfied by the self-similar measures defined by the IFSs in (1.10).

2.1. The finite type condition and measure type. Let X be a compact subset of $\mathbb{R}^d$ with a nonempty interior, and $\{S_i\}_{i\in\Lambda}$ be an IFS of contractive similitudes on X with attractor $K\subseteq\mathbb{R}^d$. To each probability vector $(p_i)_{i\in\Lambda}$ (that is, $p_i>0$ and $\sum_{i\in\Lambda}p_i=1$), let μ be the associated self-similar measure, which satisfies the *self-similar identity*

$$\mu=\sum_{i\in\Lambda}p_i\mu\circ S_i^{-1}.$$

Moreover, $\operatorname{supp}(\mu)=K$. An IFS $\{S_i\}_{i\in\Lambda}$ is said to satisfy the *open set condition (OSC)* if there exists a nonempty bounded open subset $U\subset\mathbb{R}^d$ such that $\bigcup_{i\in\Lambda}S_i(U)\subset U$ and $S_i(U)\cap S_j(U)=\emptyset$ for all $i\neq j$.

For $k\geq 1$, define

$$\Lambda^k:=\{(i_1,\dots,i_k): i_j\in\Lambda \text{ for } j=1,\dots,k\},$$

where we call $\boldsymbol{i}\in\Lambda^k$ a word of length k, and denote its length by $|\boldsymbol{i}|$. If $k=0$, we define $\Lambda^0:=\{\emptyset\}$. Also, we let $\Lambda^*:=\bigcup_{k\geq0}\Lambda^k$. We frequently write $\boldsymbol{i}:=i_1\cdots i_k$ instead of $\boldsymbol{i}=(i_1,\dots,i_k)$ if no confusion is possible; in particular, we write $\boldsymbol{i}=: i_1^k$, if $i_j=i_1$ for all $j=1,\dots,k$. For $k\geq0$ and $\boldsymbol{i}=i_1\cdots i_k\in\Lambda^k$, we use the standard notation

$$S_{\boldsymbol{i}}:=S_{i_1}\circ\cdots\circ S_{i_k},\qquad r_{\boldsymbol{i}}:=r_{i_1}\cdots r_{i_k},\qquad p_{\boldsymbol{i}}:=p_{i_1}\cdots p_{i_k},$$

with $S_\emptyset:=\mathrm{id}$, $r_\emptyset=p_\emptyset:=1$, where id is the identity map on $\mathbb{R}^d$.

For two indices $\boldsymbol{i},\boldsymbol{j}\in\Lambda^*$, we write $\boldsymbol{i}\preccurlyeq\boldsymbol{j}$ if $\boldsymbol{i}$ is a prefix of $\boldsymbol{j}$ or $\boldsymbol{i}=\boldsymbol{j}$, and denote by $\boldsymbol{i}\not\preccurlyeq\boldsymbol{j}$ if $\boldsymbol{i}\preccurlyeq\boldsymbol{j}$ does not hold. We say that $\boldsymbol{i},\boldsymbol{j}\in\Lambda^*$ are *comparable* if $\boldsymbol{i}\preccurlyeq\boldsymbol{j}$ or $\boldsymbol{j}\preccurlyeq\boldsymbol{i}$. If two elements are not comparable, we say they are *incomparable*; that is, $\boldsymbol{i}$ and $\boldsymbol{j}$ are incomparable if neither $\boldsymbol{i}\preccurlyeq\boldsymbol{j}$ nor $\boldsymbol{j}\preccurlyeq\boldsymbol{i}$. A *chain* in Λ^* is a subset of Λ^* in which each pair of elements is comparable. An *antichain* in Λ^* is a subset of Λ^* in which each pair of distinct elements is incomparable.

Let $\{\mathcal{M}_k\}_{k=1}^{\infty}$ be a sequence of index sets, where $\mathcal{M}_k\subseteq\Lambda^*$. Let

$$\underline{m}_k=\underline{m}_k(\mathcal{M}_k):=\min\{|\boldsymbol{i}|:\boldsymbol{i}\in\mathcal{M}_k\},$$

and

$$\overline{m}_k=\overline{m}_k(\mathcal{M}_k):=\max\{|\boldsymbol{i}|:\boldsymbol{i}\in\mathcal{M}_k\}.$$

We also let $\mathcal{M}_0:=\{\emptyset\}$.

Definition 2.1. We say that $\{\mathcal{M}_k\}_{k=0}^{\infty}$ is a *sequence of nested index sets* if it satisfies the following conditions:

(1) both $\{\underline{m}_k\}$ and $\{\overline{m}_k\}$ are nondecreasing, and

$$\lim_{k\to\infty}\underline{m}_k=\lim_{k\to\infty}\overline{m}_k=\infty;$$

(2) for each $k\ge1$, $\mathcal{M}_k$ is an antichain in Λ^*;

(3) for each $\boldsymbol{j}\in\Lambda^*$ with $|\boldsymbol{j}|>\overline{m}_k$ or $\boldsymbol{j}\in\mathcal{M}_{k+1}$, there exists $\boldsymbol{i}\in\mathcal{M}_k$ such that $\boldsymbol{i}\preccurlyeq\boldsymbol{j}$;

(4) for each $\boldsymbol{j}\in\Lambda^*$ with $|\boldsymbol{j}|<\underline{m}_k$ or $\boldsymbol{j}\in\mathcal{M}_{k-1}$, there exists $\boldsymbol{i}\in\mathcal{M}_k$ such that $\boldsymbol{j}\preccurlyeq\boldsymbol{i}$;

(5) there exists a positive integer L_0, independent of k, such that for all $\boldsymbol{i}\in\mathcal{M}_k$ and $\boldsymbol{j}\in\mathcal{M}_{k+1}$ with $\boldsymbol{i}\preccurlyeq\boldsymbol{j}$, we have $|\boldsymbol{j}|-|\boldsymbol{i}|\le L_0$.

To define neighborhood types, we fix a sequence of nested index sets $\{\mathcal{M}_k\}_{k=0}^{\infty}$.

Notation 2.2.

(1) For each integer $k\ge0$, let $\mathcal{V}_k$ be the set of *level-k vertices* (with respect to $\{\mathcal{M}_k\}$) defined as

$$\mathcal{V}_0:=\{(\mathrm{id},0)\},\quad \mathcal{V}_k:=\{(S_{\boldsymbol{i}},k):\boldsymbol{i}\in\mathcal{M}_k\}\quad\text{for all } k\ge1,$$

we call $(\mathrm{id},0)$ the *root vertex* and denote it by $\boldsymbol{v}_{root}$.

(2) Let $\mathcal{V}:=\bigcup_{k\ge0}\mathcal{V}_k$ be the set of all vertices.

(3) For $\boldsymbol{v}=(S_{\boldsymbol{i}},k)\in\mathcal{V}_k$, we use the convenient notation $S_{\boldsymbol{v}}:=S_{\boldsymbol{i}}$ and $r_{\boldsymbol{v}}:=r_{\boldsymbol{i}}$. It is possible to have $\boldsymbol{v}=(S_{\boldsymbol{i}},k)=(S_{\boldsymbol{j}},k)$ with $\boldsymbol{i}\neq\boldsymbol{j}$.

(4) More generally, for any $k\ge0$ and any subset $\mathcal{A}\subseteq\mathcal{V}_k$, we use the notation

$$S_{\mathcal{A}}(\Omega):=\bigcup_{\boldsymbol{v}\in\mathcal{A}}S_{\boldsymbol{v}}(\Omega).\tag{2.1}$$

Let $\Omega\subseteq X$ be a nonempty bounded open set which is *invariant* under $\{S_i\}_{i\in\Lambda}$, that is, $\bigcup_{i\in\Lambda}S_i(\Omega)\subseteq\Omega$. Such an Ω exists by our assumption; in particular, X° is such a set.

Next, we recall the definitions of neighbors and neighborhoods.

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DEFINITION 2.3. We say that two level-k vertices $\boldsymbol{v}, \boldsymbol{v}' \in \mathcal{V}_k$ (allowing $\boldsymbol{v} = \boldsymbol{v}'$) are *neighbors* (with respect to Ω and $\{\mathcal{M}_k\}$) if $S_{\boldsymbol{v}}(\Omega) \cap S_{\boldsymbol{v}'}(\Omega) \neq \emptyset$. We call the set of vertices

$$\mathfrak{N}_\Omega(\boldsymbol{v}) := \{\boldsymbol{v}' : \boldsymbol{v}' \in \mathcal{V}_k \text{ is a neighbor of } \boldsymbol{v}\}$$

the *neighborhood* of $\boldsymbol{v}$ (with respect to Ω and $\{\mathcal{M}_k\}$).

Obviously $\boldsymbol{v} \in \mathfrak{N}_\Omega(\boldsymbol{v})$. If no confusion is possible, we omit the subscript Ω in $\mathfrak{N}_\Omega(\boldsymbol{v})$.

Let $\mathscr{S} := \{S_{\boldsymbol{j}} S_{\boldsymbol{i}}^{-1} : \boldsymbol{i}, \boldsymbol{j} \in \Lambda^*\}$. We define an equivalence relation on the set of vertices $\mathcal{V}$.

DEFINITION 2.4. Two vertices $\boldsymbol{v} \in \mathcal{V}_k$ and $\boldsymbol{v}' \in \mathcal{V}_{k'}$ are said to be *equivalent*, denoted $\boldsymbol{v} \sim_\sigma \boldsymbol{v}'$ (or simply $\boldsymbol{v} \sim \boldsymbol{v}'$), if for $\sigma := S_{\boldsymbol{v}'} S_{\boldsymbol{v}}^{-1} (\in \mathscr{S}) : \bigcup_{\boldsymbol{u} \in \mathfrak{N}(\boldsymbol{v})} S_{\boldsymbol{u}}(X) \to X$, the following conditions hold:

(1) $\{S_{\boldsymbol{u}'} : \boldsymbol{u}' \in \mathfrak{N}(\boldsymbol{v}')\} = \{\sigma S_{\boldsymbol{u}} : \boldsymbol{u} \in \mathfrak{N}(\boldsymbol{v})\}$; in particular, $\sigma S_{\boldsymbol{u}}$ is defined for all $\boldsymbol{u} \in \mathfrak{N}(\boldsymbol{v})$;

(2) for $\boldsymbol{u} \in \mathfrak{N}(\boldsymbol{v})$ and $\boldsymbol{u}' \in \mathfrak{N}(\boldsymbol{v}')$ such that $S_{\boldsymbol{u}'} = \sigma S_{\boldsymbol{u}}$, and for any positive integer $\ell \geq 1$, an index $\boldsymbol{i} \in \Lambda^*$ satisfies $(S_{\boldsymbol{u}} S_{\boldsymbol{i}}, k + \ell) \in \mathcal{V}_{k+\ell}$ if and only if it satisfies $(S_{\boldsymbol{u}'} S_{\boldsymbol{i}}, k' + \ell) \in \mathcal{V}_{k'+\ell}$.

It is direct to check that $\sim$ is an equivalence relation. We denote the equivalence class containing $\boldsymbol{v}$ by $[\boldsymbol{v}]$ and call it the *(neighborhood) type* of $\boldsymbol{v}$ (with respect to Ω and $\{\mathcal{M}_k\}$).

We define an infinite graph $\mathcal{G}$ with vertex set $\mathcal{V}$ and directed edges $\mathcal{E}$ defined as follows. Let $\boldsymbol{v} \in \mathcal{V}_k$ and $\boldsymbol{u} \in \mathcal{V}_{k+1}$. Suppose there exist $\boldsymbol{i} \in \mathcal{M}_k$, $\boldsymbol{j} \in \mathcal{M}_{k+1}$, and $l \in \Lambda^*$ such that

$$\boldsymbol{v} = (S_{\boldsymbol{i}}, k), \quad \boldsymbol{u} = (S_{\boldsymbol{j}}, k + 1), \quad \boldsymbol{j} = (\boldsymbol{i}, l).$$

Then we connect a directed edge $l : \boldsymbol{v} \to \boldsymbol{u}$. We call $\boldsymbol{v}$ a *parent* of $\boldsymbol{u}$ and $\boldsymbol{u}$ an *offspring* of $\boldsymbol{v}$. We write $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{E}$ is the set of all directed edges defined above. We call $\boldsymbol{v} = (S_{\boldsymbol{i}}, k)$ a *predecessor* of $\boldsymbol{u} = (S_{\boldsymbol{j}}, k')$, and $\boldsymbol{u}$ a *descendent* of $\boldsymbol{v}$, if $\boldsymbol{i} \preccurlyeq \boldsymbol{j}$ and $k' \geq k + 1$.

DEFINITION 2.5. Let $\{S_i\}_{i \in \Lambda}$ be an IFS of contractive similitudes on a compact subset $X \subseteq \mathbb{R}^d$. We say that $\{S_i\}_{i \in \Lambda}$ is *of finite type* (or that it satisfies the *finite type condition (FTC)*) if there exist a sequence of nested index sets $\{\mathcal{M}_k\}_{k=0}^\infty$ and a nonempty bounded invariant open set $\Omega \subseteq X$ such that, with respect to Ω and $\{\mathcal{M}_k\}$, the set of equivalence classes $\mathcal{V}/_\sim := \{[\boldsymbol{v}] : \boldsymbol{v} \in \mathcal{V}\}$ is finite. We call such an Ω a *finite type condition set* (or FTC-set).

DEFINITION 2.6. A subset $\mathcal{I} \subseteq \mathcal{V}_k$ is called a *level-k island* (with respect to Ω and $\{\mathcal{M}_k\}$) if $S_{\mathcal{I}}(\Omega)$ is a connected component of $S_{\mathcal{V}_k}(\Omega)$.

REMARK 2.7.

(1) For each $\boldsymbol{v} \in \mathcal{V}_k$, there exists a unique island, denoted $\mathcal{I}(\boldsymbol{v})$, containing $\boldsymbol{v}$ and, moreover, $\mathfrak{N}(\boldsymbol{v}) \subseteq \mathcal{I}(\boldsymbol{v})$.

(2) If $\{S_i\}_{i \in \Lambda}$ satisfies OSC with Ω being an OSC-set, then $\mathcal{I}(\boldsymbol{v}) = \{\boldsymbol{v}\}$ for all $\boldsymbol{v} \in \mathcal{V}$.

NOTATION 2.8.

(1) Let

$$\mathbb{I}_k := \{\mathcal{I} : \mathcal{I} \text{ is a level-}k \text{ island}\}, \quad \mathbb{I} := \bigcup_{k\geq 0} \mathbb{I}_k$$

be the collection of all level-k islands and the collection of all islands, respectively.

(2) Generalizing (2.1), for any $k \geq 0$ and any subset $\mathbb{B} \subseteq \mathbb{I}_k$, we use the notation

$$S_{\mathbb{B}}(\Omega) := \bigcup_{\mathcal{I}\in\mathbb{B}} S_{\mathcal{I}}(\Omega).$$

DEFINITION 2.9. We say that two islands $\mathcal{I} \in \mathbb{I}_k$ and $\mathcal{I}' \in \mathbb{I}_{k'}$ are *equivalent*, and denote it by $\mathcal{I} \approx_\sigma \mathcal{I}'$ (or simply $\mathcal{I} \approx \mathcal{I}'$), if there exists some $\sigma \in \mathscr{S}$ such that $\{S_{\boldsymbol{v}'} : \boldsymbol{v}' \in \mathcal{I}'\} = \{\sigma S_{\boldsymbol{v}} : \boldsymbol{v} \in \mathcal{I}\}$ and, moreover, $\boldsymbol{v} \sim_\sigma \boldsymbol{v}'$ for any $\boldsymbol{v} \in \mathcal{I}$ and $\boldsymbol{v}' \in \mathcal{I}'$ satisfying $S_{\boldsymbol{v}'} = \sigma S_{\boldsymbol{v}}$.

NOTATION 2.10.

(1) We denote the equivalence class of $\mathcal{I}$ by $[\mathcal{I}]$ and we call $[\mathcal{I}]$ the *(island) type* of $\mathcal{I}$.

(2) For $\mathcal{I} \in \mathbb{I}_k$, $\mathcal{I}' \in \mathbb{I}_{k+1}$, $\mathcal{I}$ is said to be a *parent* of $\mathcal{I}'$ and $\mathcal{I}'$ an *offspring* of $\mathcal{I}$ if for any $\boldsymbol{v} \in \mathcal{I}'$, $\mathcal{I}$ contains some parent of $\boldsymbol{v}$. For any $k \geq 0$ and $\mathcal{I} \in \mathbb{I}_k$, let

$$O(\mathcal{I}) := \{\mathcal{J} : \mathcal{J} \text{ is an offspring of } \mathcal{I}\} \tag{2.2}$$

be the collection of all offspring of $\mathcal{I}$. Analogously, we define *predecessors* and *descendants* of an island.

DEFINITION 2.11. Let μ be a self-similar measure defined by an IFS $\{S_i\}_{i\in\Lambda}$ of finite type with Ω being an FTC-set. Two equivalent vertices $\boldsymbol{v} \in \mathcal{V}_k$ and $\boldsymbol{v}' \in \mathcal{V}_{k'}$ are *μ-equivalent*, denoted $\boldsymbol{v} \sim_{\mu,\sigma,w} \boldsymbol{v}'$ (or simply $\boldsymbol{v} \sim_\mu \boldsymbol{v}'$) if for $\sigma = S_{\boldsymbol{v}'} \circ S_{\boldsymbol{v}}^{-1}$, there exists a number $w > 0$ such that

$$\mu|_{S_{\mathfrak{N}(\boldsymbol{v}')}(\Omega)} = w \cdot \mu|_{S_{\mathfrak{N}(\boldsymbol{v})}(\Omega)} \circ \sigma^{-1}.$$

As $\sim$ is an equivalence relation, so is $\sim_\mu$. Denote the μ-equivalence class of $\boldsymbol{v}$ by $[\boldsymbol{v}]_\mu$ and call it the *(neighborhood) measure type* of $\boldsymbol{v}$ (with respect to Ω, $\{\mathcal{M}_k\}$ and μ). Intuitively, $\boldsymbol{v} \sim_\mu \boldsymbol{v}'$ means that the measures $\mu|_{S_{\mathfrak{N}(\boldsymbol{v})}(\Omega)}$ and $\mu|_{S_{\mathfrak{N}(\boldsymbol{v}')}(\Omega)}$ have the same structure. The following proposition shows that μ-equivalent vertices generate the same number of offspring of each neighborhood measure type. The proof can be found in [21].

PROPOSITION 2.12. *For two equivalent vertices $\boldsymbol{v} \in \mathcal{V}_k$ and $\boldsymbol{v}' \in \mathcal{V}_{k'}$, let $\{\boldsymbol{u}_i\}_{i\in\Lambda_1}$ and $\{\boldsymbol{u}'_i\}_{i\in\Lambda'_1}$ be the offspring of $\boldsymbol{v}$ and $\boldsymbol{v}'$ in $\mathcal{G}$, respectively. If $[\boldsymbol{v}]_\mu = [\boldsymbol{v}']_\mu$, then, counting multiplicity, $\{[\boldsymbol{u}_i]_\mu : i \in \Lambda_1\} = \{[\boldsymbol{u}'_i]_\mu : i \in \Lambda'_1\}$.*

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Definition 2.13. Let μ be a self-similar measure defined by a finite type IFS $\{S_i\}_{i\in\Lambda}$ on $\mathbb{R}^d$ with Ω being an FTC-set. Two islands $\mathcal{I} \in \mathbb{I}_k$ and $\mathcal{I}' \in \mathbb{I}_{k'}$ are said to be *μ-equivalent*, denoted $\mathcal{I} \approx_{\mu,\sigma,w} \mathcal{I}'$ (or simply $\mathcal{I} \approx_\mu \mathcal{I}'$), if $\mathcal{I} \approx_\sigma \mathcal{I}'$ and there exists some $w > 0$ such that

$$\mu|_{S_{\mathcal{I}'}(\Omega)} = w \cdot \mu|_{S_{\mathcal{I}}(\Omega)} \circ \sigma^{-1}. \tag{2.3}$$

We remark that (2.3) holds if and only if $\boldsymbol{v} \sim_{\mu,\sigma,w} \boldsymbol{v}'$ for any $\boldsymbol{v} \in \mathcal{I}$ and $\boldsymbol{v}' \in \mathcal{I}'$ satisfying $S_{\boldsymbol{v}'} = \sigma S_{\boldsymbol{v}}$. We note that $\approx_\mu$ is an equivalence relation. We denote the μ-equivalence class of $\mathcal{I}$ by $[\mathcal{I}]_\mu$, and call $[\mathcal{I}]_\mu$ the *(island) measure type* of $\mathcal{I}$ (with respect to Ω, $\{\mathcal{M}_k\}$, and μ). From the definition of $\approx_\mu$, we obtain an analog of Proposition 2.12 concerning $\approx_\mu$. That is, μ-equivalent islands generate the same number of offspring of each island measure type.

Definition 2.14. Let μ be a self-similar measure defined by a finite type IFS. Let $\mathbb{B} \subseteq \mathbb{I}_k$ for $k \geq 0$ and $\mathbb{B}_\mu := \{[\mathcal{I}]_\mu : \mathcal{I} \in \mathbb{B}\}$. We call $\mathcal{I}$ a *level-2 nonbasic island* with respect to $\mathbb{B}$ if $\mathcal{I} \in O(\mathcal{J})$ for some $\mathcal{J} \in \mathbb{B}$ and $[\mathcal{I}]_\mu \notin \mathbb{B}_\mu$. Inductively, for $\ell \geq 3$, we call $\mathcal{I}$ a *level-ℓ nonbasic island* with respect to $\mathbb{B}$ if $\mathcal{I}$ is an offspring of some level-$(\ell-1)$ nonbasic island with respect to $\mathbb{B}$ and $[\mathcal{I}]_\mu \notin \mathbb{B}_\mu$.

We remark that, by definition, for any $\ell \geq 2$, $\mathcal{I}$ is a level-ℓ nonbasic island with respect to $\mathbb{B}$ if and only if there exists a finite sequence of $\{\mathcal{I}_k\}_{k=1}^{\ell}$ such that $\mathcal{I}_1 \in \mathbb{B}$, $\mathcal{I}_\ell = \mathcal{I}$, $[\mathcal{I}_i]_\mu \notin \mathbb{B}_\mu$, and $\mathcal{I}_i$ is an offspring of $\mathcal{I}_{i-1}$ for all $i = 2, \dots, \ell$. In particular, $\mathcal{I}_i$ is a level-i nonbasic island with respect to $\mathbb{B}$ for all $i = 2, \dots, \ell$.

2.2. Measures that are essentially of finite type. We recall the definition of EFT in [21, Section 2.2]. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open subset and μ be a positive finite Borel measure with $\operatorname{supp}(\mu) \subseteq \overline{\Omega}$ and $\mu(\Omega) > 0$. We call a μ-measurable subset U of Ω a *cell (in Ω)* if $\mu(U) > 0$.

We say that two cells U and V are *μ-equivalent*, denoted $U \simeq_{\mu,\sigma,w} V$ (or simply $U \simeq_\mu V$), if there exist some similitude $\sigma : U \to V$ and some constant $w > 0$ such that $\sigma(U) = V$ and

$$\mu|_V = w\mu|_U \circ \sigma^{-1}.$$

It is easy to check that $\simeq_\mu$ is an equivalence relation.

Let $U \subseteq \Omega$ be a cell. Two cells V, W in U are *measure disjoint* with respect to μ if $\mu(V \cap W) = 0$. We call a finite family $\mathbf{P}$ of measure disjoint cells a *μ-partition* of U if $V \subseteq U$ for all $V \in \mathbf{P}$, and $\mu(U) = \sum_{V\in\mathbf{P}} \mu(V)$. A sequence of μ-partitions $\{\mathbf{P}_k\}_{k\geq1}$ is said to be *refining* if for any $V \in \mathbf{P}_k$ and any $W \in \mathbf{P}_{k+1}$, either $W \subseteq V$ or they are measure disjoint, that is, each member of $\mathbf{P}_{k+1}$ is a subset of some member of $\mathbf{P}_k$.

Remark 2.15. Let μ be a self-similar measure defined by a finite type IFS $\{S_i\}_{i\in\Lambda}$ on $\mathbb{R}^d$ with Ω being an FTC-set. The following can be verified directly.

(1) For any island $\mathcal{I} \in \mathbb{I}$, $S_{\mathcal{I}}(\Omega)$ is a cell.

(2) Let $\mathcal{I}$ and $\mathcal{I}'$ be two islands. By definition, $\mathcal{I} \approx_\mu \mathcal{I}'$ if and only if $\mathcal{I} \approx \mathcal{I}'$ and $S_{\mathcal{I}}(\Omega) \simeq_\mu S_{\mathcal{I}'}(\Omega)$.

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(3) Let $k \ge m \ge 0$. Then for any $\mathcal{I} \in \mathbb{I}_m$, $\mathbf{P} := \{S_{\mathcal{J}}(\Omega) : \mathcal{J} \in \mathbb{I}_k$ is a descendent of $\mathcal{I}\}$ is a refining μ-partition of $S_{\mathcal{I}}(\Omega)$.

Let $\mathbf{B} := \{B_{1,\ell}\}_{\ell\in\Gamma}$ be a finite family of measure disjoint cells in Ω, and for each $\ell \in \Gamma$, let $\{\mathbf{P}_{k,\ell}\}_{k\ge1}$ be a family of refining μ-partitions of $B_{1,\ell}$ with $\mathbf{P}_{1,\ell} := \{B_{1,\ell}\}$. For $k \ge 2$, we divide each $\mathbf{P}_{k,\ell}$ into two (possibly empty) subcollections, $\mathbf{P}^1_{k,\ell}$ and $\mathbf{P}^2_{k,\ell}$, with respect to $\mathbf{B}$, defined as follows:

$$\begin{aligned} \mathbf{P}^1_{k,\ell} &:= \{B \in \mathbf{P}_{k,\ell} : B \simeq_\mu B_{1,i} \text{ for some } i \in \Gamma\}, \\ \mathbf{P}^2_{k,\ell} &:= \mathbf{P}_{k,\ell} \backslash \mathbf{P}^1_{k,\ell} = \{B \in \mathbf{P}_{k,\ell} : B \notin \mathbf{P}^1_{k,\ell}\}. \end{aligned} \tag{2.4}$$

Definition 2.16. We say that a positive finite Borel measure μ on $\mathbb{R}^d$ is *essentially of finite type (EFT)* if there exist a bounded open subset $\Omega \subseteq \mathbb{R}^d$ with $\operatorname{supp}(\mu) \subseteq \overline{\Omega}$ and $\mu(\Omega) > 0$, and a finite family $\mathbf{B} := \{B_{1,\ell}\}_{\ell\in\Gamma}$ of measure disjoint cells in Ω such that for any $\ell \in \Gamma$, there is a family of refining μ-partitions $\{\mathbf{P}_{k,\ell}\}_{k\ge1}$ of $B_{1,\ell}$ satisfying the following conditions:

(1) $\mathbf{P}_{1,\ell} = \{B_{1,\ell}\}$, and there exists some $B \in \mathbf{P}^1_{2,\ell}$ such that $B \ne B_{1,\ell}$;
(2) if for some $k \ge 2$, there exists some $B \in \mathbf{P}^1_{k,\ell}$, then $B \in \mathbf{P}^1_{k+1,\ell}$ and hence $B \in \mathbf{P}^1_{m,\ell}$ for all $m \ge k$;
(3) $\lim_{k\to\infty} \sum_{B\in\mathbf{P}^2_{k,\ell}} \mu(B) = 0$.

Here $\mathbf{P}^1_{k,\ell}$ and $\mathbf{P}^2_{k,\ell}$ ($k \ge 2$) are defined as in (2.4). In this case, we call Ω an *EFT-set*, $\mathbf{B}$ a *basic family of cells* (in Ω), and $(\mathbf{B}, \mathbf{P}) := (\{B_{1,\ell}\}, \{\mathbf{P}_{k,\ell}\}_{k\ge1})_{\ell\in\Gamma}$ a *basic pair (with respect to* Ω).

Remark 2.17.

(1) We remark that conditions (1) and (2) are needed in Section 3 to derive the vector-valued renewal equation, and error estimate forces condition (3) to hold. In fact, to derive the vector-valued renewal equation, we only need condition (2) as well as (1'): the existence of some $B \in \bigcup_{k\ge2} \mathbf{P}^1_{k,\ell}$ such that $B \ne B_{1,\ell}$. Since condition (3) implies that $\bigcup_{k\ge2} \mathbf{P}^1_{k,\ell} \ne \emptyset$, we have chosen to use the more convenient condition (1).
(2) Let $(\mathbf{B}, \mathbf{P}) := (\{B_{1,\ell}\}, \{\mathbf{P}_{k,\ell}\}_{k\ge1})_{\ell\in\Gamma}$ be a basic pair. Then for some $k \ge 2$, $\mathbf{P}^2_{k,\ell} = \emptyset$ if and only if $\mathbf{P}_{m,\ell} = \mathbf{P}_{k,\ell}$ for all $m \ge k$.

The following definition of a weakly regular basic pair is weaker than that of a regular basic pair defined in [21].

Definition 2.18. Assume that μ satisfies EFT with Ω being an EFT-set and $(\mathbf{B}, \mathbf{P}) := (\{B_{1,\ell}\}, \{\mathbf{P}_{k,\ell}\}_{k\ge1})_{\ell\in\Gamma}$ being a basic pair with respect to Ω. We say that $(\mathbf{B}, \mathbf{P})$ is *weakly regular* if for any $\ell \in \Gamma$, there exists some similitude σ_ℓ such that $\sigma_\ell(\Omega) \subseteq B_{1,\ell}$. In this case, we call $\mathbf{B}$ a *weakly regular basic family of cells* (*in* Ω).

Measures studied in this paper are mainly self-similar. The following result is modified from [21, Proposition 2.15] to suit our purposes. The proof is similar. A connected FTC-set Ω is replaced by an FTC-set Ω.

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PROPOSITION 2.19 [21]. *Let μ be a self-similar measure defined by a finite type IFS on $\mathbb{R}^d$ with an FTC-set Ω. Suppose there exists some $m \geq 0$ such that the following two conditions hold.*

(1) *There exists a finite index set Γ such that $\mathbb{I}_m = \{\mathcal{I}_{1,\ell} : \ell \in \Gamma\}$; moreover, for each $\ell \in \Gamma$, there exists some constant $c(\ell) \geq 2$ (chosen to be the minimum) and descendant $\mathcal{J} \in \mathbb{I}_{m+c(\ell)-1}$ of $\mathcal{I}_{1,\ell}$ satisfying $S_{\mathcal{J}}(\Omega) \neq S_{\mathcal{I}_{1,\ell}}(\Omega)$ and $\mathcal{J} \approx_\mu \mathcal{I}_{1,i}$ for some $i \in \Gamma$.*
(2) *For $k \geq 2$ and $\ell \in \Gamma$, let $\mathbf{I}_{k,\ell}$ be the collection of all level-k nonbasic islands with respect to $\mathbb{I}_m$ that are descendants of $\mathcal{I}_{1,\ell}$. Then $\lim_{k\to\infty} \sum_{\mathcal{I}\in\mathbf{I}_{k,\ell}} \mu(S_{\mathcal{I}}(\Omega)) = 0$ for all $\ell \in \Gamma$.*

Then μ satisfies EFT with Ω being an EFT-set and with $\mathbf{B} = \{S_{\mathcal{I}_{1,\ell}}(\Omega) : \ell \in \Gamma\}$ being a basic family of cells in Ω.

In the proof of Proposition 2.19, for any $\ell \in \Gamma$, we define

$$\begin{aligned} \mathbf{P}_{1,\ell} &:= \{B_{1,\ell}\} \quad \text{and} \\ \mathbf{P}_{2,\ell} &:= \{S_{\mathcal{J}}(\Omega) : \mathcal{J} \in \mathbb{I}_{m+c(\ell)-1} \text{ is a descendant of } \mathcal{I}_{1,\ell}\}, \end{aligned} \tag{2.5}$$

where $B_{1,\ell} = S_{\mathcal{I}_{1,\ell}}(\Omega)$. For $k \geq 3$, if $\mathbf{P}^2_{k-1,\ell} = \emptyset$, define $\mathbf{P}_{k,\ell} := \mathbf{P}^1_{k-1,\ell}$; otherwise, define

$$\begin{aligned} \mathbf{P}_{k,\ell} := \mathbf{P}^1_{k-1,\ell} \cup \{S_{\mathcal{I}}(\Omega) : \mathcal{I} \in O(\mathcal{J}) &\text{ for some island } \mathcal{J} \\ &\text{satisfying } S_{\mathcal{J}}(\Omega) \in \mathbf{P}^2_{k-1,\ell}\}. \end{aligned} \tag{2.6}$$

The following two classes of examples for EFT are proved in [21].

EXAMPLE 2.20. Let μ be a self-similar measure defined by an IFS $\{S_i\}_{i\in\Lambda}$ on $\mathbb{R}^d$ satisfying OSC with Ω being an OSC-set and $\mu(\Omega) > 0$. Then μ satisfies EFT with Ω being an EFT-set and $\mathbf{B} := \{B_{1,1}\} = \{\Omega\}$ being a weakly regular basic family of cells.

Let $\{S_i\}_{i=1}^3$ be defined as in (1.6) and μ be the self-similar measure associated with a probability vector $(p_i)_{i=1}^3$. Let $w_1(k)$, $k \geq 0$, be defined as in (1.8). We remark that for $k \geq 0$,

$$\begin{aligned} p_1 p_3^{k+1} + p_2 w_1(k) = p_1 p_2^{k+1} + p_3 w_1(k) = w_1(k+1) \quad \text{and} \\ w_1(k+1) \leq w_1(k) \leq p_1. \end{aligned}$$

EXAMPLE 2.21. Let μ be the self-similar measure defined by an IFS $\{S_i\}_{i=1}^3$ in (1.6) together with a probability vector $(p_i)_{i=1}^3$. Then μ satisfies EFT with $\Omega = (0, 1)$ being an EFT-set and there exists a weakly regular basic pair with respect to Ω.

2.3. EFT for a class of IFSs on $\mathbb{R}^2$. In this subsection, we prove that any self-similar measure defined by an IFS in (1.10) satisfies EFT.

Let $\{S_i\}_{i=1}^4$ be defined as in (1.10) and μ be the self-similar measure associated with a probability vector $(p_i)_{i=1}^4$. Let $w_2(k)$, $k \geq 0$, be defined as in (1.12). We remark that for $k \geq 0$,

$$\begin{aligned} p_1 p_3^{k+1} + p_2 w_2(k) = p_1 p_2^{k+1} + p_3 w_2(k) = w_2(k+1) \quad \text{and} \\ w_2(k+1) \leq w_2(k) \leq p_1. \end{aligned} \tag{2.7}$$

Throughout this subsection we let $X = [0,1] \times [0,1]$,

$$\Omega = X^\circ, \quad W_k := \{2^{k-i}13^i : i = 0, 1, \dots, k\} \quad \text{for } k \geq 0. \tag{2.8}$$

To simplify notation we let

$$\gamma_k := 1 - r^k \quad \text{for } k \geq 0. \tag{2.9}$$

Define

$$\mathcal{I}_{1,1} := \{(S_1, 1), (S_2, 1)\}, \quad \mathcal{I}_{1,2} := \{(S_3, 1)\}, \quad \mathcal{I}_{1,3} := \{(S_4, 1)\} \tag{2.10}$$

(see Figure 2(a)) and

$$\begin{aligned} B_{1,1} := S_{\mathcal{I}_{1,1}}(\Omega) = S_1(\Omega) \cup S_2(\Omega) = (0, \rho\gamma_1) \times (0, \rho) \cup (\rho\gamma_1, \rho\gamma_1 + r) \times (0, r), \\ B_{1,2} := S_{\mathcal{I}_{1,2}}(\Omega) = S_3(\Omega) = (\gamma_1, 1) \times (0, r), \\ B_{1,3} := S_{\mathcal{I}_{1,3}}(\Omega) = S_4(\Omega) = (0, r) \times (\gamma_1, 1), \end{aligned} \tag{2.11}$$

where $\mathcal{I}_{1,i}, i = 1, 2, 3$, are defined in (2.10).

Example 2.22. Let μ be a self-similar measure defined by an IFS $\{S_i\}_{i=1}^4$ in (1.10) together with a probability vector $(p_i)_{i=1}^4$. Let Ω and W_k be as in (2.8). Then μ satisfies EFT with $\Omega = (0,1) \times (0,1)$ being an EFT-set and there exists a weakly regular basic pair with respect to Ω.

To prove Example 2.22, we first summarize without proof some elementary properties. Proposition 2.23(1) below implies that all multi-indices in W_k correspond to the same vertex.

Proposition 2.23. *Let $\{S_i\}_{i=1}^4$ be as in (1.10) and $\{\mathcal{I}_{1,i}\}_{i=1}^3$ be as in (2.10). The following relations hold:*

(1) $S_{13} = S_{21}$. *Moreover, for any* $\mathbf{i}, \mathbf{j} \in W_k$, $S_{\mathbf{i}} = S_{\mathbf{j}}$;

(2) $\mathbb{I}_1 = \{\mathcal{I}_{1,1}, \mathcal{I}_{1,2}, \mathcal{I}_{1,3}\}$.

Proposition 2.24. *Assume the hypotheses of Example 2.22 and $\{B_{1,i}\}_{i=1}^3$ defined as in (2.11). Then (1)–(3) below hold, and (4)–(6) hold for all $k \geq 0$:*

(1)
$$\begin{aligned} S_3(B_{1,1}) &= (\gamma_1, (1+\rho r)\gamma_1) \times (0, \rho r) \\ &\quad \cup ((1+\rho r)\gamma_1, (1+\rho r)\gamma_1 + r^2) \times (0, r^2), \\ S_4(B_{1,1}) &= (0, \rho r\gamma_1) \times (\gamma_1, \gamma_1 + \rho r) \\ &\quad \cup (\rho r\gamma_1, \rho r\gamma_1 + r^2) \times (\gamma_1, \gamma_1 + r^2); \end{aligned}$$

(2)
$$\begin{aligned} S_1(B_{1,2}) &= (\rho\gamma_1, \rho) \times (0, \rho r), \\ S_3(B_{1,2}) &= (\gamma_2, 1) \times (0, r^2), \\ S_4(B_{1,2}) &= (r\gamma_1, r) \times (\gamma_1, \gamma_1 + r^2); \end{aligned}$$

(3) $S_3(B_{1,3}) = (\gamma_1, \gamma_1 + r^2) \times (r\gamma_1, r)$ *and* $S_4(B_{1,3}) = (0, r^2) \times (\gamma_2, 1)$;

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(4)
$$S_{2^k1}(B_{1,1}) = (\rho\gamma_k, \rho\gamma_k + \rho^2 r^k\gamma_1) \times (0, \rho^2 r^k) \\ \cup (\rho\gamma_k + \rho^2 r^k\gamma_1, \rho\gamma_k + \rho^2 r^k\gamma_1 + \rho r^{k+1}) \times (0, \rho r^{k+1}), \\ S_{2^k1}(B_{1,3}) = (\rho\gamma_k, \rho\gamma_k + \rho r^{k+1}) \times (\rho r^k\gamma_1, \rho r^k);$$

(5)
$$S_{2^k}(B_{1,1}) = (\rho\gamma_k, \rho\gamma_{k+1}) \times (0, \rho r^k) \cup (\rho\gamma_{k+1}, \rho\gamma_{k+1} + r^{k+1}) \times (0, r^{k+1}), \\ S_{2^k}(B_{1,2}) = (r^k\gamma_1 + \rho\gamma_k, r^k + \rho\gamma_k) \times (0, r^{k+1}), \\ S_{2^k}(B_{1,3}) = (\rho\gamma_k, \rho\gamma_k + r^{k+1}) \times (r^k\gamma_1, r^k);$$

(6) $S_{2^k1}(\Omega) = (\rho\gamma_k, \rho) \times (0, \rho r^k)$ *and* $S_{2^k}(\Omega) = (\rho\gamma_k, \rho\gamma_k + r^k) \times (0, r^k)$.

Proof. (1)–(3) follow from (2.11), and (4)–(6) can be proved directly by induction; we omit the details. □

Lemma 2.25. *Assume the hypotheses of Proposition 2.24. Then for* $k \geq 1$,

$$\mu(S_1(\Omega) \cap S_{2^k}(\Omega)) = \mu\Big(\bigcup_{i=1}^{3} S_1(B_{1,i}) \cap S_{2^k}(\Omega)\Big) = \mu(S_{2^k1}(\Omega)). \tag{2.12}$$

Proof. First, we prove the first equality in (2.12). Since $\mu(S_1(\Omega)) = \mu(\bigcup_{i=1}^3 S_1(B_{1,i}))$, $\mu(S_1(\Omega) \cap A) = \mu((\bigcup_{i=1}^3 S_1(B_{1,i})) \cap A)$ for any $A \subseteq \Omega$. Hence $\mu(S_1(\Omega) \cap S_{2^k}(\Omega)) = \mu(\bigcup_{i=1}^3 S_1(B_{1,i}) \cap S_{2^k}(\Omega))$.

Next, we show that

$$\bigcup_{i=1}^{3} S_1(B_{1,i}) \cap S_{2^k}(\Omega) = S_{2^k1}(\Omega) \quad \text{for } k \geq 1. \tag{2.13}$$

By Proposition 2.24(2,4,6),

$$S_1(B_{1,1}) = (0, \rho^2\gamma_1) \times (0, \rho^2) \cup (\rho^2\gamma_1, \rho^2\gamma_1 + \rho r) \times (0, \rho r), \\ S_1(B_{1,2}) = (\rho\gamma_1, \rho) \times (0, \rho r), \quad S_1(B_{1,3}) = (0, \rho r) \times (\rho\gamma_1, \rho),$$

and $S_2(\Omega) = (\rho\gamma_1, \rho\gamma_1 + r) \times (0, r)$. It follows from (1.11) that $\rho r + \rho^2\gamma_1 \leq \rho\gamma_1$ and hence $S_1(B_{1,1}) \cap S_2(\Omega) = \emptyset$. Since $\rho < r + \rho\gamma_1$, $S_1(B_{1,2}) \cap S_2(\Omega) = (\rho\gamma_1, \rho) \times (0, \rho r) = S_{21}(\Omega)$, where Proposition 2.24(6) is used in the last equality. Since $r < \gamma_1$, $S_1(B_{1,3}) \cap S_2(\Omega) = \emptyset$. Hence $\bigcup_{i=1}^3 S_1(B_{1,i}) \cap S_2(\Omega) = S_{21}(\Omega)$. Assume that the stated inequality holds for $k = m$, that is, $\bigcup_{i=1}^3 S_1(B_{1,i}) \cap S_{2^m}(\Omega) = S_{2^m1}(\Omega)$. Then $S_1(B_{1,2}) \cap S_{2^m}(\Omega) = S_{2^m1}(\Omega)$ and $S_1(B_{1,i}) \cap S_{2^m}(\Omega) = \emptyset$ for $i = 1, 3$. For $k = m+1$, since $S_1(B_{1,i}) \cap S_{2^{m+1}}(\Omega) \subseteq S_1(B_{1,i}) \cap S_{2^m}(\Omega)$, $S_1(B_{1,i}) \cap S_{2^{m+1}}(\Omega) = \emptyset$ for $i = 1, 3$. By (2.11) and Proposition 2.23(1),

$$S_1(B_{1,2}) \cap S_{2^{m+1}}(\Omega) = S_{13}(\Omega) \cap S_{2^{m+1}}(\Omega) = S_{21}(\Omega) \cap S_{2^{m+1}}(\Omega) \\ = S_2(S_1(\Omega) \cap S_{2^m}(\Omega)) = S_2\Big(\Big(\bigcup_{i=1}^{3} S_1(B_{1,i})\Big) \cap S_{2^m}(\Omega)\Big)$$

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$$= S_2\Big(\bigcup_{i=1}^{3} S_1(B_{1,i}) \cap S_{2^m}(\Omega)\Big)$$
$$= S_2(S_{2^m1}(\Omega)) = S_{2^{m+1}1}(\Omega).$$

This proves (2.13). Hence the second inequality in (2.12) holds. □

For any $k \geq 0$, $w_2(k)$ denotes the sum of probability weights corresponding to all multi-indices in W_k. Part (1) of the following lemma explains the meaning of the factor $w_2(k)$.

LEMMA 2.26. *Assume the hypotheses of Proposition 2.24 and let $w_2(k)$ be defined as in (1.12). Then:*

(1) *for $k \geq 0$ and $i = 1, 3$, $\mu|_{S_{2^k1}(B_{1,i})} = w_2(k)\mu \circ S_{2^k1}^{-1}$;*
(2) *for $k \geq 1$, $\mu|_{S_{2^k}(B_{1,1})} = w_2(k-1)\mu \circ S_{2^{k-1}1}^{-1} + p_2^k\mu \circ S_{2^k}^{-1}$;*
(3) *for $k \geq 1$ and $i = 2, 3$, $\mu|_{S_{2^k}(B_{1,i})} = p_2^k\mu \circ S_{2^k}^{-1}$;*
(4) *for $i = 1, 2, 3$ and $j = 3, 4$, $\mu|_{S_j(B_{1,i})} = p_j\mu \circ S_j^{-1}$.*

PROOF. We only prove (1) for $i = 1$ as an example. By Proposition 2.24(4), $S_1(B_{1,1}) = (0, \rho^2\gamma_1) \times (0, \rho^2) \cup (\rho^2\gamma_1, \rho r + \rho^2\gamma_1) \times (0, \rho r)$. Note that $S_2(\Omega) = (\rho\gamma_1, \rho\gamma_1 + r) \times (0, r)$. Moreover, since $\rho\gamma_1 - (\rho^2\gamma_1 + \rho r) = \rho(1 - 2r - \rho + \rho r) \geq 0$, $S_1(B_{1,1}) \subseteq S_1(\Omega)\backslash S_2(\Omega)$. Hence $\mu(A) = p_1\mu \circ S_1^{-1}(A)$ for any $A \subseteq S_1(B_{1,1})$. Assume that the stated equality holds for $k = m$, that is, $\mu|_{S_{2^m1}(B_{1,1})} = w_2(m)\mu \circ S_{2^m1}^{-1}$. For $k = m+1$, by Proposition 2.23(1), $S_{2^{m+1}1}(B_{1,1}) = S_{13^{m+1}}(B_{1,1})$. Then $S_1^{-1}(A) \subseteq S_{3^{m+1}}(B_{1,1})$ and $S_2^{-1}(A) \subseteq S_{2^m1}(B_{1,1})$ for any $A \subseteq S_{2^{m+1}1}(B_{1,1})$. It follows that $\mu(S_1^{-1}(A)) = p_3^{m+1}\mu \circ S_{3^{m+1}}^{-1}(S_1^{-1}(A))$ and $\mu(S_2^{-1}(A)) = w_2(m)\mu \circ S_{2^m1}^{-1}(S_2^{-1}(A))$. Thus,

$$\begin{aligned}
\mu(A) &= p_1\mu \circ S_1^{-1}(A) + p_2\mu \circ S_2^{-1}(A) \\
&= p_1p_3^{m+1}\mu \circ S_{3^{m+1}}^{-1}(S_1^{-1}(A)) + p_2w_2(m)\mu \circ S_{2^m1}^{-1}(S_2^{-1}(A)) \\
&= p_1p_3^{m+1}\mu \circ S_{13^{m+1}}^{-1}(A) + p_2w_2(m)\mu \circ S_{2^{m+1}1}^{-1}(A) \\
&= (p_1p_3^{m+1} + p_2w_2(m))\mu \circ S_{2^{m+1}1}^{-1}(A) \\
&= w_2(m+1)\mu \circ S_{2^{m+1}1}^{-1}(A).
\end{aligned}$$

The last equality follows from (2.7). This proves part (1) for $i = 1$. For the proof of part (3) in the case $i = 3$, we use Lemma 2.25. □

PROOF (EXAMPLE 2.22). It suffices to show that for $m = 1$, all the assumptions of Proposition 2.19 are satisfied. By (2.8), $\Omega = (0, 1) \times (0, 1)$. For each $k \geq 0$, let $\mathcal{M}_k = \{1, 2, 3, 4\}^k$. Let $\mathcal{I}_{1,\ell}$ be defined as in (2.10). Thus, $\mathbb{I}_1 = \{\mathcal{I}_{1,1}, \mathcal{I}_{1,2}, \mathcal{I}_{1,3}\}$. Let $\mathbf{I}_{1,\mu} := \{[\mathcal{I}_{1,1}]_\mu, [\mathcal{I}_{1,2}]_\mu, [\mathcal{I}_{1,3}]_\mu\}$. Next, we show that for any $k \geq 2$, $\mathcal{I}_{k,1,3} := \{(S_{2^{k-1}1}, k), (S_{2^k}, k)\}$ is the only level-k nonbasic island with respect to $\mathbb{I}_1$ (see Figure 2(b)). For $\ell = 2, 3$, since $\mathcal{I}(v_{\text{root}}) \approx_\mu \mathcal{I}_{1,\ell}$, none of the $\mathcal{I} \in O(\mathcal{I}_{1,\ell})$ are nonbasic islands with respect to $\mathbb{I}_1$ (see Figure 4) and hence assumption (1) of Proposition 2.19 holds for $\ell = 2, 3$ with $c(\ell) = 2$. Upon iterating the IFS once, $\mathcal{I}_{1,1}$ generates the following five islands:

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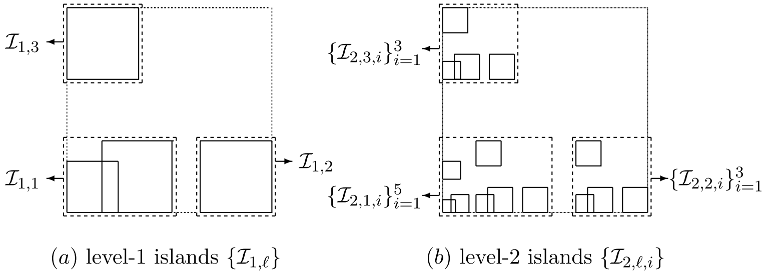


FIGURE 2. (a) First level iterations containing $\{\mathcal{I}_{1,\ell}\}_{\ell=1}^3$. (b) Second level iterations containing $\{\mathcal{I}_{2,1,i}\}_{i=1}^5$ and $\{\mathcal{I}_{2,\ell,i}\}_{i=1}^3$ for $\ell = 2,3$. The figures are drawn with $\rho = 1/4$ and $r = 7/20$.

$$\mathcal{I}_{2,1,1} := \{(S_{11}, 2), (S_{12}, 2)\}, \quad \mathcal{I}_{2,1,2} := \{(S_{14}, 2)\},$$
$$\mathcal{I}_{2,1,3} := \{(S_{21}, 2), (S_{22}, 2)\}, \quad \mathcal{I}_{2,1,4} := \{(S_{23}, 2)\}, \quad \mathcal{I}_{2,1,5} := \{(S_{24}, 2)\}$$

(see Figure 3). Lemma 2.26 implies that $[\mathcal{I}_{2,1,i}]_\mu \in \mathbf{I}_{1,\mu}$ for $i = 1, 2, 4, 5$, and $[\mathcal{I}_{2,1,3}]_\mu \notin \mathbf{I}_{1,\mu}$. Thus, assumption (1) of Proposition 2.19 holds for $\ell = 1$ with $c(1) = 2$ and $\mathcal{I}_{2,1,3}$ is the only level-2 nonbasic island with respect to $\mathbb{I}_1$. Assume that $\mathcal{I}_{k,1,3} := \{(S_{2^{k-1}1}, k), (S_{2^k}, k)\}$ is the only level-k nonbasic island with respect to $\mathbb{I}_1$. Similarly, $\mathcal{I}_{k,1,3}$ generates five islands, namely,

$$\mathcal{I}_{k+1,1,1} := \{(S_{2^{k-1}11}, k+1), (S_{2^{k-1}12}, k+1)\}, \quad \mathcal{I}_{k+1,1,2} := \{(S_{2^{k-1}14}, k+1)\},$$
$$\mathcal{I}_{k+1,1,3} := \{(S_{2^k1}, k+1), (S_{2^{k+1}}, k+1)\}, \quad \mathcal{I}_{k+1,1,4} := \{(S_{2^k3}, k+1)\},$$
$$\mathcal{I}_{k+1,1,5} := \{(S_{2^k4}, k+1)\}.$$

Lemma 2.26 again implies that $[\mathcal{I}_{k+1,1,i}]_\mu \in \mathbf{I}_{1,\mu}$ for $i = 1, 2, 4, 5$, and $[\mathcal{I}_{k+1,1,3}]_\mu \notin \mathbf{I}_{1,\mu}$. Thus, $\mathcal{I}_{k+1,1,3}$ is the only level-$(k+1)$ nonbasic island with respect to $\mathbb{I}_1$. Since the closure of $S_{\mathcal{I}_{k,1,3}}(\Omega)$ converges to a point, $\lim_{k\to\infty} \mu(S_{\mathcal{I}_{k,1,3}}(\Omega)) = 0$. Thus, assumption (2) in Proposition 2.19 holds. Equation (2.11) implies that $S_1(\Omega) \subseteq B_{1,1}$, $S_3(\Omega) = B_{1,2}$ and $S_4(\Omega) = B_{1,3}$, and hence $\mathbf{B} := \{B_{1,\ell} : \ell = 1, 2, 3\}$ is weakly regular. □

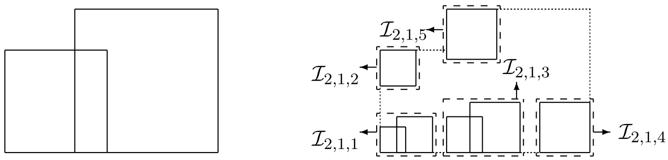
3. Renewal equation and proof of Theorem 1.1

Let $\{S_i\}_{i\in\Lambda}$ be a finite type IFS on a compact subset $X \subseteq \mathbb{R}^d$ with $\Omega \subseteq X$ being an FTC-set and let μ be the self-similar measure defined by $\{S_i\}_{i\in\Lambda}$ together with a probability vector $(p_i)_{i\in\Lambda}$. To compute $\tau(q)$ for $q \geq 0$, we will use the equivalent definition in (1.1).

In the rest of this section, we assume that μ satisfies EFT with Ω being an EFT-set and $(\mathbf{B}, \mathbf{P}) := (\{B_{1,\ell}\}, \{\mathbf{P}_{k,\ell}\}_{k\geq 1})_{\ell\in\Gamma}$ being a weakly regular basic pair with respect to Ω. Let $\varphi_\ell(\delta)$ and $\Phi_\ell^{(\alpha)}(\delta)$ be defined as in (1.2).

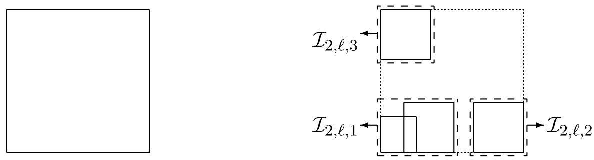
PROPOSITION 3.1. *Assume the above hypotheses and let $q \geq 0$. Then there exist constants $c_1, c_2 > 0$ such that*

$$c_1 \int_X \mu(B_{c_2\delta}(x))^q \, dx \leq \sum_{\ell\in\Gamma} \int_{B_{1,\ell}} \mu(B_\delta(x))^q \, dx \leq \int_X \mu(B_\delta(x))^q \, dx. \tag{3.1}$$



(a) $\mathcal{I}_{1,1}$ (b) $\{\mathcal{I}_{2,1,i}\}_{i=1}^{5}$

FIGURE 3. $\mathcal{I}_{1,1}$ and its offspring $\{\mathcal{I}_{2,1,i}\}_{i=1}^{5}$.



(a) $\mathcal{I}_{2,\ell}$ (b) $\{\mathcal{I}_{2,\ell,i}\}_{i=1}^{3}$

FIGURE 4. $\mathcal{I}_{2,\ell}$ and its offspring $\{\mathcal{I}_{2,\ell,i}\}_{i=1}^{3}$ for $\ell = 2, 3$.

Consequently,

$$\tau(q) = \inf\Big\{\alpha \geq 0 : \overline{\lim_{\delta\to 0^+}} \sum_{\ell\in\Gamma} \Phi_\ell^{(\alpha)}(\delta) > 0\Big\}$$
$$= \sup\Big\{\alpha \geq 0 : \overline{\lim_{\delta\to 0^+}} \sum_{\ell\in\Gamma} \Phi_\ell^{(\alpha)}(\delta) < \infty\Big\}. \tag{3.2}$$

PROOF. Since $\mathbf{B}$ is a finite family of measure disjoint cells in Ω and $\Omega \subseteq X$,

$$\sum_{\ell\in\Gamma} \int_{B_{1,\ell}} \mu(B_\delta(x))^q\,dx \leq \int_\Omega \mu(B_\delta(x))^q\,dx \leq \int_X \mu(B_\delta(x))^q\,dx,$$

proving the second inequality in (3.1).

To prove the first inequality in (3.1), we note that by the weak regularity of $\mathbf{B}$, for any $\ell \in \Gamma$, there exists some similitude σ_ℓ such that $\sigma_\ell(\Omega) \subseteq B_{1,\ell}$. For $\ell \in \Gamma$, let $r_{\boldsymbol{i}_\ell} := \max\{r_{\boldsymbol{i}} : S_{\boldsymbol{i}}(\Omega) \subseteq B_{1,\ell},\ \boldsymbol{i} \in \Lambda^k \text{ and } k \geq 1\}$ and $\sigma_\ell := S_{\boldsymbol{i}_\ell}$. Then

$$\int_{B_{1,\ell}} \mu(B_\delta(x))^q\,dx \geq \int_{B_{1,\ell}} (p_{\boldsymbol{i}_\ell}\mu \circ S_{\boldsymbol{i}_\ell}^{-1}(B_\delta(x)))^q\,dx$$
$$= p_{\boldsymbol{i}_\ell}^q r_{\boldsymbol{i}_\ell}^d \int_{S_{\boldsymbol{i}_\ell}^{-1}(B_{1,\ell})} \mu(B_{r_{\boldsymbol{i}_\ell}^{-1}\delta}(y))^q\,dy \quad (\text{let } S_{\boldsymbol{i}_\ell}^{-1}(x) = y)$$
$$\geq p_{\boldsymbol{i}_m}^q r_{\boldsymbol{i}_m}^d \int_\Omega \mu(B_{r_{\boldsymbol{i}_M}^{-1}\delta}(y))^q\,dy, \tag{3.3}$$

where $p_{\boldsymbol{i}_m} = \min\{p_{\boldsymbol{i}_\ell} : \ell \in \Gamma\}$, $r_{\boldsymbol{i}_m} = \min\{r_{\boldsymbol{i}_\ell} : \ell \in \Gamma\}$, and $r_{\boldsymbol{i}_M} = \max\{r_{\boldsymbol{i}_\ell} : \ell \in \Gamma\}$.

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For convenience, let $Y := X \backslash \overline{\Omega}$. Since $\Omega, \partial\Omega, Y$ are mutually disjoint and $X = \Omega \cup \partial\Omega \cup Y$,

$$\int_X \mu(B_\delta(x))^q\,dx = \int_\Omega \mu(B_\delta(x))^q\,dx + \int_{\partial\Omega} \mu(B_\delta(x))^q\,dx + \int_Y \mu(B_\delta(x))^q\,dx. \tag{3.4}$$

For any $x \in \partial\Omega$, there exists $y \in B_\delta(x) \cap \Omega \subseteq \Omega$ such that $B_\delta(x) \subseteq B_{2\delta}(y)$. Hence

$$\int_{\partial\Omega} \mu(B_\delta(x))^q\,dx \le \int_\Omega \mu(B_{2\delta}(y))^q\,dy. \tag{3.5}$$

Let $Q_\delta(Y)$ be the largest subset of Y satisfying $B_\delta(x) \subseteq Y$ for any $x \in Q_\delta(Y)$. Combining this with the fact that $\mu(Y) = 0$, we see that $\mu(B_\delta(x)) = 0$ for any $x \in Q_\delta(Y)$. Let $R_\delta(Y) := Y \backslash Q_\delta(Y)$. Then

$$\int_Y \mu(B_\delta(x))^q\,dx = \int_{R_\delta(Y)} \mu(B_\delta(x))^q\,dx. \tag{3.6}$$

Since $B_\delta(x) \cap \overline{\Omega} \neq \emptyset$ for any $x \in R_\delta(Y)$, there exists $y \in B_\delta(x) \cap \Omega \subseteq \Omega$ such that $B_\delta(x) \cap \overline{\Omega} \subseteq B_{2\delta}(y)$, and thus $\mu(B_\delta(x)) = \mu(B_\delta(x) \cap \overline{\Omega}) \le \mu(B_{2\delta}(y))$. Combining this with (3.4), (3.5), and (3.6),

$$\int_X \mu(B_\delta(x))^q\,dx \le 3 \int_\Omega \mu(B_{2\delta}(x))^q\,dx. \tag{3.7}$$

Equations (3.3) and (3.7) imply that

$$1/3 \cdot p_{i_m}^q r_{i_m}^d \int_X \mu(B_{1/2 \cdot r_{i_M}^{-1}\delta}(y))^q\,dy \le \sum_{\ell \in \Gamma} \int_{B_{1,\ell}} \mu(B_\delta(x))^q\,dx,$$

and hence the first inequality of (3.1) holds with $c_1 = 1/3 \cdot p_{i_m}^q r_{i_m}^d$ and $c_2 = 1/2 \cdot r_{i_M}^{-1}$.

Multiplying both sides of (3.1) by $\delta^{-(d+\alpha)}$, and using (1.2),

$$\frac{c_1}{\delta^{d+\alpha}} \int_X \mu(B_{c_2\delta}(x))^q\,dx \le \sum_{\ell \in \Gamma} \Phi_\ell^{(\alpha)}(\delta) \le \frac{1}{\delta^{d+\alpha}} \int_X \mu(B_\delta(x))^q\,dx. \tag{3.8}$$

Taking $\overline{\lim}_{\delta\to 0^+}$ in (3.8),

$$\begin{aligned}\varlimsup_{\delta\to 0^+} \frac{c_1}{\delta^{d+\alpha}} \int_X \mu(B_{c_2\delta}(x))^q\,dx &\le \varlimsup_{\delta\to 0^+} \sum_{\ell\in\Gamma} \Phi_\ell^{(\alpha)}(\delta) \\ &\le \varlimsup_{\delta\to 0^+} \frac{1}{\delta^{d+\alpha}} \int_X \mu(B_\delta(x))^q\,dx. \end{aligned} \tag{3.9}$$

Note that $c_2 > 0$. Letting $\delta' := c_2\delta$,

$$\varlimsup_{\delta\to 0^+} \frac{c_1}{\delta^{d+\alpha}} \int_X \mu(B_{c_2\delta}(x))^q\,dx = c_1 c_2^{d+\alpha} \varlimsup_{\delta'\to 0^+} \frac{1}{\delta'^{(d+\alpha)}} \int_X \mu(B_{\delta'}(x))^q\,dx. \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$\begin{aligned}\inf\Big\{\alpha \ge 0 : \overline{\lim_{\delta\to 0^+}} \sum_{\ell\in\Gamma} \Phi_\ell^{(\alpha)}(\delta) > 0\Big\}\\ = \inf\Big\{\alpha \ge 0 : \overline{\lim_{\delta\to 0^+}} \frac{1}{\delta^{d+\alpha}} \int_X \mu(B_\delta(x))^q\,dx > 0\Big\},\end{aligned}$$

and

$$\begin{aligned}\sup\Big\{\alpha \ge 0 : \underline{\lim_{\delta\to 0^+}} \sum_{\ell\in\Gamma} \Phi_\ell^{(\alpha)}(\delta) < \infty\Big\}\\ = \sup\Big\{\alpha \ge 0 : \underline{\lim_{\delta\to 0^+}} \frac{1}{\delta^{d+\alpha}} \int_X \mu(B_\delta(x))^q\,dx < \infty\Big\}.\end{aligned}$$

Equation (3.2) holds by combining these with (1.1). □

We denote the contraction ratio of a contractive similitude σ by r_σ. In view of Proposition 3.1, it suffices to study $\Phi_\ell^{(\alpha)}(\delta)$ for $\ell \in \Gamma$.

Step 1. *Derivation of a functional equation for* $\Phi_\ell^{(\alpha)}(\delta)$ *for* $\ell \in \Gamma$. For $\ell \in \Gamma$ and $k \ge 2$, let $\mathbf{P}^1_{k,\ell}$ and $\mathbf{P}^2_{k,\ell}$ be defined as in (2.4). Without loss of generality, we assume that Γ can be partitioned into two (possibly empty) sub-collections, Γ_* and Γ'_*, defined as follows. For $\ell \in \Gamma$, we say $\ell \in \Gamma_*$ if there exists some integer κ_ℓ satisfying $\mathbf{P}^2_{\kappa_\ell,\ell} = \emptyset$, where we choose κ_ℓ to be the smallest number satisfying the above condition. Let $\Gamma'_* := \Gamma\backslash\Gamma_*$. Define $\kappa_\ell := \infty$ for $\ell \in \Gamma'_*$.

Fix $\ell \in \Gamma$, by the definition of EFT, for any $2 \le k \le \kappa_\ell$, there exist two finite disjoint subsets $G_{k,\ell}, G'_{k,\ell} \subseteq \mathbb{N}$ such that

$$\mathbf{P}^1_{k,\ell} = \bigcup_{j=2}^{k} \{B_{j,\ell,i} : i \in G_{j,\ell}\}, \quad \mathbf{P}^2_{k,\ell} = \{B_{k,\ell,i} : i \in G'_{k,\ell}\}.$$

Define

$$B_{k,\ell,i} := S_{\mathcal{I}_{k,\ell,i}}(\Omega) \quad \text{for } 2 \le k \le \kappa_\ell \text{ and } i \in G_{k,\ell} \cup G'_{k,\ell}.$$

Condition (1) of EFT implies that $G_{2,\ell} \ne \emptyset$ for all $\ell \in \Gamma$. If $\ell \in \Gamma'_*$, condition (3) of EFT implies that $\lim_{k\to\infty} \sum_{i\in G'_{k,\ell}} \mu(B_{k,\ell,i}) = 0$. Thus, for all $\ell \in \Gamma_*$,

$$\varphi_\ell(\delta) = \sum_{j=2}^{\kappa_\ell} \sum_{i\in G_{j,\ell}} \int_{B_{j,\ell,i}} \mu(B_\delta(x))^q\,dx, \tag{3.11}$$

while for all $\ell \in \Gamma'_*$ and $n \ge 2$,

$$\begin{aligned}\varphi_\ell(\delta) = \sum_{j=2}^{n} \sum_{i\in G_{j,\ell}} \int_{B_{j,\ell,i}} \mu(B_\delta(x))^q\,dx\\ + \sum_{i\in G'_{n,\ell}} \int_{B_{n,\ell,i}} \mu(B_\delta(x))^q\,dx.\end{aligned} \tag{3.12}$$

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For $\ell \in \Gamma, 2 \le k \le \kappa_\ell, i \in G_{k,\ell}$ and $\delta > 0$, let $\widetilde{B}_{k,\ell,i}(\delta)$ be the largest subset of $B_{k,\ell,i}$ satisfying $B_\delta(x) \subseteq B_{k,\ell,i}$ for any $x \in \widetilde{B}_{k,\ell,i}(\delta)$. We denote $\widehat{B}_{k,\ell,i}(\delta) := B_{k,\ell,i} \backslash \widetilde{B}_{k,\ell,i}(\delta)$. So for $\ell \in \Gamma_*$, (3.11) can be written as

$$\varphi_\ell(\delta) = \sum_{j=2}^{\kappa_\ell} \sum_{i \in G_{j,\ell}} \int_{\widetilde{B}_{j,\ell,i}(\delta)} \mu(B_\delta(x))^q \, dx + \sum_{j=2}^{\kappa_\ell} \sum_{i \in G_{j,\ell}} \int_{\widehat{B}_{j,\ell,i}(\delta)} \mu(B_\delta(x))^q \, dx,$$

while for $\ell \in \Gamma'_*$ and $n \ge 2$, (3.12) can be expressed as

$$\varphi_\ell(\delta) = \sum_{j=2}^{n} \sum_{i \in G_{j,\ell}} \int_{\widetilde{B}_{j,\ell,i}(\delta)} \mu(B_\delta(x))^q \, dx + \sum_{j=2}^{n} \sum_{i \in G_{j,\ell}} \int_{\widehat{B}_{j,\ell,i}(\delta)} \mu(B_\delta(x))^q \, dx + \sum_{i \in G'_{n,\ell}} \int_{B_{n,\ell,i}} \mu(B_\delta(x))^q \, dx.$$

For $\ell \in \Gamma, 2 \le k \le \kappa_\ell$ and $i \in G_{k,\ell}$, there exist unique $\sigma(k,\ell,i) \in \mathscr{S}, w(k,\ell,i) > 0$ and $c(k,\ell,i) \in \Gamma$ such that $\mathcal{I}_{1,c(k,\ell,i)} \approx_{\mu,\sigma(k,\ell,i),w(k,\ell,i)} \mathcal{I}_{k,\ell,i}$. By Definition 2.13,

$$\mu|_{S_{\mathcal{I}_{k,\ell,i}}(\Omega)} = w(k,\ell,i) \cdot \mu|_{S_{\mathcal{I}_{1,c(k,\ell,i)}}(\Omega)} \circ \sigma(k,\ell,i)^{-1}.$$

For $\widetilde{B}_{k,\ell,i}(\delta) \subseteq B_{k,\ell,i}$, let $\widetilde{B}_{1,c(k,\ell,i)}(\delta/r_{\sigma(k,\ell,i)})$ be the largest subset of $B_{1,c(k,\ell,i)}$ satisfying $B_{\delta/r_{\sigma(k,\ell,i)}}(x) \subseteq B_{1,c(k,\ell,i)}$ for any $x \in \widetilde{B}_{1,c(k,\ell,i)}(\delta/r_{\sigma(k,\ell,i)})$, where $B_{1,c(k,\ell,i)} = S_{\mathcal{I}_{1,c(k,\ell,i)}}(\Omega)$. Thus,

$$\mu|_{\widetilde{B}_{k,\ell,i}(\delta)} = w(k,\ell,i) \cdot \mu|_{\widetilde{B}_{1,c(k,\ell,i)}(\delta/r_{\sigma(k,\ell,i)})} \circ \sigma(k,\ell,i)^{-1}.$$

We denote $\widehat{B}_{1,c(k,\ell,i)}(\delta/r_{\sigma(k,\ell,i)}) = B_{1,c(k,\ell,i)} \backslash \widetilde{B}_{1,c(k,\ell,i)}(\delta/r_{\sigma(k,\ell,i)})$. Hence for $\ell \in \Gamma_*$,

$$\varphi_\ell(\delta) = \sum_{j=2}^{\kappa_\ell} \sum_{i \in G_{j,\ell}} w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^d \int_{B_{1,c(j,\ell,i)}} \mu(B_{\delta/r_{\sigma(j,\ell,i)}}(x))^q \, dx + \sum_{j=2}^{\kappa_\ell} (e_j^\ell(\delta) - \tilde{e}_j^\ell(\delta)), \tag{3.13}$$

where

$$e_j^\ell(\delta) = \sum_{i \in G_{j,\ell}} \int_{\widehat{B}_{j,\ell,i}(\delta)} \mu(B_\delta(x))^q \, dx,$$

$$\tilde{e}_j^\ell(\delta) = \sum_{i \in G_{j,\ell}} w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^d \int_{\widehat{B}_{1,c(j,\ell,i)}(\delta/r_{\sigma(j,\ell,i)})} \mu(B_{\delta/r_{\sigma(j,\ell,i)}}(x))^q \, dx.$$

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For $\ell \in \Gamma'_*$ and $n \geq 2$,

$$\varphi_\ell(\delta) = \sum_{j=2}^{n} \sum_{i \in G_{j,\ell}} w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^d \int_{B_{1,c(j,\ell,i)}} \mu(B_{\delta/r_{\sigma(j,\ell,i)}}(x))^q \, dx$$
$$+ \sum_{j=2}^{n} (e_j^\ell(\delta) - \tilde{e}_j^\ell(\delta)) + \sum_{i \in G'_{n,\ell}} \int_{B_{n,\ell,i}} \mu(B_\delta(x))^q \, dx, \tag{3.14}$$

where

$$e_j^\ell(\delta) = \sum_{i \in G_{j,\ell}} \int_{\widehat{B}_{j,\ell,i}(\delta)} \mu(B_\delta(x))^q \, dx,$$
$$\tilde{e}_j^\ell(\delta) = \sum_{i \in G_{j,\ell}} w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^d \int_{\widehat{B}_{1,c(j,\ell,i)}(\delta/r_{\sigma(j,\ell,i)})} \mu(B_{\delta/r_{\sigma(j,\ell,i)}}(x))^q \, dx.$$

Multiplying both sides of (3.13) and (3.14) by $\delta^{-(d+\alpha)}$, and using (1.2), we have for $\ell \in \Gamma_*$,

$$\Phi_\ell^{(\alpha)}(\delta) = \sum_{j=2}^{\kappa_\ell} \sum_{i \in G_{j,\ell}} w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^{-\alpha} \Phi_{c(j,\ell,i)}^{(\alpha)}(\delta/r_{\sigma(j,\ell,i)})$$
$$+ E_\ell^{(\alpha)}(\delta), \tag{3.15}$$

where

$$E_\ell^{(\alpha)}(\delta) = \sum_{j=2}^{\kappa_\ell} \delta^{-(d+\alpha)} (e_j^\ell(\delta) - \tilde{e}_j^\ell(\delta))$$

and

$$\Phi_\ell^{(\alpha)}(\delta) = \sum_{j=2}^{n} \sum_{i \in G_{j,\ell}} w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^{-\alpha} \Phi_{c(j,\ell,i)}^{(\alpha)}(\delta/r_{\sigma(j,\ell,i)})$$
$$+ \sum_{j=2}^{n} \delta^{-(d+\alpha)} (e_j^\ell(\delta) - \tilde{e}_j^\ell(\delta))$$
$$+ \delta^{-(d+\alpha)} \sum_{i \in G'_{n,\ell}} \int_{B_{n,\ell,i}} \mu(B_\delta(x))^q \, dx \quad \text{for } \ell \in \Gamma'_* \text{ and } n \geq 2. \tag{3.16}$$

For $\delta > 0$ and $\ell \in \Gamma'_*$, let $N = N(\ell) := \max\{n \in \mathbb{N} : \delta \leq \min\{r_{\sigma(j,\ell,i)} : i \in G_{j,\ell}$ for all $j \leq n\}\}$. Taking $n := N$ in (3.16) for $\ell \in \Gamma'_*$ and $N \geq 2$,

$$\Phi_\ell^{(\alpha)}(\delta) = \sum_{j=2}^{\infty} \sum_{i \in G_{j,\ell}} w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^{-\alpha} \Phi_{c(j,\ell,i)}^{(\alpha)}(\delta/r_{\sigma(j,\ell,i)})$$
$$+ E_\ell^{(\alpha)}(\delta) - E_{\ell,\infty}^{(\alpha)}(\delta), \tag{3.17}$$

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where

$$E_\ell^{(\alpha)}(\delta) = \sum_{j=2}^{N} \delta^{-(d+\alpha)}(e_j^\ell(\delta) - \tilde{e}_j^\ell(\delta)) + \delta^{-(d+\alpha)} \sum_{i \in G'_{N,\ell}} \int_{B_{N,\ell,i}} \mu(B_\delta(x))^q \, dx$$

$$E_{\ell,\infty}^{(\alpha)}(\delta) = \sum_{j=N+1}^{\infty} \sum_{i \in G_{j,\ell}} w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^{-\alpha} \Phi_{c(j,\ell,i)}^{(\alpha)}(\delta/r_{\sigma(j,\ell,i)}).$$

Step 2. Derivation of the vector-valued equation. For each $\ell \in \Gamma$, define

$$f_\ell(x) = f_\ell^{(\alpha)}(x) := \Phi_\ell^{(\alpha)}(e^{-x}). \tag{3.18}$$

If we let $\delta = e^{-x}$, then $\Phi_\ell^{(\alpha)}(\beta\delta) = f_\ell(x - \ln\beta)$ for any $\beta > 0$. Combining (3.15) and (3.17), we have for $\ell \in \Gamma_*$,

$$f_\ell(x) = \sum_{j=2}^{\kappa_\ell} \sum_{i \in G_{j,\ell}} w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^{-\alpha} f_{c(j,\ell,i)}(x + \ln(r_{\sigma(j,\ell,i)})) + z_\ell^{(\alpha)}(x), \tag{3.19}$$

where $z_\ell^{(\alpha)}(x) = E_\ell^{(\alpha)}(e^{-x})$. For $\ell \in \Gamma'_*$ and $N \geq 2$,

$$f_\ell(x) = \sum_{j=2}^{\infty} \sum_{i \in G_{j,\ell}} w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^{-\alpha} f_{c(j,\ell,i)}(x + \ln(r_{\sigma(j,\ell,i)})) + z_\ell^{(\alpha)}(x), \tag{3.20}$$

where $z_\ell^{(\alpha)}(x) = E_\ell^{(\alpha)}(e^{-x}) - E_{\ell,\infty}^{(\alpha)}(e^{-x})$.

For $\ell, m \in \Gamma$, let $\mu_{m\ell}^{(\alpha)}$ be the discrete measure such that for $2 \leq j \leq \kappa_\ell, i \in G_{j,\ell}, c(j,\ell,i) = m$,

$$\mu_{m\ell}^{(\alpha)}(-\ln(r_{\sigma(j,\ell,i)})) := w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^{-\alpha}. \tag{3.21}$$

Then (see (1.4) and (1.5))

$$\mu_{m\ell}^{(\alpha)}(\mathbb{R}) = \sum_{j=2}^{\kappa_\ell} \sum_{i \in G_{j,\ell}} w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^{-\alpha},$$

and

$$F_\ell(\alpha) = \sum_{m \in \Gamma} \sum_{j=2}^{\kappa_\ell} \sum_{i \in G_{j,\ell}} w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^{-\alpha}.$$

We summarize the above derivations in the following theorem.

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Theorem 3.2. *Let μ be a self-similar measure defined by an IFS $\{S_i\}_{i\in\Lambda}$ of finite type. Assume that μ satisfies EFT. Let $\mathbf{f}, \mathbf{M}_\alpha$, and $\mathbf{z}$ be defined as in (1.3). Then $\mathbf{f}$ satisfies the vector-valued renewal equation $\mathbf{f} = \mathbf{f} * \mathbf{M}_\alpha + \mathbf{z}$.*

Proof (Theorem 1.1). We use a similar argument as that in [20, Theorem 1.1]. (1) We observe that each $F_\ell(\alpha)$ is a strictly increasing continuous positive function of α and

$$\lim_{\alpha\to-\infty} F_\ell(\alpha) = 0, \qquad \lim_{\alpha\to\infty} F_\ell(\alpha) = \infty. \tag{3.22}$$

Thus, there exists a unique α such that the spectral radius of $\mathbf{M}_\alpha(\infty)$ is 1.

(2) Let α be the unique number in part (1). Let $\mathbf{m} := [m_{k\ell}^{(\alpha)}] = [\int_0^\infty x\, d\mu_{k\ell}^{(\alpha)}]$ be the moment matrix. Following the proof of [20, Theorem 1.1(b)], we need to show that some moment condition holds, and it suffices to show that

$$0 < \sum_{k\in\Gamma} m_{k\ell}^{(\alpha)} < \infty.$$

It is easy to check that for $\ell \in \Gamma$, $\sum_{k\in\Gamma} m_{k\ell}^{(\alpha)}$ takes the following values:

$$\sum_{k\in\Gamma}\sum_{j=2}^{\kappa_\ell}\sum_{i\in G_{j,\ell}} w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^{-\alpha} |\ln(r_{\sigma(j,\ell,i)})|.$$

Equation (3.22) implies that there exists $\epsilon > 0$ such that $0 < F_\ell(\alpha+\epsilon) < \infty$. Thus,

$$\begin{aligned} 0 &< \sum_{k\in\Gamma}\sum_{j=2}^{\kappa_\ell}\sum_{i\in G_{j,\ell}} w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^{-\alpha} |\ln(r_{\sigma(j,\ell,i)})| \\ &= \sum_{k\in\Gamma}\sum_{j=2}^{\kappa_\ell}\sum_{i\in G_{j,\ell}} w(j,\ell,i)^q r_{\sigma(j,\ell,i)}^{-(\alpha+\epsilon)} r_{\sigma(j,\ell,i)}^{\epsilon} |\ln(r_{\sigma(j,\ell,i)})| \\ &< \infty. \end{aligned}$$

Moreover, it follows from (3.21) that $\sum_{m\in\Gamma} \mu_{m\ell}^{(\alpha)}(0) = 0 < \sum_{m\in\Gamma} \mu_{m\ell}^{(\alpha)}(\infty)$, that is, each column of $\mathbf{M}_\alpha$ is nondegenerate at 0. From Theorem 3.2, $\mathbf{f} = \mathbf{f} * \mathbf{M}_\alpha + \mathbf{z}$, where, by assumption, $\mathbf{z}$ is directly Riemann integrable on $\mathbb{R}$.

We first consider the case $\mathbf{M}_\alpha(\infty)$ is irreducible. It follows from the above observations and [20, Theorem 4.1] that there exist positive constants C_1 and C_2 such that $0 < C_1 \le \underline{\lim}_{x\to\infty} f_\ell(x) \le C_2 < \infty$ for all $\ell \in \Gamma$. By (3.18), $0 < C_1 \le \overline{\lim}_{\delta\to 0^+} \Phi_\ell^{(\alpha)}(\delta) \le C_2 < \infty$ for all $\ell \in \Gamma$. Consequently, $\Phi_\ell^{(\alpha)}(\delta) \le \sum_{\ell\in\Gamma} \Phi_\ell^{(\alpha)}(\delta)$ and thus,

$$\begin{aligned} 0 < C_1 \le \overline{\lim_{\delta\to 0^+}}\, \Phi_\ell^{(\alpha)}(\delta) &\le \overline{\lim_{\delta\to 0^+}} \sum_{\ell\in\Gamma} \Phi_\ell^{(\alpha)}(\delta) \\ &\le \sum_{\ell\in\Gamma} \overline{\lim_{\delta\to 0^+}}\, \Phi_\ell^{(\alpha)}(\delta) \le C_2 \# \Gamma < \infty. \end{aligned}$$

It now follows from (3.2) that $\tau(q) = \alpha$.

It remains to consider the case $\mathbf{M}_\alpha(\infty)$ is reducible. As in the proof of [20, Theorem 1.1(b), Case 2],

$$\lim_{x\to\infty} f_\ell^{(\beta)}(x) = 0 \quad \text{for all } \ell \in \Gamma \text{ and all } \beta < \alpha. \tag{3.23}$$

Moreover, there exists some $\ell_0 \in \Gamma$ such that

$$\underline{\lim}_{x\to\infty} f_{\ell_0}^{(\alpha)}(x) > 0. \tag{3.24}$$

Combining (3.23) with (3.18), we see that for all $\ell \in \Gamma$ and all $\beta < \alpha$, $\lim_{\delta\to 0^+} \Phi_\ell^{(\beta)}(\delta) = 0$. Thus, Proposition 3.1 implies that $\tau(q) \geq \alpha$. Similarly, combining (3.24) and (3.18),

$$0 < \underline{\lim}_{\delta\to 0^+} \Phi_{\ell_0}^{(\alpha)}(\delta) \leq \overline{\lim}_{\delta\to 0^+} \sum_{\ell\in\Gamma} \Phi_\ell^{(\alpha)}(\delta).$$

It follows from Proposition 3.1 again that $\tau(q) \leq \alpha$, which completes the proof. $\square$

4. A class of one-dimensional IFSs with overlaps

In this section, we derive renewal equations and compute the L^q-spectrum of self-similar measures μ defined by the IFSs in (1.6). Let $X := [0, 1]$ and $\Omega = (0, 1)$. Define

$$\mathcal{I}_{1,1} = \{(S_1, 1), (S_2, 1)\}, \quad \mathcal{I}_{1,2} = \{(S_3, 1)\},$$

and

$$B_{1,\ell} := S_{\mathcal{I}_{1,\ell}}(\Omega) \quad \text{for } \ell \in \Gamma,$$

where $\Gamma = \{1, 2\}$. For $\ell \in \Gamma$ and $k \geq 1$, let $\mathbf{P}_{k,\ell}$ be defined as in (2.5) and (2.6). It follows from Example 2.21 that μ satisfies EFT with $\Omega = (0, 1)$ being an EFT-set, $\mathbf{B} := \{B_{1,\ell}\}_{\ell\in\Gamma}$ being a weakly regular basic family of cells in Ω, and $(\mathbf{B}, \mathbf{P}) := (\{B_{1,\ell}\}, \{\mathbf{P}_{k,\ell}\}_{k\geq 1})_{\ell\in\Gamma}$ being a weakly regular basic pair with respect to Ω.

In the rest of this section, we use the notation defined in Section 3. For $\mathcal{I} \in \mathbb{I}$, let $S_{\mathcal{I}}(\Omega)$ and $O(\mathcal{I})$ be defined as in (2.1) and (2.2), respectively. For $\ell \in \Gamma$, $i = 1, 2$, and $k \geq 2$, let $\mathbf{P}_{k,\ell}^i$ be defined as in (2.4). We first observe that $O(\mathcal{I}_{1,2}) = \{\mathcal{I}_{2,2,1}, \mathcal{I}_{2,2,2}\}$, where $\mathcal{I}_{2,2,1} := \{(S_{31}, 2), (S_{32}, 2)\}$ and $\mathcal{I}_{2,2,2} := \{(S_{33}, 2)\}$ (see Figure 5). Since $\mathcal{I}(\boldsymbol{v}_{\text{root}}) \approx_{\mu,S_3,p_3} \mathcal{I}_{1,2}$, $\mathcal{I}_{1,i} \approx_{\mu,S_3,p_3} \mathcal{I}_{2,2,i}$ for $i = 1, 2$. Define

$$B_{2,2,i} := S_{\mathcal{I}_{2,2,i}}(\Omega) = S_3(B_{1,i}) \quad \text{for } i = 1, 2. \tag{4.1}$$

Thus, $\mathbf{P}_{2,2} = \mathbf{P}_{2,2}^1 = \{B_{2,2,1}, B_{2,2,2}\}$ and $\mathbf{P}_{2,2}^2 = \emptyset$. It follows that $\mathbf{P}_{k,2} = \mathbf{P}_{2,2}$ for all $k \geq 2$; in particular, $2 \in \Gamma_*$, $\kappa_2 = 2$, and $G_{2,2} = \{1, 2\}$.

Define

$$\mathcal{I}_{k,1,1} := \{(S_{2^{k-2}11}, k), (S_{2^{k-2}12}, k)\},$$
$$\mathcal{I}_{k,1,2} := \{(S_{2^{k-1}1}, k), (S_{2^k}, k)\}, \quad \mathcal{I}_{k,1,3} := \{(S_{2^{k-1}3}, k)\}$$

for $k \geq 2$. By the proof of [21, Example 3.3],

$$O(\mathcal{I}_{1,1}) = \{\mathcal{I}_{2,1,i} : i = 1, 2, 3\}, \quad O(\mathcal{I}_{k,1,2}) = \{\mathcal{I}_{k+1,1,i} : i = 1, 2, 3\},$$

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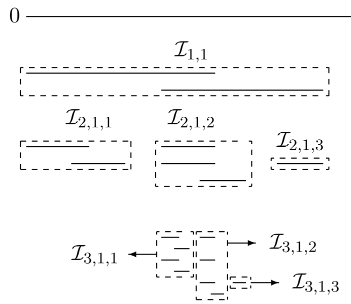

FIGURE 5. First, second, and third levels of iterations containing $\{\mathcal{I}_{1,\ell}\}$, $\{\mathcal{I}_{2,\ell,i}\}$, and $\{\mathcal{I}_{3,1,i}\}$. The figure is drawn with $\rho = 1/3$ and $r = 2/7$.

and $\mathcal{I}_{k,1,2}$ is the only level-k nonbasic island with respect to $\mathbb{I}_1$. For $k \geq 2$, let

$$\begin{aligned} B_{k,1,1} &:= S_{\mathcal{I}_{k,1,1}}(\Omega) = S_{2^{k-2}1}(B_{1,1}), \\ B_{k,1,2} &:= S_{\mathcal{I}_{k,1,2}}(\Omega) = S_{2^{k-1}}(B_{1,1}), \\ B_{k,1,3} &:= S_{\mathcal{I}_{k,1,3}}(\Omega) = S_{2^{k-1}}(B_{1,2}). \end{aligned} \tag{4.2}$$

Thus, $\mathbf{P}^1_{k,1} = \bigcup_{j=2}^k \{B_{j,1,1}, B_{j,1,3}\}$ and $\mathbf{P}^2_{k,1} = \{B_{k,1,2}\}$ for all $k \geq 2$. Consequently, $1 \in \Gamma'_*$, $\kappa_1 = \infty$, $G_{k,1} = \{1, 3\}$, and $G'_{k,1} = \{2\}$ for $k \geq 2$.

In the rest of this section, fix $q \geq 0$ and let $w_1(k)$ be defined as in (1.8).

First, we derive functional equations for $\Phi^{(\alpha)}_\ell(\delta)$ for $\ell = 1, 2$. Combining (3.12), (3.11), (4.2) and (4.1),

$$\varphi_1(\delta) = \sum_{j=2}^n \Big(\int_{B_{j,1,1}} + \int_{B_{j,1,3}}\Big)\mu(B_\delta(x))^q\,dx + \int_{B_{n,1,2}} \mu(B_\delta(x))^q\,dx$$

and

$$\varphi_2(\delta) = \Big(\int_{B_{2,2,1}} + \int_{B_{2,2,2}}\Big)\mu(B_\delta(x))^q\,dx.$$

For $\ell \in \Gamma$, $2 \leq k \leq \kappa_\ell$, $i \in G_{k,\ell}$ and $\delta > 0$, let $\widetilde{B}_{k,\ell,i}(\delta)$, $\widehat{B}_{k,\ell,i}(\delta)$, $\widetilde{B}_{1,c(k,\ell,i)}(\delta/r_{\sigma(k,\ell,i)})$ and $\widehat{B}_{1,c(k,\ell,i)}(\delta/r_{\sigma(k,\ell,i)})$ be defined as in Section 3. Combining (4.2) and (4.1), we have for $j \geq 2$,

$$\begin{aligned} \widetilde{B}_{j,1,1}(\delta) &= (S_{2^{j-2}11}(0) + \delta, S_{2^{j-2}12}(1) - \delta), \\ \widehat{B}_{j,1,1}(\delta) &= (S_{2^{j-2}11}(0), S_{2^{j-2}11}(0) + \delta) \cup (S_{2^{j-2}12}(1) - \delta, S_{2^{j-2}12}(1)), \\ \widetilde{B}_{j,1,3}(\delta) &= (S_{2^{j-1}3}(0) + \delta, S_{2^{j-1}3}(1) - \delta), \\ \widehat{B}_{j,1,3}(\delta) &= (S_{2^{j-1}3}(0), S_{2^{j-1}3}(0) + \delta) \cup (S_{2^{j-1}3}(1) - \delta, S_{2^{j-1}3}(1)), \\ \widetilde{B}_{2,2,1}(\delta) &= (S_{31}(0) + \delta, S_{32}(1) - \delta), \\ \widehat{B}_{2,2,1}(\delta) &= (S_{31}(0), S_{31}(0) + \delta) \cup (S_{32}(1) - \delta, S_{32}(1)), \\ \widetilde{B}_{2,2,2}(\delta) &= (S_{33}(0) + \delta, S_{33}(1) - \delta), \\ \widehat{B}_{2,2,2}(\delta) &= (S_{33}(0), S_{33}(0) + \delta) \cup (S_{33}(1) - \delta, S_{33}(1)). \end{aligned} \tag{4.3}$$

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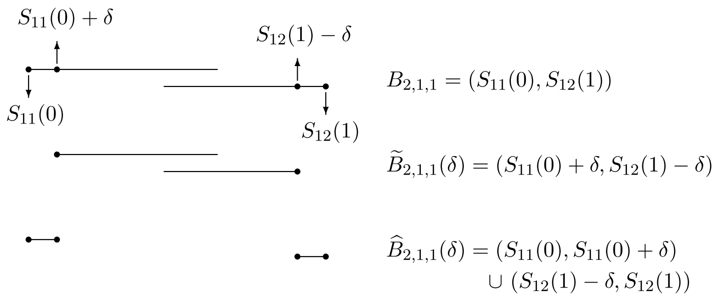

FIGURE 6. Figure showing the sets $B_{2,1,1}$, $\widetilde{B}_{2,1,1}(\delta)$, and $\widehat{B}_{2,1,1}(\delta)$.

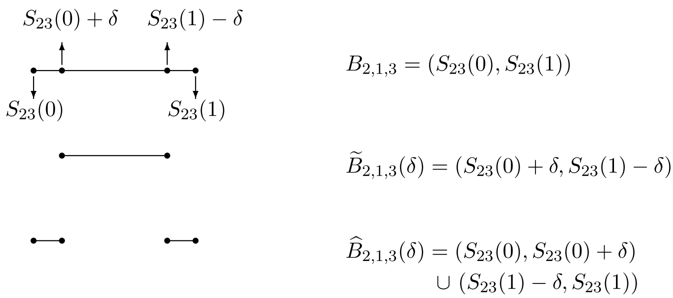

FIGURE 7. Figure showing the sets $B_{2,1,3}$, $\widetilde{B}_{2,1,3}(\delta)$, and $\widehat{B}_{2,1,3}(\delta)$.

(See Figures 6 and 7.)

It follows from (4.2), (4.1), and [21, Lemma 2.14] that for $j \geq 2$,

$$\mu(B_{j,1,1}) = w_1(j-2)\mu(B_{1,1}), \quad \mu(B_{j,1,2}) = p_2^{j-1}\mu(B_{1,2}).$$

Thus,

$$\begin{gathered} w(j,1,1) = w_1(j-2), \quad c(j,1,1) = 1, \quad \sigma(j,1,1) = S_{2^{j-2}1}, \quad r_{\sigma(j,1,1)} = \rho r^{j-2}, \\ w(j,1,3) = p_2^{j-1}, \quad c(j,1,3) = 2, \quad \sigma(j,1,3) = S_{2^{j-1}}, \quad r_{\sigma(j,1,3)} = r^{j-1}, \end{gathered}$$

and

$$\begin{gathered} \widetilde{B}_{1,1}(\delta/\rho r^{j-2}) = (S_1(0)+\delta/\rho r^{j-2}, S_2(1)-\delta/\rho r^{j-2}), \\ \widetilde{B}_{1,2}(\delta/r^{j-1}) = (S_3(0)+\delta/r^{j-1}, S_3(1)-\delta/r^{j-1}), \\ \widehat{B}_{1,1}(\delta/\rho r^{j-2}) = (S_1(0), S_1(0)+\delta/\rho r^{j-2}) \cup (S_2(1)-\delta/\rho r^{j-2}, S_2(1)), \\ \widehat{B}_{1,2}(\delta/r^{j-1}) = (S_3(0), S_3(0)+\delta/r^{j-1}) \cup (S_3(1)-\delta/r^{j-1}, S_3(1)). \end{gathered}$$

Since $\mu|_{S_3(B_{1,i})} = p_3\mu \circ S_3^{-1}$ on $S_3(B_{1,i})$ for $i = 1, 2$, by using (4.1), we have $\mu(B_{2,2,i}) = p_3\mu(B_{1,i})$. Hence $w(2,2,i) = p_3$, $c(2,2,i) = i$, $\sigma(2,2,i) = S_3$, $r_{\sigma(2,2,i)} = r$ and

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$$\begin{aligned}\widetilde{B}_{1,1}(\delta/r) &= (S_1(0)+\delta/r, S_2(1)-\delta/r),\\ \widetilde{B}_{1,2}(\delta/r) &= (S_3(0)+\delta/r, S_3(1)-\delta/r),\\ \widehat{B}_{1,1}(\delta/r) &= (S_1(0), S_1(0)+\delta/r)\cup(S_2(1)-\delta/r, S_2(1)),\\ \widehat{B}_{1,2}(\delta/r) &= (S_3(0), S_3(0)+\delta/r)\cup(S_3(1)-\delta/r, S_3(1)).\end{aligned}$$

By (3.14) and (3.13),

$$\begin{aligned}\varphi_1(\delta) = \sum_{j=2}^{n}\Big(& w_1(j-2)^q\rho r^{j-2}\int_{B_{1,1}}\mu(B_{\delta/\rho r^{j-2}}(x))^q\,dx\\ &+(p_2^q r)^{j-1}\int_{B_{1,2}}\mu(B_{\delta/r^{j-1}}(x))^q\,dx\Big)\\ &+\sum_{j=2}^{n}(e_j^1(\delta)-\tilde{e}_j^1(\delta))+\int_{B_{n,1,2}}\mu(B_\delta(x))^q\,dx,\end{aligned}\tag{4.4}$$

and

$$\varphi_2(\delta) = p_3^q r\Big(\int_{B_{1,1}}+\int_{B_{1,2}}\Big)\mu(B_{\delta/r}(x))^q\,dx + e_2^2(\delta)-\tilde{e}_2^2(\delta),\tag{4.5}$$

where

$$\begin{aligned}e_j^1(\delta) &= \Big(\int_{\widehat{B}_{j,1,1}(\delta)}+\int_{\widehat{B}_{j,1,3}(\delta)}\Big)\mu(B_\delta(x))^q\,dx,\\ \tilde{e}_j^1(\delta) &= w_1(j-2)^q\rho r^{j-2}\int_{\widehat{B}_{1,1}(\delta/\rho r^{j-2})}\mu(B_{\delta/\rho r^{j-2}}(x))^q\,dx\\ &\quad+(p_2^q r)^{j-1}\int_{\widehat{B}_{1,2}(\delta/r^{j-1})}\mu(B_{\delta/r^{j-1}}(x))^q\,dx,\\ e_2^2(\delta) &= \Big(\int_{\widehat{B}_{2,2,1}(\delta)}+\int_{\widehat{B}_{2,2,2}(\delta)}\Big)\mu(B_\delta(x))^q\,dx,\\ \tilde{e}_2^2(\delta) &= p_3^q r\Big(\int_{\widehat{B}_{1,1}(\delta/r)}+\int_{\widehat{B}_{1,2}(\delta/r)}\Big)\mu(B_{\delta/r}(x))^q\,dx.\end{aligned}\tag{4.6}$$

Multiplying both sides of (4.4) and (4.5) by $\delta^{-(1+\alpha)}$ and using (1.2),

$$\begin{aligned}\Phi_1^{(\alpha)}(\delta) = \sum_{j=2}^{n}(& w_1(j-2)^q(\rho r^{j-2})^{-\alpha}\Phi_1^{(\alpha)}(\delta/\rho r^{j-2})\\ &+(p_2^q r^{-\alpha})^{j-1}\Phi_2^{(\alpha)}(\delta/r^{j-1}))\\ &+\sum_{j=2}^{n}\delta^{-1-\alpha}(e_j^1(\delta)-\tilde{e}_j^1(\delta))+\delta^{-1-\alpha}\int_{B_{n,1,2}}\mu(B_\delta(x))^q\,dx\end{aligned}\tag{4.7}$$

and

$$\Phi_2^{(\alpha)}(\delta) = p_3^q r^{-\alpha}(\Phi_1^{(\alpha)}(\delta/r)+\Phi_2^{(\alpha)}(\delta/r))+\delta^{-1-\alpha}(e_2^2(\delta)-\tilde{e}_2^2(\delta)).$$

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Let $N := \max\{n \in \mathbb{N} : \delta \le \min\{\rho r^{n-2}, r^{n-1}\}\}$. Substituting $n = N$ in (4.7),

$$\begin{aligned}\Phi_1^{(\alpha)}(\delta) = \sum_{j=2}^{\infty}&(w_1(j-2)^q(\rho r^{j-2})^{-\alpha}\Phi_1^{(\alpha)}(\delta/\rho r^{j-2})\\ &+(p_2^q r^{-\alpha})^{j-1}\Phi_2^{(\alpha)}(\delta/r^{j-1}))\\ &+E_1^{(\alpha)}(\delta) - E_{1,\infty}^{(\alpha)}(\delta),\end{aligned} \tag{4.8}$$

where

$$E_1^{(\alpha)}(\delta) = \sum_{j=2}^{N}\delta^{-1-\alpha}(e_j^1(\delta) - \tilde{e}_j^1(\delta)) + \delta^{-1-\alpha}\int_{B_{N,1,2}}\mu(B_\delta(x))^q\,dx,$$

$$\begin{aligned}E_{1,\infty}^{(\alpha)}(\delta) = \sum_{j=N+1}^{\infty}&(w_1(j-2)^q(\rho r^{j-2})^{-\alpha}\Phi_1^{(\alpha)}(\delta/\rho r^{j-2})\\ &+(p_2^q r^{-\alpha})^{j-1}\Phi_2^{(\alpha)}(\delta/r^{j-1})).\end{aligned}$$

Let

$$\Phi_2^{(\alpha)}(\delta) = p_3^q r^{-\alpha}(\Phi_1^{(\alpha)}(\delta/r) + \Phi_2^{(\alpha)}(\delta/r)) + E_2^{(\alpha)}(\delta), \tag{4.9}$$

where

$$E_2^{(\alpha)}(\delta) = \delta^{-1-\alpha}(e_2^2(\delta) - \tilde{e}_2^2(\delta)).$$

Next, we derive a vector-valued equation. It follows from (3.20), (3.19), (4.8), and (4.9) that

$$\begin{aligned}f_1(x) = \sum_{j=2}^{\infty}&(w_1(j-2)^q(\rho r^{j-2})^{-\alpha}f_1(x+\ln(\rho r^{j-2}))\\ &+(p_2^q r^{-\alpha})^{j-1}f_2(x+\ln(r^{j-1}))) + z_1^{(\alpha)}(x)\end{aligned}$$

and

$$f_2(x) = p_3^q r^{-\alpha}\sum_{i=1}^{2}f_i(x+\ln r) + z_2^{(\alpha)}(x),$$

where $z_1^{(\alpha)}(x) = E_1^{(\alpha)}(e^{-x}) - E_{1,\infty}^{(\alpha)}(e^{-x})$, $z_2^{(\alpha)}(x) = E_2^{(\alpha)}(e^{-x})$. For $\ell, m = 1,2$, let $\mu_{m\ell}^{(\alpha)}$ be the discrete measures such that for $j \ge 2$,

$$\begin{aligned}\mu_{11}^{(\alpha)}(-\ln(\rho r^{j-2})) &= (w_1(j-2))^q(\rho r^{j-2})^{-\alpha},\\ \mu_{21}^{(\alpha)}(-\ln(r^{j-1})) &= (p_2^q r^{-\alpha})^{j-1},\\ \mu_{12}^{(\alpha)}(-\ln r) = \mu_{22}^{(\alpha)}(-\ln r) &= p_3^q r^{-\alpha}.\end{aligned}$$

Then

$$\mu_{11}^{(\alpha)}(\mathbb{R}) = \sum_{j=2}^{\infty}w_1(j-2)^q(\rho r^{j-2})^{-\alpha},$$

$$\mu_{21}^{(\alpha)}(\mathbb{R}) = \sum_{j=2}^{\infty}(p_2^q r^{-\alpha})^{j-1},$$

$$\mu_{12}^{(\alpha)}(\mathbb{R}) = \mu_{22}^{(\alpha)}(\mathbb{R}) = p_3^q r^{-\alpha}.$$

For fixed $q \geq 0$, let

$$\begin{aligned} F_1(\alpha) &:= \sum_{j=2}^{\infty} w_1(j-2)^q (\rho r^{j-2})^{-\alpha} + \sum_{j=2}^{\infty} (p_2^q r^{-\alpha})^{j-1}, \\ F_2(\alpha) &:= 2p_3^q r^{-\alpha}, \\ D_\ell &:= \{\alpha \in \mathbb{R} : F_\ell(\alpha) < \infty\} \quad \text{for } \ell = 1, 2, \end{aligned} \tag{4.10}$$

and

$$\mathbf{M}_\alpha(\infty) = \begin{pmatrix} \sum_{j=2}^{\infty} w_1(j-2)^q (\rho r^{j-2})^{-\alpha} & p_3^q r^{-\alpha} \\ \sum_{j=2}^{\infty} (p_2^q r^{-\alpha})^{j-1} & p_3^q r^{-\alpha} \end{pmatrix}.$$

Finally, we show that the error terms $z_\ell^{(\alpha)}(x) = o(e^{-\epsilon x})$ as $x \to \infty$, that is, $E_\ell^{(\alpha)}(\delta) = o(\delta^\epsilon)$ and $E_{1,\infty}^{(\alpha)}(\delta) = o(\delta^\epsilon)$ as $\delta \to 0$ for some $\epsilon > 0$ and $\ell = 1, 2$.

Proposition 4.1.

(1) $\Phi_1^{(\alpha)}(\delta/\rho r^k) \leq 1$ *for any* $k \geq N-1$;
(2) $\Phi_2^{(\alpha)}(\delta/r^k) \leq 1$ *for any* $k \geq N$.

Proof.
(1) It follows from the definition of N that $\delta \geq \rho r^k$ for any $k \geq N-1$. Hence

$$\Phi_1^{(\alpha)}(\delta/\rho r^k) = \frac{1}{(\delta/\rho r^k)^{1+\alpha}} \int_{B_{1,1}} \mu(B_{\delta/\rho r^k}(x))^q \, dx \leq (\rho r^k/\delta)^{1+\alpha} \leq 1.$$

Hence $\Phi_1^{(\alpha)}(\delta/\rho r^k) \leq 1$ for any $k \geq N-1$.
(2) The proof is similar to that of (1). □

The following proposition can be proved directly by using induction; we omit the details.

Proposition 4.2.

(1) $S_{2^k}(1) = r^k + \rho(1-r^k)$ *for any* $k \geq 1$;
(2) $S_{2^{k-1}1}(0) = \rho(1-r^{k-1})$ *for any* $k \geq 1$.

Proposition 4.3. *For $q \geq 0$, let $F_1(\alpha)$ and D_1 be defined as in (4.10). Then D_1 is open.*

Proof. Let $p := \max\{p_2, p_3\}$. In view of (1.8), we consider the following two cases for $w_1(k)$.

Case 1. $p_2 = p_3$. Then $w_1(k) = (k+1)p_1 p_2^k$; moreover,

$$p_1 p^k \leq w_1(k) = (k+1)p_1 p^k. \tag{4.11}$$

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Thus,

$$\begin{aligned}\lim_{k\to\infty} \sqrt[k]{w_1(k)^q(\rho r^k)^{-\alpha}} &= \lim_{k\to\infty} \sqrt[k]{((k+1)p_1p^k)^q(\rho r^k)^{-\alpha}} \\ &= \lim_{k\to\infty} \sqrt[k]{(k+1)^q p_1^q \rho^{-\alpha}} \cdot p^q/r^\alpha \\ &= p^q/r^\alpha. \end{aligned} \tag{4.12}$$

Case 2. $p_2 \neq p_3$. Assume $p_2 > p_3$. Then

$$w_1(k) = p_1p_2^k \sum_{j=0}^{k} (p_3/p_2)^j = p_1p_2^k \frac{1-(p_3/p_2)^{k+1}}{1-p_3/p_2}.$$

Note that

$$1 \le \frac{1-(p_3/p_2)^{k+1}}{1-p_3/p_2} < \frac{1}{1-p_3/p_2} = \frac{p_2}{p_2-p_3} =: c.$$

Thus, $p_1p_2^k \le w_1(k) \le cp_1p_2^k$. Similarly, if $p_3 > p_2$, $p_1p_3^k \le w_1(k) \le cp_1p_3^k$. So if $p_2 \neq p_3$,

$$p_1p^k \le w_1(k) \le cp_1p^k. \tag{4.13}$$

Hence

$$\lim_{k\to\infty} \sqrt[k]{w_1(k)^q(\rho r^k)^{-\alpha}} = p^q/r^\alpha \quad \text{if } p_2 \neq p_3. \tag{4.14}$$

Combining (4.12) and (4.14), $\lim_{k\to\infty} \sqrt[k]{w_1(k)^q(\rho r^k)^{-\alpha}} = p^q/r^\alpha$. By the root test, the series $\sum_{k=0}^{\infty} w_1(k)^q(\rho r^k)^{-\alpha}$ is convergent if $p^q/r^\alpha < 1$, that is, $\sum_{k=0}^{\infty} w_1(k)^q(\rho r^k)^{-\alpha}$ and $\sum_{k=0}^{\infty}(p^q/r^\alpha)^k$ have the same radius of convergence. If $p^q/r^\alpha = 1$, then $\sum_{k=0}^{\infty}(p^q/r^\alpha)^k = \infty$. It follows from (4.11) and (4.13) that $(p_1p^k)^q \le w_1(k)^q$ for $q \ge 0$. For $k \ge 0$, $(p_1p^k)^q(\rho r^k)^{-\alpha} \le w_1(k)^q(\rho r^k)^{-\alpha}$. Thus,

$$\infty = p_1^q\rho^{-\alpha} \sum_{k=0}^{\infty} (p^q/r^\alpha)^k \le \sum_{k=0}^{\infty} w_1(k)^q(\rho r^k)^{-\alpha}.$$

Hence D_1 is open. $\square$

PROPOSITION 4.4. *For $q \ge 0$, assume that $\alpha \in D_\ell$ for $\ell = 1, 2$. Then there exists $\epsilon > 0$ such that:*

(1) $\sum_{j=N+1}^{\infty} w_1(j-2)^q(\rho r^{j-2})^{-\alpha}\Phi_1^{(\alpha)}(\delta/\rho r^{j-2}) = o(\delta^\epsilon)$;
(2) $\sum_{j=N+1}^{\infty} (p_2^q r^{-\alpha})^{j-1}\Phi_2^{(\alpha)}(\delta/r^{j-1}) = o(\delta^\epsilon)$;
(3) $\sum_{j=2}^{N} \delta^{-1-\alpha}(e_j^1(\delta) - \tilde{e}_j^1(\delta)) = o(\delta^\epsilon)$;
(4) $\delta^{-1-\alpha}\int_{B_{N,1,2}} \mu(B_\delta(x))^q\,dx = o(\delta^\epsilon)$;
(5) $\delta^{-1-\alpha}(e_2^2(\delta) - \tilde{e}_2^2(\delta)) = o(\delta^\epsilon)$.

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PROOF.
(1) By Proposition 4.3, $D_1 = \{\alpha \in \mathbb{R} : F_1(\alpha) < \infty\}$ is open. Thus, there exists $\epsilon > 0$ sufficiently small such that $\alpha + \epsilon \in D_1$. So there exists a positive constant C such that

$$\sum_{j=N+1}^{\infty} w_1(j-2)^q (\rho r^{j-2})^{-\alpha-\epsilon} + \sum_{j=N+1}^{\infty} (p_2^q r^{-\alpha-\epsilon})^{j-1} \le C.$$

Since

$$(\rho r^{N-1})^{-\epsilon} \sum_{j=N+1}^{\infty} w_1(j-2)^q (\rho r^{j-2})^{-\alpha} \le \sum_{j=N+1}^{\infty} w_1(j-2)^q (\rho r^{j-2})^{-\alpha-\epsilon},$$

$\sum_{j=N+1}^{\infty} w_1(j-2)^q(\rho r^{j-2})^{-\alpha} \le C(\rho r^{N-1})^{\epsilon} \le C\delta^{\epsilon}$, where the last inequality follows from the definition of N. Combining these with Proposition 4.1(1), $\sum_{j=N+1}^{\infty} w_1(j-2)^q(\rho r^{j-2})^{-\alpha}\Phi_1^{(\alpha)}(\delta/\rho r^{j-2}) = o(\delta^{\epsilon})$.

(2) The proof is similar to that of (1).

(3) It suffices to show that $e_j^1(\delta) = o(\delta^{1+\alpha+\epsilon})$ and $\tilde{e}_j^1(\delta) = o(\delta^{1+\alpha+\epsilon})$ for $2 \le j \le N$. It follows from (4.6) and (4.3) that

$$\begin{aligned} e_j^1(\delta) = \Big(&\int_{S_{2^{j-2}1_1}(0)}^{S_{2^{j-2}1_1}(0)+\delta} + \int_{S_{2^{j-2}1_2}(1)-\delta}^{S_{2^{j-2}1_2}(1)} \\ &+ \int_{S_{2^{j-1}3}(0)}^{S_{2^{j-1}3}(0)+\delta} + \int_{S_{2^{j-1}3}(1)-\delta}^{S_{2^{j-1}3}(1)}\Big)\mu(B_\delta(x))^q\,dx. \end{aligned}$$

As an example we only prove $\int_{S_{2^{j-2}1_1}(0)}^{S_{2^{j-2}1_1}(0)+\delta} \mu(B_\delta(x))^q\,dx = o(\delta^{1+\alpha+\epsilon})$. It follows from (1) and (2) that

$$w_1(N-1)^q \le C\delta^{\alpha+\epsilon}, \quad p_2^{Nq} \le C\delta^{\alpha+\epsilon}. \tag{4.15}$$

Since $B_\delta(x) \subseteq B_{2\delta}(S_{2^{j-2}1_1}(0))$ for any $x \in (S_{2^{j-2}1_1}(0), S_{2^{j-2}1_1}(0)+\delta)$ and $\mu(B_{2\delta}(S_{2^{j-2}1_1}(0))) = p_1 w_1(j-2)\mu(B_{2\delta/\rho^2 r^{j-2}}(0)) \le p_1 w_1(j-2)$,

$$\begin{aligned} \int_{S_{2^{j-2}1_1}(0)}^{S_{2^{j-2}1_1}(0)+\delta} \mu(B_\delta(x))^q\,dx &\le (\mu(B_{2\delta}(S_{2^{j-2}1_1}(0))))^q\delta \le p_1^q w_1(j-2)^q\delta \\ &\le (p_1 p_2^{1-N})^q w_1(N-1)^q\delta \le C(p_1p_2^{1-N})^q\delta^{1+\alpha+\epsilon}, \end{aligned}$$

where the third inequality holds because for $0 \le k \le N-2$

$$\begin{aligned} w_1(k) &= \frac{p_1(p_2^{N-1} + p_2^{N-2}p_3 + \cdots + p_3^{N-1})(p_2^k + p_2^{k-1}p_3 + \cdots + p_3^k)}{(p_2^{N-1} + p_2^{N-2}p_3 + \cdots + p_3^{N-1})} \\ &\le \frac{w_1(N-1)(p_2+p_3)^k}{p_2^{N-1} + p_2^{N-2}p_3 + \cdots + p_3^{N-1}} \le p_2^{1-N} w_1(N-1), \end{aligned} \tag{4.16}$$

and the last inequality follows from (4.15). The estimate $\tilde{e}_j^1(\delta) = o(\delta^{1+\alpha+\epsilon})$ can be established as that for $e_j^1(\delta) = o(\delta^{1+\alpha+\epsilon})$.

(4) By (4.2),

$$\int_{B_{N,1,2}} \mu(B_\delta(x))^q\,dx = \bigg(\int_{S_{2^{N-1}1}(0)}^{S_{2^{N-1}1}(0)+\delta} + \int_{S_{2^{N-1}1}(0)+\delta}^{S_{2^N}(1)-\delta} + \int_{S_{2^N}(1)-\delta}^{S_{2^N}(1)}\bigg)\mu(B_\delta(x))^q\,dx$$
$$=: (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}).$$

We first show that $\delta^{-1-\alpha}(\mathrm{I}) = o(\delta^{\epsilon/2})$. For any $x \in (S_{2^{N-1}1}(0), S_{2^{N-1}1}(0)+\delta)$, $B_\delta(x) \subseteq B_{2\delta}(S_{2^{N-1}1}(0))$ and

$$\mu(B_{2\delta}(S_{2^{N-1}1}(0))) = w_1(N-1)\mu(B_{2\delta/\rho r^{N-1}}(0)) \le w_1(N-1).$$

Combining these with (4.15),

$$(\mathrm{I}) \le \mu(B_{2\delta}(S_{2^{N-1}1}(0)))^q\delta \le w_1(N-1)^q\delta \le C\delta^{1+\alpha+\epsilon}.$$

It follows that $\delta^{-1-\alpha}(\mathrm{I}) = o(\delta^{\epsilon/2})$.

Next, we show that $\delta^{-1-\alpha}(\mathrm{II}) = o(\delta^{\epsilon/2})$. It follows from [21, Lemma 2.14] that

$$\mu|_{S_{2^{N-1}}(B_{1,1})} = w_1(N-2)\mu\circ S^{-1}_{2^{N-2}1} + p_2^{N-1}\mu\circ S^{-1}_{2^{N-1}} \quad \text{on } S_{2^{N-1}}(B_{1,1}).$$

Thus, $\mu(B_\delta(x)) \le w_1(N-2) + p_2^{N-1}$ for $x \in (S_{2^{N-1}1}(0)+\delta, S_{2^N}(0)-\delta)$. Combining Proposition 4.2, (4.16), (4.15), and (1.7),

$$\begin{aligned}
(\mathrm{II}) &\le (S_{2^N}(1) - S_{2^{N-1}1}(0) - 2\delta)(w_1(N-2) + p_2^{N-1})^q\\
&\le r^{N-1}(2r+\rho(1-r))(p_2^{1-N}w_1(N-1) + p_2^{-1}p_2^N)^q\\
&\le r^{N-1}(2r+\rho(1-r))((C\delta^{\alpha+\epsilon})^{1/q} + (C\delta^{\alpha+\epsilon})^{1/q})^q\\
&\le C'r^{N-1}\delta^{\alpha+\epsilon} \le C'r^{-1}\delta^{1+\alpha+\epsilon};
\end{aligned}$$

that is, $\delta^{-1-\alpha}(\mathrm{II}) = o(\delta^{\epsilon/2})$.

The proof of $\delta^{-1-\alpha}(\mathrm{III}) = o(\delta^{\epsilon/2})$ is similar to that for $\delta^{-1-\alpha}(\mathrm{I}) = o(\delta^{\epsilon/2})$. Hence

$$\delta^{-1-\alpha}\int_{S_{2^{N-1}}(B_{1,1})} \mu(B_\delta(x))^q\,dx = o(\delta^{\epsilon/2}).$$

(5) The proof is similar to that of (3). □

Proof (Theorem 1.2). Combining Theorem 1.1 and Proposition 4.4 yields $\tau(q) = \alpha$. Let

$$G(q,\alpha) := (1 - p_2^q r^{-\alpha})(1 - p_3^q r^{-\alpha})\sum_{k=0}^{\infty} w_1(k)^q(\rho r^k)^{-\alpha}$$
$$+ r^{-\alpha}(p_2^q + p_3^q) - 1.$$

We show that $G(q,\alpha)$ is C^1. It follows from Proposition 4.3 that

$$\sum_{k=0}^{\infty} w_1(k)^q(\rho r^k)^{-\alpha} < \infty \qquad \text{for any } (q,\alpha) \in (0,\infty)\times D_1.$$

Since $w_1(k) \le p_1 < 1$, $\sum_{k=0}^{\infty} w_1(k)^q(\rho r^k)^{-\alpha}$ is strictly decreasing in q and strictly increasing in α. Thus, for any $(q_0, \alpha_0) \in (0, \infty) \times D_1$, the series converges uniformly on $\{(q, \alpha) : q \ge q_0, \alpha \le \alpha_0\}$. Moreover, it follows from (4.11) and (4.13) that

$$\lim_{k\to\infty} w_1(k) = 0.$$

Hence, for any $(q, \alpha) \in (0, \infty) \times D_1$,

$$\begin{aligned} G_q(q,\alpha) = (&-p_2^q r^{-\alpha}(1 - p_3^q r^{-\alpha}) \ln p_2 \\ &- p_3^q r^{-\alpha}(1 - p_2^q r^{-\alpha}) \ln p_3) \sum_{k=0}^{\infty} w_1(k)^q(\rho r^k)^{-\alpha} \\ &+ (1 - p_2^q r^{-\alpha})(1 - p_3^q r^{-\alpha}) \sum_{k=0}^{\infty} w_1(k)^q(\rho r^k)^{-\alpha} \ln w_1(k) \\ &+ r^{-\alpha} \sum_{i=2}^{3} p_i^q \ln p_i \end{aligned}$$

and

$$\begin{aligned} G_\alpha(q,\alpha) = (p_2^q(1 - p_3^q r^{-\alpha}) &+ p_3^q(1 - p_2^q r^{-\alpha}))r^{-\alpha} \ln r \sum_{k=0}^{\infty} w_1(k)^q(\rho r^k)^{-\alpha} \\ &+ (1 - p_2^q r^{-\alpha})(1 - p_3^q r^{-\alpha}) \sum_{k=0}^{\infty} w_1(k)^q(\rho r^k)^{-\alpha} \ln(\rho r^k)^{-1} \\ &+ r^{-\alpha} \sum_{i=2}^{3} p_i^q \ln r^{-1}. \end{aligned}$$

A similar argument as above shows that $G(q, \alpha)$ is C^1.

We now show that $G_\alpha(\tilde{q}, \tilde{\alpha}) \neq 0$ for any $(\tilde{q}, \tilde{\alpha}) \in (0, \infty) \times D_1$ satisfying $G(\tilde{q}, \tilde{\alpha}) = 0$. Since $\tau(q)$ is convex, we can let $\{q_n\}$ be an increasing sequence of positive numbers such that $\lim_{n\to\infty} q_n = \tilde{q}$ and that τ is differentiable at each q_n. Then (1.9) implies that

$$G_q(q_n, \alpha_n) + G_\alpha(q_n, \alpha_n) \cdot \alpha'(q_n) = 0 \quad \text{for all } n,$$

and thus,

$$G_q(\tilde{q}, \tilde{\alpha}) + G_\alpha(\tilde{q}, \tilde{\alpha}) \cdot \alpha'_-(\tilde{q}) = 0,$$

where $\alpha'_-(\tilde{q})$ denotes the left-hand derivative of $\alpha(q)(= \tau(q))$ at $\tilde{q}$.

Suppose, in contrast, that $G_\alpha(\tilde{q}, \tilde{\alpha}) = 0$. Then $G_q(\tilde{q}, \tilde{\alpha}) = 0$. So $G_\alpha(\tilde{q}, \tilde{\alpha}) - G_q(\tilde{q}, \tilde{\alpha}) = 0$. It follows from $G(\tilde{q}, \tilde{\alpha}) = 0$ that

$$\sum_{k=0}^{\infty} w_1(k)^{\tilde{q}}(\rho r^k)^{-\tilde{\alpha}} = \frac{1 - (p_2^{\tilde{q}} + p_3^{\tilde{q}})r^{-\tilde{\alpha}}}{(1 - p_2^{\tilde{q}} r^{-\tilde{\alpha}})(1 - p_3^{\tilde{q}} r^{-\tilde{\alpha}})}. \tag{4.17}$$

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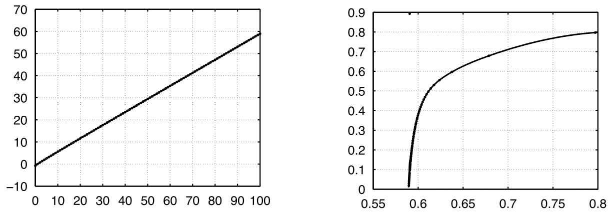

(a) $\tau(q)$ (b) $f(\alpha)$

FIGURE 8. Graphs of $\tau(q)$ and $f(\alpha)$ for the self-similar measure in Example 2.21 with $\rho = 1/3$, $r = 2/7$, $p_1 = 1/2$, $p_2 = 1/4$, and $p_3 = 1/4$.

Substituting (4.17) into the above expressions for G_q and G_α, simplifying the result, and using the fact that $0 < p_i^{\tilde{q}} r^{-\tilde{\alpha}} < 1$ for $i = 2, 3$,

$$\begin{aligned} 0 &= G_\alpha(\tilde{q}, \tilde{\alpha}) - G_q(\tilde{q}, \tilde{\alpha}) \\ &= p_2^{\tilde{q}} r^{-\tilde{\alpha}} (\ln r^{-1} - \ln p_2) \frac{p_3^{\tilde{q}} r^{-\tilde{\alpha}}}{1 - p_2^{\tilde{q}} r^{-\tilde{\alpha}}} + p_3^{\tilde{q}} r^{-\tilde{\alpha}} (\ln r^{-1} - \ln p_3) \frac{p_2^{\tilde{q}} r^{-\tilde{\alpha}}}{1 - p_3^{\tilde{q}} r^{-\tilde{\alpha}}} \\ &\qquad + (1 - p_2^{\tilde{q}} r^{-\tilde{\alpha}})(1 - p_3^{\tilde{q}} r^{-\tilde{\alpha}}) \sum_{k=0}^{\infty} w_1(k)^{\tilde{q}} (\rho r^k)^{-\tilde{\alpha}} (\ln(\rho r^k)^{-1} - \ln w_1(k)) \\ &> 0, \end{aligned}$$

a contradiction. Hence $G_\alpha(q, \alpha) \neq 0$ for any $(q, \alpha) \in (0, \infty) \times D_1$ satisfying $G(q, \alpha) = 0$. The implicit function theorem now implies that τ is differentiable on $(0, \infty)$ and the stated formula for $\dim_{\mathrm{H}}(\mu)$ follows by computing $\tau'(1) = -G_q(1, 0) G_\alpha(1, 0)^{-1}$ (see [9, 19]). This completes the proof. $\square$

Figure 8 shows the graphs of $\tau(q)$ and $f(\alpha)$, $q \geq 0$, for some measure in the family. For this example, $\dim_{\mathrm{H}}(\mu) = \tau'(1) \approx 0.720268$ and $\dim_{\mathrm{H}}(K) = -\tau(0) \approx 0.797012$, where K is the self-similar set corresponding to the IFS in (1.6).

5. A class of examples in $\mathbb{R}^2$

In this section, we derive renewal equations and compute the L^q-spectrum of self-similar measure μ defined by the IFSs in (1.10) together with a probability vector $(p_i)_{i=1}^4$. Let $X := [0, 1] \times [0, 1]$, $\Omega = (0, 1) \times (0, 1)$. Define

$$\mathcal{I}_{1,1} = \{(S_1, 1), (S_2, 1)\}, \quad \mathcal{I}_{1,2} = \{(S_3, 1)\}, \quad \mathcal{I}_{1,3} = \{(S_4, 1)\},$$

and

$$B_{1,\ell} := S_{\mathcal{I}_{1,\ell}}(\Omega) \quad \text{for } \ell \in \Gamma,$$

where $\Gamma = \{1, 2, 3\}$. For $\ell \in \Gamma$ and $k \geq 1$, let $\mathbf{P}_{k,\ell}$ be defined as in (2.5) and (2.6). It follows from Example 2.22 that μ satisfies EFT with $\Omega = (0, 1) \times (0, 1)$ being

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an EFT-set, $\mathbf{B} := \{B_{1,\ell}\}_{\ell\in\Gamma}$ being a weakly regular basic family of cells in Ω, and $(\mathbf{B},\mathbf{P}) := (\{B_{1,\ell}\},\{\mathbf{P}_{k,\ell}\}_{k\geq 1})_{\ell\in\Gamma}$ being a weakly regular basic pair with respect to Ω.

In the rest of this section, we use the notation defined in Section 3. For $\mathcal{I}\in\mathbb{I}$, let $S_{\mathcal{I}}(\Omega)$ and $O(\mathcal{I})$ be defined as in (2.1) and (2.2), respectively. For $\ell\in\Gamma$, $i=1,2$, and $k\geq 2$, let $\mathbf{P}^i_{k,\ell}$ be defined as in (2.4). We first observe that for $\ell=2,3$, $O(\mathcal{I}_{1,\ell}) = \{\mathcal{I}_{2,\ell,i}, i=1,2,3\}$, where $\mathcal{I}_{2,\ell,1} := \{(S_{(\ell+1)1},2),(S_{(\ell+1)2},2)\}$, $\mathcal{I}_{2,\ell,2} := \{(S_{(\ell+1)3},2)\}$, and $\mathcal{I}_{2,\ell,3} := \{(S_{(\ell+1)4},2)\}$ (see Figure 2). Since for $\ell=2,3$, $\mathcal{I}(\boldsymbol{v}_{\mathrm{root}}) \approx_{\mu,S_{\ell+1},p_{\ell+1}} \mathcal{I}_{1,\ell}$, we have $\mathcal{I}_{1,i}\approx_{\mu,S_{\ell+1},p_{\ell+1}} \mathcal{I}_{2,\ell,i}$ for $i=1,2,3$. For $\ell=2,3$, define

$$B_{2,\ell,i} := S_{\mathcal{I}_{2,\ell,i}}(\Omega) = S_{\ell+1}(B_{1,i}), \quad i=1,2,3. \tag{5.1}$$

Thus, $\mathbf{P}_{2,\ell} = \mathbf{P}^1_{2,\ell} = \{B_{2,\ell,i}, i=1,2,3\}$ and $\mathbf{P}^2_{2,\ell}=\emptyset$. It follows that $\mathbf{P}_{k,\ell}=\mathbf{P}_{2,\ell}$ for all $k\geq 2$; in particular, for $\ell=2,3$, $\ell\in\Gamma_*$, $\kappa_\ell=2$, and $G_{2,\ell}=\{1,2,3\}$.

By the proof of Example 2.22,

$$O(\mathcal{I}_{1,1}) = \{\mathcal{I}_{2,1,i}, i=1,\dots,5\}, \quad O(\mathcal{I}_{k,1,3}) = \{\mathcal{I}_{k+1,1,i}, i=1,\dots,5\},$$

and $\mathcal{I}_{k,1,3}$ is the only level-k nonbasic island with respect to $\mathbb{I}_1$. For $k\geq 2$, define

$$\begin{aligned}
B_{k,1,1} &:= S_{\mathcal{I}_{k,1,1}}(\Omega) = S_{2^{k-2}1}(B_{1,1}),\\
B_{k,1,2} &:= S_{\mathcal{I}_{k,1,2}}(\Omega) = S_{2^{k-2}1}(B_{1,3}),\\
B_{k,1,3} &:= S_{\mathcal{I}_{k,1,3}}(\Omega) = S_{2^{k-1}}(B_{1,1}),\\
B_{k,1,4} &:= S_{\mathcal{I}_{k,1,4}}(\Omega) = S_{2^{k-1}}(B_{1,2}),\\
B_{k,1,5} &:= S_{\mathcal{I}_{k,1,5}}(\Omega) = S_{2^{k-1}}(B_{1,3}).
\end{aligned} \tag{5.2}$$

Thus, $\mathbf{P}^1_{k,1} = \bigcup_{j=2}^k\{B_{j,1,i}, i=1,2,4,5\}$ and $\mathbf{P}^2_{k,1}=\{B_{k,1,3}\}$ for all $k\geq 2$. Consequently, $1\in\Gamma'_*$, $\kappa_1=\infty$, $G_{k,1}=\{1,2,4,5\}$, and $G'_{k,1}=\{3\}$ for $k\geq 2$.

In the rest of this section, let $w_2(k)$ be defined as in (1.12). First, we derive functional equations for $\Phi^{(\alpha)}_\ell(\delta)$ for $\ell=1,2,3$. Combining (3.11), (3.12), (5.2), and (5.1),

$$\varphi_1(\delta) = \bigg(\sum_{j=2}^n\bigg(\int_{B_{j,1,1}}+\int_{B_{j,1,2}}+\int_{B_{j,1,4}}+\int_{B_{j,1,5}}\bigg)+\int_{B_{n,1,3}}\bigg)\mu(B_\delta(\mathbf{x}))^q\,d\mathbf{x},$$

and

$$\varphi_\ell(\delta) = \sum_{i=1}^3\int_{B_{2,\ell,i}}\mu(B_\delta(\mathbf{x}))^q\,d\mathbf{x}\quad\text{for } \ell=2,3.$$

For $\ell\in\Gamma$, $2\leq k\leq\kappa_\ell$, $i\in G_{k,\ell}$, and $\delta>0$, let $\widetilde{B}_{k,\ell,i}(\delta)$, $\widehat{B}_{k,\ell,i}(\delta)$, $\widetilde{B}_{1,c(k,\ell,i)}(\delta/r_{\sigma(k,\ell,i)})$, and $\widehat{B}_{1,c(k,\ell,i)}(\delta/r_{\sigma(k,\ell,i)})$ be defined as in Section 3. Recall from (2.9) that $\gamma_k := 1-r^k$. Combining (5.2), (5.1), and Proposition 2.24, we have for $j\geq 2$,

$$\begin{aligned}
\widetilde{B}_{j,1,1}(\delta) = {} & (\rho\gamma_{j-2}+\delta, \rho\gamma_{j-2}+\rho^2r^{j-2}\gamma_1+\delta)\times(\delta,\rho^2r^{j-2}-\delta)\\
&\cup(\rho\gamma_{j-2}+\rho^2r^{j-2}\gamma_1+\delta, \rho\gamma_{j-2}+\rho^2r^{j-2}\gamma_1+\rho r^{j-1}-\delta)\\
&\quad\times(\delta,\rho r^{j-1}-\delta),
\end{aligned}$$

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$$\begin{aligned}
\widehat{B}_{j,1,1}(\delta) = {} & (\rho\gamma_{j-2}, \rho\gamma_{j-2} + \rho^2 r^{j-2}\gamma_1 + \rho\gamma_{j-1}) \times (0, \delta) \\
& \cup (\rho\gamma_{j-2}, \rho\gamma_{j-2} + \rho^2 r^{j-2}\gamma_1 + \delta) \times (\rho^2 r^{j-2} - \delta, \rho^2 r^{j-2}) \\
& \cup (\rho\gamma_{j-2} + \rho^2 r^{j-2}\gamma_1, \rho\gamma_{j-2} + \rho^2 r^{j-2}\gamma_1 + \rho r^{j-1}) \\
& \qquad \times (\rho r^{j-1} - \delta, \rho r^{j-1}) \\
& \cup (\rho\gamma_{j-2}, \rho\gamma_{j-2} + \delta) \times (\delta, \rho^2 r^{j-2} - \delta) \\
& \cup (\rho\gamma_{j-2} + \rho^2 r^{j-2}\gamma_1, \rho\gamma_{j-2} + \rho^2 r^{j-2}\gamma_1 + \delta) \\
& \qquad \times (\rho^2 r^{j-2}, \rho r^{j-1} - \delta) \\
& \cup (\rho\gamma_{j-2} + \rho^2 r^{j-2}\gamma_1 + \rho r^{j-1} - \delta, \rho\gamma_{j-2} + \rho^2 r^{j-2}\gamma_1 + \rho r^{j-1}) \\
& \qquad \times (\delta, \rho r^{j-1} - \delta),
\end{aligned}$$

$$\begin{aligned}
\widetilde{B}_{j,1,2}(\delta) = {} & (\rho\gamma_{j-2} + \delta, \rho\gamma_{j-2} + \rho r^{j-1} - \delta) \times (\rho r^{j-2}\gamma_1 + \delta, \rho r^{j-2} - \delta), \\
\widehat{B}_{j,1,2}(\delta) = {} & (\rho\gamma_{j-2}, \rho\gamma_{j-2} + \rho r^{j-1}) \\
& \qquad \times ((\rho r^{j-2}\gamma_1, \rho r^{j-2}\gamma_1 + \delta) \cup (\rho r^{j-2} - \delta, \rho r^{j-2})) \\
& \cup ((\rho\gamma_{j-2}, \rho\gamma_{j-2} + \delta) \cup (\rho\gamma_{j-2} + \rho r^{j-1} - \delta, \rho\gamma_{j-2} + \rho r^{j-1})) \\
& \qquad \times (\rho r^{j-2}\gamma_1 + \delta, \rho r^{j-2} - \delta),
\end{aligned}$$

$$\begin{aligned}
\widetilde{B}_{j,1,4}(\delta) = {} & (r^{j-1}\gamma_1 + \rho\gamma_{j-1} + \delta, r^{j-1} + \rho\gamma_{j-1} - \delta) \times (\delta, r^j - \delta), \\
\widehat{B}_{j,1,4}(\delta) = {} & (r^{j-1}\gamma_1 + \rho\gamma_{j-1}, r^{j-1} + \rho\gamma_{j-1}) \times ((0, \delta) \cup (r^j - \delta, r^j)) \\
& \cup ((r^{j-1}\gamma_1 + \rho\gamma_{j-1}, r^{j-1}\gamma_1 + \rho\gamma_{j-1} + \delta) \\
& \cup (r^{j-1} + \rho\gamma_{j-1} - \delta, r^{j-1} + \rho\gamma_{j-1})) \times (\delta, r^k - \delta),
\end{aligned}$$

$$\begin{aligned}
\widetilde{B}_{j,1,5}(\delta) = {} & (\rho\gamma_{j-1} + \delta, r^j + \rho\gamma_{j-1} - \delta) \times (r^{j-1}\gamma_1 + \delta, r^{j-1} - \delta), \\
\widehat{B}_{j,1,5}(\delta) = {} & (\rho\gamma_{j-1}, r^j + \rho\gamma_{j-1}) \times ((r^{j-1}\gamma_1, r^{j-1}\gamma_1 + \delta) \cup (r^{j-1} - \delta, r^{j-1})) \\
& \cup ((\rho\gamma_{j-1}, \rho\gamma_{j-1} + \delta) \cup (r^j + \rho\gamma_{j-1} - \delta, r^j + \rho\gamma_{j-1})) \\
& \qquad \times (r^{j-1}\gamma_1 + \delta, r^{j-1} - \delta),
\end{aligned}$$

$$\begin{aligned}
\widetilde{B}_{2,2,1}(\delta) = {} & (\gamma_1 + \delta, (1 + \rho r)\gamma_1 + \delta) \times (\delta, \rho r - \delta) \\
& \cup ((1 + \rho r)\gamma_1 + \delta, (1 + \rho r)\gamma_1 + r^2 - \delta) \times (\delta, r^2 - \delta), \\
\widehat{B}_{2,2,1}(\delta) = {} & (\gamma_1, (1 + \rho r)\gamma_1 + r^2) \times (0, \delta) \cup (\gamma_1, (1 + \rho r)\gamma_1 + \delta) \times (\rho r - \delta, \rho r) \\
& \cup ((1 + \rho r)\gamma_1, (1 + \rho r)\gamma_1 + r^2) \times (r^2 - \delta, r^2) \\
& \cup (\gamma_1, \gamma_1 + \delta) \times (\delta, \rho r - \delta) \\
& \cup ((1 + \rho r)\gamma_1, (1 + \rho r)\gamma_1 + \delta) \times (\rho r, r^2 - \delta) \\
& \cup ((1 + \rho r)\gamma_1 + r^2 - \delta, (1 + \rho r)\gamma_1 + r^2) \times (\delta, r^2 - \delta),
\end{aligned}$$

$$\begin{aligned}
\widetilde{B}_{2,2,2}(\delta) = {} & (\gamma_2 + \delta, 1 - \delta) \times (\delta, r^2 - \delta), \\
\widehat{B}_{2,2,2}(\delta) = {} & (\gamma_2, 1) \times ((0, \delta) \cup (r^2 - \delta, r^2)) \\
& \cup ((\gamma_2, \gamma_2 + \delta) \cup (1 - \delta, 1)) \times (\delta, r^2 - \delta),
\end{aligned}$$

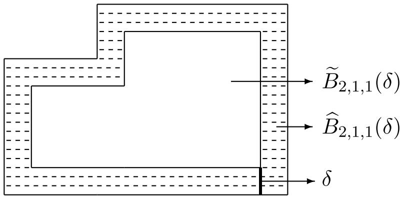

FIGURE 9. The middle part and the shaded region are $\widetilde{B}_{2,1,1}(\delta)$ and $\widehat{B}_{2,1,1}(\delta)$, respectively. The union is $B_{2,1,1}$.

$$\begin{aligned}
\widetilde{B}_{2,2,3}(\delta) &= (\gamma_1+\delta, r^2+\gamma_1-\delta)\times(r\gamma_1+\delta, r-\delta),\\
\widehat{B}_{2,2,3}(\delta) &= ((\gamma_1,\gamma_1+\delta)\cup(r^2+\gamma_1-\delta, r^2+\gamma_1))\times(r\gamma_1+\delta, r-\delta)\\
&\quad\cup(\gamma_1, r^2+\gamma_1)\times((r\gamma_1, r\gamma_{+}\delta)\cup(r-\delta, r)),
\end{aligned}$$

$$\begin{aligned}
\widetilde{B}_{2,3,1}(\delta) &= (\delta, \rho r\gamma_1+\delta)\times(\gamma_1+\delta, \rho r+\gamma_1-\delta)\\
&\quad\cup(\rho r\gamma_1+\delta, \rho r\gamma_1+r^2-\delta)\times(\gamma_1+\delta, r^2+\gamma_1-\delta),\\
\widehat{B}_{2,3,1}(\delta) &= (0, \rho r\gamma_1+r^2)\times(\gamma_1, \gamma_1+\delta)\\
&\quad\cup(0, \rho r\gamma_1+\delta)\times(\rho r+\gamma_1-\delta, \rho r+\gamma_1)\\
&\quad\cup(\rho r\gamma_1, \rho r\gamma_1+r^2)\times(r^2+\gamma_1-\delta, r^2+\gamma_1)\\
&\quad\cup(0,\delta)\times(\gamma_1+\delta, \rho r+\gamma_1-\delta)\\
&\quad\cup(\rho r\gamma_1, \rho r\gamma_1+\delta)\times(\rho r+\gamma_1, r^2+\gamma_1-\delta)\\
&\quad\cup(\rho r\gamma_1+r^2-\delta, \rho r\gamma_1+r^2)\times(\gamma_1+\delta, r^2+\gamma_1-\delta),
\end{aligned}$$

$$\begin{aligned}
\widetilde{B}_{2,3,2}(\delta) &= (r\gamma_1+\delta, r-\delta)\times(\gamma_1+\delta, r^2+\gamma_1-\delta),\\
\widehat{B}_{2,3,2}(\delta) &= (r\gamma_1, r)\times((\gamma_1,\gamma_1+\delta)\cup(r^2+\gamma_1-\delta, r^2+\gamma_1))\\
&\quad\cup(\gamma_1+\delta, r^2+\gamma_1-\delta)\times((r\gamma_1, r\gamma_1+\delta)\cup(r-\delta, r)),
\end{aligned}$$

$$\begin{aligned}
\widetilde{B}_{2,3,3}(\delta) &= (\delta, r^2-\delta)\times(\gamma_2+\delta, 1-\delta),\\
\widehat{B}_{2,3,3}(\delta) &= (0, r^2)\times((\gamma_2,\gamma_2+\delta)\cup(1-\delta, 1))\\
&\quad\cup((0,\delta)\cup(r^2-\delta, r^2))\times(\gamma_2+\delta, 1-\delta).
\end{aligned}$$

(See Figures 9 and 10.)

It follows from (5.2), (5.1), and Lemma 2.26 that for $i = 1,2,3$ and $j \geq 2$,

$$\begin{aligned}
\mu(B_{j,1,1}) &= w_2(j-2)\mu(B_{1,1}), &\quad \mu(B_{j,1,2}) &= w_2(j-2)\mu(B_{1,3}),\\
\mu(B_{j,1,4}) &= p_2^{j-1}\mu(B_{1,2}), &\quad \mu(B_{j,1,5}) &= p_2^{j-1}\mu(B_{1,3}),\\
\mu(B_{2,2,i}) &= p_3\mu(B_{1,i}), &\quad \mu(B_{2,3,i}) &= p_4\mu(B_{1,i}).
\end{aligned}$$

Thus,

$$\begin{aligned}
&w(j,1,1) = w_2(j-2), \quad c(j,1,1) = 1, \quad \sigma(j,1,1) = S_{2^{j-2}1}, \quad r_{\sigma(j,1,1)} = \rho r^{j-2},\\
&w(j,1,2) = w_2(j-2), \quad c(j,1,2) = 3, \quad \sigma(j,1,2) = S_{2^{j-2}1}, \quad r_{\sigma(j,1,2)} = \rho r^{j-2},
\end{aligned}$$

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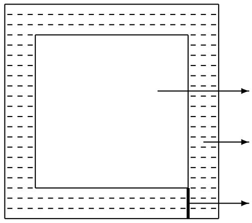

FIGURE 10. The middle part and the shaded region are $\widetilde{B}_{2,1,2}(\delta)$ and $\widehat{B}_{2,1,2}(\delta)$, respectively. The union is $B_{2,1,2}$.

$$
\begin{aligned}
&w(j,1,4)=p_2^{j-1}, \quad c(j,1,4)=2, \quad \sigma(j,1,4)=S_{2^{j-1}}, \quad r_{\sigma(j,1,1)}=r^{j-1},\\
&w(j,1,5)=p_2^{j-1}, \quad c(j,1,5)=3, \quad \sigma(j,1,5)=S_{2^{j-1}}, \quad r_{\sigma(j,1,2)}=r^{j-1},\\
&w(2,2,i)=p_3, \quad c(2,2,i)=i, \quad \sigma(2,2,i)=S_3, \quad r_{\sigma(2,2,i)}=r,\\
&w(2,3,i)=p_4, \quad c(2,3,i)=i, \quad \sigma(2,3,i)=S_4, \quad r_{\sigma(2,3,i)}=r,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{B}_{1,1}(\delta/\rho r^{j-2}) &= (\delta/\rho r^{j-2}, \rho\gamma_1+\delta/\rho r^{j-2})\times(\delta/\rho r^{j-2}, \rho-\delta/\rho r^{j-2})\\
&\quad \cup (\rho\gamma_1+\delta/\rho r^{j-2}, \rho\gamma_1+r-\delta/\rho r^{j-2})\\
&\qquad \times(\delta/\rho r^{j-2}, r-\delta/\rho r^{j-2}),\\
\widehat{B}_{1,1}(\delta/\rho r^{j-2}) &= (0,\rho\gamma_1+r)\times(0,\delta/\rho r^{j-2})\\
&\quad \cup (0,\rho\gamma_1+\delta/\rho r^{j-2})\times(\rho-\delta/\rho r^{j-2},\rho)\\
&\quad \cup (\rho\gamma_1,\rho\gamma_1+r)\times(r-\delta/\rho r^{j-2}, r)\\
&\quad \cup (0,\delta/\rho r^{j-2})\times(\delta/\rho r^{j-2},\rho-\delta/\rho r^{j-2})\\
&\quad \cup (\rho\gamma_1,\rho\gamma_1+\delta/\rho r^{j-2})\times(\rho, r-\delta/\rho r^{j-2})\\
&\quad \cup (\rho\gamma_1+r-\delta/\rho r^{j-2},\rho\gamma_1+r)\times(\delta/\rho r^{j-2}, r-\delta/\rho r^{j-2}),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{B}_{1,3}(\delta/\rho r^{j-2}) &= (\delta/\rho r^{j-2}, r-\delta/\rho r^{j-2})\times(\gamma_1+\delta/\rho r^{j-2}, 1-\delta/\rho r^{j-2}),\\
\widehat{B}_{1,3}(\delta/\rho r^{j-2}) &= (0,r)\times((\gamma_1,\gamma_1+\delta/\rho r^{j-2})\cup(1-\delta/\rho r^{j-2},1))\\
&\quad \cup ((0,\delta/\rho r^{j-2})\cup(r-\delta/\rho r^{j-2},r))\\
&\qquad \times(\gamma_1+\delta/\rho r^{j-2}, 1-\delta/\rho r^{j-2}),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{B}_{1,2}(\delta/r^{j-1}) &= (\gamma_1+\delta/r^{j-1}, 1-\delta/r^{j-1})\times(\delta/r^{j-1}, r-\delta/r^{j-1}),\\
\widehat{B}_{1,2}(\delta/r^{j-1}) &= (\gamma_1,1)\times((0,\delta/r^{j-1})\cup(r-\delta/r^{j-1},r))\\
&\quad \cup ((\gamma_1,\gamma_1+\delta/r^{j-1})\cup(1-\delta/r^{j-1},1))\\
&\qquad \times(\delta/r^{j-1}, r-\delta/r^{j-1}),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{B}_{1,3}(\delta/r^{j-1}) &= (\delta/r^{j-1}, r-\delta/r^{j-1})\times(\gamma_1+\delta/r^{j-1}, 1-\delta/r^{j-1}),\\
\widehat{B}_{1,3}(\delta/r^{j-1}) &= (0,r)\times((\gamma_1,\gamma_1+\delta/r^{j-1})\cup(1-\delta/r^{j-1},1))\\
&\quad \cup ((0,\delta/r^{j-1})\cup(r-\delta/r^{j-1},r))\\
&\qquad \times(\gamma_1+\delta/r^{j-1}, 1-\delta/r^{j-1}),
\end{aligned}
$$

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$$
\begin{aligned}
\widetilde{B}_{1,1}(\delta/r) &= (\delta/r, \rho\gamma_1 + \delta/r) \times (\delta/r, \rho - \delta/r) \\
&\quad \cup (\rho\gamma_1 + \delta/r, \rho\gamma_1 + r - \delta/r) \times (\delta/r, r - \delta/r), \\
\widehat{B}_{1,1}(\delta/r) &= (0, \rho\gamma_1 + r) \times (0, \delta/r) \cup (0, \rho\gamma_1 + \delta/r) \times (\rho - \delta/r, \rho) \\
&\quad \cup (\rho\gamma_1, \rho\gamma_1 + r) \times (r - \delta/r, r) \cup (0, \delta/r) \times (\delta/r, \rho - \delta/r) \\
&\quad \cup (\rho\gamma_1, \rho\gamma_1 + \delta/r) \times (\rho, r - \delta/r) \\
&\quad \cup (\rho\gamma_1 + r - \delta/r, \rho\gamma_1 + r) \times (\delta/r, r - \delta/r),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{B}_{1,2}(\delta/r) &= (\gamma_1 + \delta/r, 1 - \delta/r) \times (\delta/r, r - \delta/r), \\
\widehat{B}_{1,2}(\delta/r) &= (\gamma_1, 1) \times ((0, \delta/r) \cup (r - \delta/r, r)) \\
&\quad \cup ((\gamma_1, \gamma_1 + \delta/r) \cup (1 - \delta/r, 1)) \times (\delta/r, r - \delta/r),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{B}_{1,3}(\delta/r) &= (\delta/r, r - \delta/r) \times (\gamma_1 + \delta/r, 1 - \delta/r), \\
\widehat{B}_{1,3}(\delta/r) &= (0, r) \times ((\gamma_1, \gamma_1 + \delta/r) \cup (1 - \delta/r, 1)) \\
&\quad \cup ((0, \delta/r) \cup (r - \delta/r, r)) \times (\gamma_1 + \delta/r, 1 - \delta/r).
\end{aligned}
$$

By (3.14) and (3.13),

$$
\begin{aligned}
\varphi_1(\delta) = \sum_{j=2}^{n} w_2(j-2)^q (\rho r^{j-2})^2 \Big(\int_{B_{1,1}} + \int_{B_{1,3}} \Big) \mu(B_{\delta/\rho r^{j-2}}(\mathbf{x}))^q \, d\mathbf{x} \\
+ \sum_{j=2}^{n} (p_2^q r^2)^{j-1} \Big(\int_{B_{1,2}} + \int_{B_{1,3}} \Big) \mu(B_{\delta/r^{j-1}}(\mathbf{x}))^q \, d\mathbf{x} \\
+ \sum_{j=2}^{n} (e_j^1(\delta) - \tilde{e}_j^1(\delta)) + \int_{B_{n,1,3}} \mu(B_\delta(\mathbf{x}))^q \, d\mathbf{x}
\end{aligned}
\tag{5.3}
$$

and

$$
\varphi_\ell(\delta) = p_{\ell+1}^q r^2 \sum_{i=1}^{3} \int_{B_{1,i}} \mu(B_{\delta/r}(\mathbf{x}))^q \, d\mathbf{x} + e_2^\ell(\delta) - \tilde{e}_2^\ell(\delta) \quad \text{for } \ell = 2, 3, \tag{5.4}
$$

where

$$
\begin{aligned}
e_j^1(\delta) &= \Big(\int_{\widehat{B}_{j,1,1}(\delta)} + \int_{\widehat{B}_{j,1,2}(\delta)} + \int_{\widehat{B}_{j,1,4}(\delta)} + \int_{\widehat{B}_{j,1,5}(\delta)} \Big) \mu(B_\delta(\mathbf{x}))^q \, d\mathbf{x}, \\
\tilde{e}_j^1(\delta) &= w_2(j-2)^q (\rho r^{j-2})^2 \Big(\int_{\widehat{B}_{1,1}(\delta/\rho r^{j-2})} + \int_{\widehat{B}_{1,3}(\delta/\rho r^{j-2})} \Big) \mu(B_{\delta/\rho r^{j-2}}(\mathbf{x}))^q \, d\mathbf{x} \\
&\quad + (p_2^q r^2)^{j-1} \Big(\int_{\widehat{B}_{1,2}(\delta/r^{j-1})} + \int_{\widehat{B}_{1,3}(\delta/r^{j-1})} \Big) \mu(B_{\delta/r^{j-1}}(\mathbf{x}))^q \, d\mathbf{x}, \qquad (5.5) \\
e_2^\ell(\delta) &= \sum_{i=1}^{3} \int_{\widehat{B}_{2,\ell,i}(\delta)} \mu(B_\delta(\mathbf{x}))^q \, d\mathbf{x}, \\
\tilde{e}_2^\ell(\delta) &= p_{\ell+1}^q r^2 \sum_{i=1}^{3} \int_{\widehat{B}_{1,i}(\delta/r)} \mu(B_{\delta/r}(\mathbf{x}))^q \, d\mathbf{x} \quad \text{for } \ell = 2, 3.
\end{aligned}
$$

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Multiplying both sides of (5.3) and (5.4) by $\delta^{-(2+\alpha)}$, and using (1.2),

$$\begin{aligned}\Phi_1^{(\alpha)}(\delta) = \sum_{j=2}^{n} w_2(j-2)^q(\rho r^{j-2})^{-\alpha} \sum_{i=1,3} \Phi_i^{(\alpha)}(\delta/\rho r^{j-2}) \\ + \sum_{j=2}^{n}(p_2^q r^{-\alpha})^{j-1} \sum_{i=2,3} \Phi_i^{(\alpha)}(\delta/r^{j-1}) \\ + \sum_{j=2}^{n} \delta^{-2-\alpha}(e_j^1(\delta) - \tilde{e}_j^1(\delta)) + \delta^{-2-\alpha} \int_{B_{n,1,3}} \mu(B_\delta(\mathbf{x}))^q \, d\mathbf{x},\end{aligned} \tag{5.6}$$

and

$$\Phi_\ell^{(\alpha)}(\delta) = p_{\ell+1}^q r^{-\alpha} \sum_{i=1}^{3} \Phi_i^{(\alpha)}(\delta/r) + \delta^{-2-\alpha}(e_2^\ell(\delta) - \tilde{e}_2^\ell(\delta)) \quad \text{for } \ell = 2, 3.$$

Let $N := \max\{n \in \mathbb{N} : \delta \le \min\{\rho r^{n-2}, r^{n-1}\}\}$. Letting $n = N$ in (5.6),

$$\begin{aligned}\Phi_1^{(\alpha)}(\delta) = \sum_{j=2}^{\infty} w_2(j-2)^q(\rho r^{j-2})^{-\alpha} \sum_{i=1,3} \Phi_i^{(\alpha)}(\delta/\rho r^{j-2}) \\ + \sum_{j=2}^{\infty}(p_2^q r^{-\alpha})^{j-1} \sum_{i=2,3} \Phi_i^{(\alpha)}(\delta/r^{j-1}) \\ + E_1^{(\alpha)}(\delta) - E_{1,\infty}^{(\alpha)}(\delta),\end{aligned} \tag{5.7}$$

where

$$\begin{aligned}E_1^{(\alpha)}(\delta) &:= \sum_{j=2}^{N} \delta^{-2-\alpha}(e_j^1(\delta) - \tilde{e}_j^1(\delta)) + \delta^{-2-\alpha} \int_{B_{N,1,3}} \mu(B_\delta(\mathbf{x}))^q \, d\mathbf{x}, \\ E_{1,\infty}^{(\alpha)}(\delta) &:= \sum_{j=N+1}^{\infty} w_2(j-2)^q(\rho r^{j-2})^{-\alpha} \sum_{i=1,3} \Phi_i^{(\alpha)}(\delta/\rho r^{j-2}) \\ &\quad + \sum_{j=N+1}^{\infty} (p_2^q r^{-\alpha})^{j-1} \sum_{i=2,3} \Phi_i^{(\alpha)}(\delta/r^{j-1}).\end{aligned}$$

Let

$$\Phi_\ell^{(\alpha)}(\delta) = p_{\ell+1}^q r^{-\alpha} \sum_{i=1}^{3} \Phi_i^{(\alpha)}(\delta/r) + E_\ell^{(\alpha)}(\delta) \quad \text{for } \ell = 2, 3, \tag{5.8}$$

where

$$E_\ell^{(\alpha)}(\delta) := \delta^{-2-\alpha}(e_2^\ell(\delta) - \tilde{e}_2^\ell(\delta)).$$

Next, we derive a vector-valued renewal equation. It follows from (3.19), (3.20), (5.7), and (5.8) that

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$$f_1(x) = \sum_{j=2}^{\infty} w_2(j-2)^q(\rho r^{j-2})^{-\alpha} \sum_{i=1,3} f_i(x+\ln(\rho r^{j-2}))$$
$$+ \sum_{j=2}^{\infty}(p_2^q r^{-\alpha})^{j-1} \sum_{i=2,3} f_i(x+\ln(r^{j-1})) + z_1^{(\alpha)}(x),$$

and

$$f_\ell(x) = p_{\ell+1}^q r^{-\alpha} \sum_{i=1}^{3} f_i(x+\ln(r)) + z_\ell^{(\alpha)}(x) \quad \text{for } \ell = 2,3,$$

where

$$z_1^{(\alpha)}(x) = E_1^{(\alpha)}(e^{-x}) - E_{1,\infty}^{(\alpha)}(e^{-x}), \quad z_\ell^{(\alpha)}(x) = E_\ell^{(\alpha)}(e^{-x}).$$

For $\ell, m = 1,2$, let $\mu_{m\ell}^{(\alpha)}$ be the discrete measures such that for $j \geq 2$,

$$\mu_{m1}^{(\alpha)}(-\ln(\rho r^{j-2})) = w_2(j-2)^q(\rho r^{j-2})^{-\alpha} \quad \text{for } m = 1,3,$$
$$\mu_{m1}^{(\alpha)}(-\ln(r^{j-1})) = (p_2^q r^{-\alpha})^{j-1} \quad \text{for } m = 2,3,$$
$$\mu_{m\ell}^{(\alpha)}(-\ln(r)) = p_{\ell+1}^q r^{-\alpha} \quad \text{for } m = 1,2,3 \text{ and } \ell = 2,3.$$

Then

$$\mu_{11}^{(\alpha)}(\mathbb{R}) = \sum_{j=2}^{\infty} w_2(j-2)^q(\rho r^{j-2})^{-\alpha}, \quad \mu_{21}^{(\alpha)}(\mathbb{R}) = \sum_{j=2}^{\infty}(p_2^q r^{-\alpha})^{j-1},$$
$$\mu_{31}^{(\alpha)}(\mathbb{R}) = \sum_{j=2}^{\infty} w_2(j-2)^q(\rho r^{j-2})^{-\alpha} + \sum_{j=2}^{\infty}(p_2^q r^{-\alpha})^{j-1},$$
$$\mu_{m\ell}^{(\alpha)}(\mathbb{R}) = p_{\ell+1}^q r^{-\alpha} \quad \text{for } m = 1,2,3 \text{ and } \ell = 2,3.$$

For fixed $q \geq 0$,

$$\begin{aligned} F_1(\alpha) &= 2\Big(\sum_{j=2}^{\infty} w_2(j-2)^q(\rho r^{j-2})^{-\alpha} + \sum_{j=2}^{\infty}(p_2^q r^{-\alpha})^{j-1}\Big), \\ F_\ell(\alpha) &= 3p_{\ell+1}^q r^{-\alpha} \quad \text{for } \ell = 2,3, \\ D_\ell &= \{\alpha \in \mathbb{R} : F_\ell(\alpha) < \infty\} \quad \text{for } \ell = 1,2,3, \end{aligned} \tag{5.9}$$

and

$$\mathbf{M}_\alpha(\infty) = \begin{pmatrix} a & p_3^q r^{-\alpha} & p_4^q r^{-\alpha} \\ b & p_3^q r^{-\alpha} & p_4^q r^{-\alpha} \\ a+b & p_3^q r^{-\alpha} & p_4^q r^{-\alpha} \end{pmatrix},$$

where $a := \sum_{j=2}^{\infty} w_2(j-2)^q(\rho r^{j-2})^{-\alpha}$ and $b := \sum_{j=2}^{\infty}(p_2^q r^{-\alpha})^{j-1}$.

Finally, we need to show that the error terms $z_\ell^{(\alpha)}(x) = o(e^{-\epsilon x})$ as $x \to \infty$, that is, $E_\ell^{(\alpha)}(\delta) = o(\delta^\epsilon)$ and $E_{1,\infty}^{(\alpha)}(\delta) = o(\delta^\epsilon)$ as $\delta \to 0$ for some $\epsilon > 0$ and $\ell = 1,2,3$.

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PROPOSITION 5.1.

(1) $\Phi_i^{(\alpha)}(\delta/\rho r^k) \le 1$ *for* $i = 1, 3$ *and any* $k \ge N - 1$*;*

(2) $\Phi_i^{(\alpha)}(\delta/r^k) \le 1$ *for* $i = 2, 3$ *and any* $k \ge N$*.*

PROOF.

(1) It follows from the definition of N that $\delta \ge \rho r^k$ for any $k \ge N - 1$. Thus, for $i = 1, 3$,

$$\Phi_i^{(\alpha)}(\delta/\rho r^k) = \frac{\varphi_i(\delta/\rho r^k)}{(\delta/\rho r^k)^{2+\alpha}} \le \int_{B_{1,i}} \mu(B_{\delta/\rho r^k}(\mathbf{x}))^q\, d\mathbf{x} \le \int_{B_{1,i}} d\mathbf{x} \le 1.$$

This proves part (1).

(2) The proof is similar to that of (1). □

PROPOSITION 5.2. *For* $q \ge 0$*, let* $F_1(\alpha)$ *and* D_1 *be defined as in (5.9). Then* D_1 *is open.*

PROOF. The proof is similar to that of Proposition 4.3. □

PROPOSITION 5.3. *For* $q \ge 0$*, assume that* $\alpha \in D_\ell$ *for* $\ell = 1, 2, 3$*. Then there exists* $\epsilon > 0$ *such that:*

(1) $\sum_{j=N+1}^{\infty} w_2(j-2)^q(\rho r^{j-2})^{-\alpha} \sum_{i=1,3} \Phi_i^{(\alpha)}(\delta/\rho r^{j-2}) = o(h^\epsilon)$*;*

(2) $\sum_{j=N+1}^{\infty} (p_2^q r^{-\alpha})^{j-1} \sum_{i=2,3} \Phi_i^{(\alpha)}(\delta/r^{j-1}) = o(\delta^\epsilon)$*;*

(3) $\sum_{j=2}^{N} \delta^{-2-\alpha}(e_j^1(\delta) - \tilde{e}_j^1(\delta)) = o(\delta^\epsilon)$*;*

(4) $\delta^{-2-\alpha} \int_{B_{N,1,3}} \mu(B_\delta(\mathbf{x}))^q\, d\mathbf{x} = o(\delta^\epsilon)$*;*

(5) $\delta^{-2-\alpha}(e_2^\ell(\delta) - \tilde{e}_2^\ell(\delta)) = o(\delta^\epsilon)$ *for* $\ell = 2, 3$*.*

PROOF.

(1) By Proposition 5.2, $D_1 = \{\alpha \in \mathbb{R} : F_1(\alpha) < \infty\}$ is open. Thus, there exists $\epsilon > 0$ such that $\alpha + \epsilon \in D_1$. So there exists a positive constant C such that

$$\sum_{j=N+1}^{\infty} w_2(j-2)^q(\rho r^{j-2})^{-\alpha-\epsilon} + \sum_{j=N+1}^{\infty} (p_2^q r^{-\alpha-\epsilon})^{j-1} \le C.$$

Since

$$(\rho r^{N-1})^{-\epsilon} \sum_{j=N+1}^{\infty} w_2(j-2)^q(\rho r^{j-2})^{-\alpha} \le \sum_{j=N+1}^{\infty} w_2(j-2)^q(\rho r^{j-2})^{-\alpha-\epsilon},$$

$\sum_{j=N+1}^{\infty} w_2(j-2)^q(\rho r^{j-2})^{-\alpha} \le C(\rho r^{N-1})^\epsilon \le C\delta^\epsilon$, where the last inequality follows from the definition of N. Combining this with Proposition 5.1(1),

$$\sum_{j=N+1}^{\infty} w_2(j-2)^q(\rho r^{j-2})^{-\alpha} \sum_{i=1,3} \Phi_i^{(\alpha)}(\delta/\rho r^{j-2}) \le 2C\delta^\epsilon.$$

This proves part (1).

(2) The proof is similar to that of (1).

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(3) It suffices to show that $e_j^1(\delta) = o(\delta^{2+\alpha+\epsilon})$ and $\tilde{e}_j^1(\delta) = o(\delta^{2+\alpha+\epsilon})$ for $2 \le j \le N$. In order to estimate the remaining error terms, we will need the following facts. It follows from (1) and (2) that

$$w_2(N-1)^q \le 2C\delta^{\alpha+\epsilon}, \quad p_2^{Nq} \le 2C\delta^{\alpha+\epsilon}. \tag{5.10}$$

By (5.5),

$$e_j^1(\delta) = \sum_{i=1,2,4,5} \int_{\widehat{B}_{j,1,i}(\delta)} \mu(B_\delta(x,y))^q \,dx\,dy.$$

As an example we only prove that $\int_{\widehat{B}_{j,1,1}(\delta)} \mu(B_\delta(x,y))^q \,dxdy = o(\delta^{2+\alpha+\epsilon})$. Note that

$$\begin{aligned}
&\int_{\widehat{B}_{j,1,1}(\delta)} \mu(B_\delta(x,y))^q \,dx\,dy\\
&\quad= \bigg(\int_{\rho\gamma_{j-2}}^{\rho\gamma_{j-2}+\rho^2r^{j-2}\gamma_1+\rho\gamma_{j-1}} \int_0^\delta + \int_{\rho\gamma_{j-2}}^{\rho\gamma_{j-2}+\rho^2r^{j-2}\gamma_1+\delta} \int_{\rho^2r^{j-2}-\delta}^{\rho^2r^{j-2}}\\
&\qquad+ \int_{\rho\gamma_{j-2}+\rho^2r^{j-2}\gamma_1}^{\rho\gamma_{j-2}+\rho^2r^{j-2}\gamma_1+\rho r^{j-1}} \int_{\rho r^{j-1}-\delta}^{\rho r^{j-1}} + \int_{\rho\gamma_{j-2}}^{\rho\gamma_{j-2}+\delta} \int_\delta^{\rho^2r^{j-2}-\delta}\\
&\qquad+ \int_{\rho\gamma_{j-2}+\rho^2r^{j-2}\gamma_1}^{\rho\gamma_{j-2}+\rho^2r^{j-2}\gamma_1+\delta} \int_{\rho^2r^{j-2}}^{\rho r^{j-1}-\delta}\\
&\qquad+ \int_{\rho\gamma_{j-2}+\rho^2r^{j-2}\gamma_1+\rho r^{j-1}-\delta}^{\rho\gamma_{j-2}+\rho^2r^{j-2}\gamma_1+\rho r^{j-1}} \int_\delta^{\rho r^{j-1}-\delta}\bigg) \mu(B_\delta(x,y))^q \,dx\,dy\\
&\quad=: \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 + \mathcal{E}_5 + \mathcal{E}_6.
\end{aligned}$$

For $\mathcal{E}_1$, since

$$\sqrt{(\rho^2r^{j-2}\gamma_1 + \rho\gamma_{j-1})^2 + \delta^2} \le 2\rho + \delta,$$

$B_\delta(x,y) \subseteq B_{2\rho+2\delta}(S_{2^{j-2}11}(0,0))$ for $(x,y) \in (\rho\gamma_{j-2}, \rho\gamma_{j-2} + \rho^2r^{j-2}\gamma_1 + \rho\gamma_{j-1}) \times (0,\delta)$. Note that

$$\mu(B_{2\rho+2\delta}(S_{2^{j-2}11}(0,0))) = p_1w_2(j-2)\mu(B_{(2\rho+2\delta)/\rho^2r^{j-2}}(0,0)) \le p_1w_2(j-2)$$

and for $0 \le k \le N-2$,

$$\begin{aligned}
w_2(k) &= \frac{p_1(p_2^{N-1} + p_2^{N-2}p_3 + \cdots + p_3^{N-1})(p_2^k + p_2^{k-1}p_3 + \cdots + p_3^k)}{(p_2^{N-1} + p_2^{N-2}p_3 + \cdots + p_3^{N-1})}\\
&\le \frac{w_1(N-1)(p_2+p_3)^k}{p_2^{N-1} + p_2^{N-2}p_3 + \cdots + p_3^{N-1}} \le p_2^{1-N}w_2(N-1).
\end{aligned} \tag{5.11}$$

Combining these with the definition of N,

$$\begin{aligned}
\mathcal{E}_1 &\le (p_1w_2(j-2))^q(\rho^2r^{j-2}\gamma_1 + \rho\gamma_{j-1})\delta \le 2\rho p_1^q w_2(j-2)^q\delta\\
&\le 2p_1^q p_2^{(1-N)q} w_2(N-1)^q \rho r^{N-1} r^{1-N} \le 2(p_1p_2^{1-N})^q r^{1-N}\delta^{2+\alpha+\epsilon}.
\end{aligned}$$

The proofs for $\mathcal{E}_2 \le C\delta^{2+\alpha+\epsilon}$ and $\mathcal{E}_3 \le C\delta^{2+\alpha+\epsilon}$ are similar.

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For $\mathcal{E}_4$, since

$$\mathcal{E}_4 \le \int_{\rho\gamma_{j-2}}^{\rho\gamma_{j-2}+\delta}\int_0^{\rho^2 r^{j-2}-\delta} \mu(B_\delta(x,y))^q\,dx\,dy$$

and $\sqrt{(\delta^2+(\rho^2 r^{j-2}-\delta)^2)} \le \rho^2 r^{j-2} \le \rho^2$, then,

$$B_\delta(x,y) \subseteq B_{\rho^2+\delta}(S_{2^{j-2}12}(0,0))$$

for $(x,y) \in (\rho\gamma_{j-2}, \rho\gamma_{j-2}+\delta) \times (0, \rho^2 r^{j-2}-\delta)$. Note that

$$\mu(B_{\rho^2+\delta}(S_{2^{j-2}12}(0,0))) \le p_2 w_2(j-2).$$

Combining these with (5.10) and (5.11), and using the definition of N,

$$\begin{aligned}\mathcal{E}_4 &\le (p_2 w_2(j-2))^q(\rho^2 r^{j-2}-\delta)\delta \le p_2^q w_2(j-2)^q \rho^2\delta\\ &\le p_2^{(2-N)q} w_2(N-1)^q \rho r^{1-N}\rho r^{N-1}\delta\\ &\le 2C\rho r^{1-N} p_2^{(2-N)q}\delta^{2+\alpha+\epsilon}.\end{aligned}$$

The proofs for $\mathcal{E}_5 \le C\delta^{2+\alpha+\epsilon}$ and $\mathcal{E}_6 \le C\delta^{2+\alpha+\epsilon}$ are similar. Combining the estimates for $\mathcal{E}_1,\dots,\mathcal{E}_6$, we have $\int_{\widehat{B}_{j,1,1}(\delta)} \mu(B_\delta(x,y))^q\,dx\,dy \le C\delta^{2+\alpha+\epsilon}$.

Next, we will show that $\tilde{e}_j^1(\delta) = o(\delta^{2+\alpha+\epsilon})$. By (5.5),

$$\begin{aligned}\tilde{e}_j^1(\delta) &= w_2(j-2)^q(\rho r^{j-2})^2\Big(\int_{\widehat{B}_{1,1}(\delta/\rho r^{j-2})} + \int_{\widehat{B}_{1,3}(\delta/\rho r^{j-2})}\Big)\mu(B_{\delta/\rho r^{j-2}}(\mathbf{x}))^q\,d\mathbf{x}\\ &\quad + (p_2^q r^2)^{j-1}\Big(\int_{\widehat{B}_{1,2}(\delta/r^{j-1})} + \int_{\widehat{B}_{1,3}(\delta/r^{j-1})}\Big)\mu(B_{\delta/r^{j-1}}(\mathbf{x}))^q\,d\mathbf{x}.\end{aligned}$$

As an example, we only prove

$$w_2(j-2)^q(\rho r^{j-2})^2\int_{\widehat{B}_{1,1}(\delta/\rho r^{j-2})}\mu(B_{\delta/\rho r^{j-2}}(\mathbf{x}))^q\,d\mathbf{x} = o(\delta^{2+\alpha+\epsilon}).$$

Note that

$$\begin{aligned}&w_2(j-2)^q(\rho r^{j-2})^2\int_{\widehat{B}_{1,1}(\delta/\rho r^{j-2})}\mu(B_{\delta/\rho r^{j-2}}(\mathbf{x}))^q\,d\mathbf{x}\\ &\quad= w_2(j-2)^q(\rho r^{j-2})^2\Big(\int_0^{\rho\gamma_1+r}\int_0^{\delta/\rho r^{j-2}} + \int_0^{\rho\gamma_1+\delta/\rho r^{j-2}}\int_{\rho-\delta/\rho r^{j-2}}^{\rho}\\ &\qquad + \int_{\rho\gamma_1}^{\rho\gamma_1+r}\int_{r-\delta/\rho r^{j-2}}^{r} + \int_0^{\delta/\rho r^{j-2}}\int_{\delta/\rho r^{j-2}}^{\rho-\delta/\rho r^{j-2}} + \int_{\rho\gamma_1}^{\rho\gamma_1+\delta/\rho r^{j-2}}\int_{\rho}^{r-\delta/\rho r^{j-2}}\\ &\qquad + \int_{\rho\gamma_1+r-\delta/\rho r^{j-2}}^{\rho\gamma_1+r}\int_{\delta/\rho r^{j-2}}^{r-\delta/\rho r^{j-2}}\Big)\mu(B_{\delta/\rho r^{j-2}}(x,y))^q\,dx\,dy\\ &\quad=: \widetilde{\mathcal{E}}_1+\widetilde{\mathcal{E}}_2+\widetilde{\mathcal{E}}_3+\widetilde{\mathcal{E}}_4+\widetilde{\mathcal{E}}_5+\widetilde{\mathcal{E}}_6.\end{aligned}$$

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Since

$$\sqrt{(\rho\gamma_1 + r)^2 + (\delta/\rho r^{j-2})^2} \le \rho + r + \delta/\rho r^{j-2},$$

$B_{\delta/\rho r^{j-2}}(x,y) \subseteq B_{\rho+r+2\delta/\rho r^{j-2}}(S_1(0,0))$ for $(x,y) \in (0, \rho\gamma_1 + r) \times (0, \delta/\rho r^{j-2})$. Note that $\mu(B_{\rho+r+2\delta/\rho r^{j-2}})(S_1(0,0)) \le p_1$. Thus,

$$\begin{aligned}
\widetilde{\mathcal{E}}_1 &\le w_2(j-2)^q(\rho r^{j-2})^2 p_1^q(\rho\gamma_1 + r)\cdot \delta/\rho r^{j-2}\\
&\le w_2(j-2)^q \rho r^{j-2} p_1^q(\rho + r)\delta\\
&\le p_1^q p_2^{(1-N)q} w_2(N-1)^q \rho(\rho + r)\delta\\
&\le r^{1-N}(p_1 p_2^{1-N})^q w_2(N-1)^q \rho(\rho r^{N-1} + r^N)\delta\\
&\le 2C\rho r^{1-N}(p_1 p_2^{1-N})^q \delta^{2+\alpha+\epsilon}.
\end{aligned}$$

The proofs for $\widetilde{\mathcal{E}}_2 \le C\delta^{2+\alpha+\epsilon}$ and $\widetilde{\mathcal{E}}_3 \le C\delta^{2+\alpha+\epsilon}$ are similar. For $\widetilde{\mathcal{E}}_4$,

$$\begin{aligned}
\widetilde{\mathcal{E}}_4 &\le w_2(j-2)^q(\rho r^{j-2})^2 \int_0^{\delta/\rho r^{j-2}} \int_0^{\rho-\delta/\rho r^{j-2}} \mu(B_{\delta/\rho r^{j-2}}(x,y))^q \,dx\,dy\\
&\le w_2(j-2)^q(\rho r^{j-2})^2 \mu(B_{\rho+\delta/\rho r^{j-2}}(S_1(0,0)))^q(\rho - \delta/\rho r^{j-2})\delta/\rho r^{j-2}\\
&\le p_1^q p_2^{(1-N)q} w_2(N-1)^q(\rho^2 r^{j-2} - \delta)\delta\\
&\le (p_1 p_2^{1-N})^q w_2(N-1)^q \rho r^{N-1} \rho r^{1-N}\delta\\
&\le 2C\rho r^{1-N}(p_1 p_2^{1-N})^q \delta^{2+\alpha+\epsilon}.
\end{aligned}$$

The proofs for $\widetilde{\mathcal{E}}_5 \le C\delta^{2+\alpha+\epsilon}$ and $\widetilde{\mathcal{E}}_6 \le C\delta^{2+\alpha+\epsilon}$ are similar. Hence,

$$w_2(j-2)^q(\rho r^{j-2})^2 \int_{\widehat{B}_{1,1}(\delta/\rho r^{j-2})} \mu(B_{\delta/\rho r^{j-2}}(\mathbf{x}))^q \,d\mathbf{x} = o(\delta^{2+\alpha+\epsilon}).$$

Similarly, we can derive analogous results for the second, third, and fourth terms of $\tilde{e}_j^1(\delta)$. Thus, $\tilde{e}_j^1(\delta) = o(\delta^{2+\alpha+\epsilon})$. This proves part (3).

(4) It suffices to show that $\int_{B_{N,1,3}} \mu(B_\delta(\mathbf{x}))^q \,d\mathbf{x} \le C\delta^{2+\alpha+\epsilon}$. It follows from (5.2) and Proposition 2.24(4) that

$$\begin{aligned}
&\int_{B_{N,1,3}} \mu(B_\delta(\mathbf{x}))^q \,d\mathbf{x}\\
&\quad= \Big(\int_{\rho\gamma_{N-1}}^{\rho\gamma_N} \int_0^{\rho r^{N-1}} + \int_{\rho\gamma_N}^{\rho\gamma_N + r^N} \int_0^{r^N}\Big) \mu(B_\delta(x,y))^q \,dx\,dy\\
&\quad= \Big(\int_{\rho\gamma_{N-1}+\delta}^{\rho\gamma_N+\delta} \int_\delta^{\rho r^{N-1}-\delta} + \int_{\rho\gamma_N+\delta}^{\rho\gamma_N + r^N-\delta} \int_\delta^{r^N-\delta}\\
&\qquad + \int_{\rho\gamma_{N-1}}^{\rho\gamma_N + r^N} \int_0^{\delta} + \int_{\rho\gamma_{N-1}}^{\rho\gamma_N+\delta} \int_{\rho r^{N-1}-\delta}^{\rho r^{N-1}}\\
&\qquad + \int_{\rho\gamma_N}^{\rho\gamma_N + r^N} \int_{r^N-\delta}^{r^N} + \int_{\rho\gamma_{N-1}}^{\rho\gamma_{N-1}+\delta} \int_\delta^{\rho r^{N-1}-\delta}
\end{aligned}$$

$$+\int_{\rho\gamma_N}^{\rho\gamma_N+\delta}\int_{\rho r^{N-1}}^{r^N-\delta}+\int_{\rho\gamma_N+r^N-\delta}^{\rho\gamma_N+r^N}\int_{\delta}^{r^N-\delta}\Big)\mu(B_\delta(x,y))^q\,dx\,dy$$
$$=: \mathcal{E}_1^N+\mathcal{E}_2^N+\mathcal{E}_3^N+\mathcal{E}_4^N+\mathcal{E}_5^N+\mathcal{E}_6^N+\mathcal{E}_7^N+\mathcal{E}_8^N.$$

By Lemma 2.26(2), $\mu|_{S_{2^{N-1}}(B_{1,1})}=w_2(N-2)\mu\circ S_{2^{N-2}1}^{-1}+p_2^{N-1}\mu\circ S_{2^{N-1}}^{-1}$, and hence $\mu(S_{2^{N-1}}(B_{1,1}))\le w_2(N-2)+p_2^{N-1}$. Since $B_\delta(x,y)\subseteq S_{2^{N-1}}(B_{1,1})$ for $(x,y)\in(\rho\gamma_{N-1}+\delta,\rho\gamma_N+\delta)\times(\delta,\rho r^{N-1}-\delta)\cup(\rho\gamma_N+\delta,\rho\gamma_N+r^N-\delta)\times(\delta,r^N-\delta)$, (5.10) implies

$$\begin{aligned}\mathcal{E}_1^N+\mathcal{E}_2^N&\le(w_2(N-2)+p_2^{N-1})^q((\rho\gamma_N-\rho\gamma_{N-1})(\rho r^{N-1}-2\delta)+(r^N-2\delta)^2)\\&\le(p_2^{1-N}w_2(N-1)+p_2^{-1}p_2^N)^q((\rho r^{N-1})^2+r^{2N})\\&\le 2C(p_2^{1-N}+p_2^{-1})\delta^{2+\alpha+\epsilon}.\end{aligned}$$

For the other six terms,

$$\begin{aligned}\mathcal{E}_3^N&\le\mu(B_{\rho r+2\delta}(S_{2^{N-1}1}(0,0)))^q(\rho\gamma_N+r^N-\rho\gamma_{N-1})\delta\\&\le w_2(N-1)^q(r^N+\rho r^{N-1})\delta\le 2C\delta^{2+\alpha+\epsilon}.\end{aligned}$$

The proofs for $\mathcal{E}_4^N\le C\delta^{2+\alpha+\epsilon}$ and $\mathcal{E}_5^N\le C\delta^{2+\alpha+\epsilon}$ are similar. $\mathcal{E}_6^N$ can be estimated as follows:

$$\begin{aligned}\mathcal{E}_6^N&\le\int_{\rho\gamma_{N-1}}^{\rho\gamma_{N-1}+\delta}\int_0^{\rho r^{N-1}-\delta}\mu(B_\delta(x,y))^q\,dx\,dy\\&\le\mu(B_{\rho r+\delta}(S_1(0,0)))^q(\rho r^{N-1}-\delta)\delta\le p_1^q\rho r^{N-1}\delta\\&\le p_1^q p_2^{-Nq}p_2^{Nq}\delta^2\le 2C(p_1p_2^{-N})^q\delta^{2+\alpha+\epsilon}.\end{aligned}$$

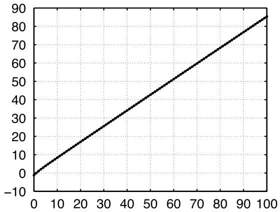
The proofs for $\mathcal{E}_7^N\le C\delta^{2+\alpha+\epsilon}$ and $\mathcal{E}_8^N\le C\delta^{2+\alpha+\epsilon}$ are similar. This proves part (4); part (5) can be proved similarly. □

Proof (Theorem 1.4). Combining Theorem 1.1 and Proposition 5.3, we have $\tau(q)=\alpha$. Let

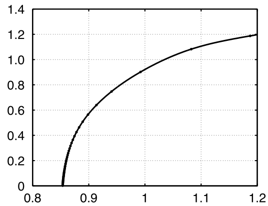
$$G(q,\alpha):=(1-p_2^qr^{-\alpha})(1-p_3^qr^{-\alpha})\sum_{k=0}^{\infty}w_2(k)^q(\rho r^k)^{-\alpha}+r^{-\alpha}\sum_{i=2}^{4}p_i^q-1.$$

Similar to the proof of Theorem 1.2, we can show that $G(q,\alpha)$ is C^1 and that $G_\alpha(q,\alpha)\ne0$ for any (q,α) satisfying $G(a,\alpha)=0$. The implicit function theorem now implies that τ is differentiable on $(0,\infty)$ and the formula for $\dim_{\mathrm{H}}(\mu)$ follows by computing $\tau'(1)=-G_q(1,0)G_\alpha(1,0)^{-1}$. This completes the proof. □

Figure 11 shows graphs of $\tau(q)$ and $f(\alpha)$ for one of the measures. For this example, $\dim_{\mathrm{H}}(\mu)=\tau'(1)\approx1.13748$ and $\dim_{\mathrm{H}}(K)=-\tau(0)\approx1.18726$, where K is the self-similar set.



(a) $\tau(q)$



(b) $f(\alpha)$

FIGURE 11. Graphs of $\tau(q)$ and $f(\alpha)$ for a self-similar measure in Example 2.22, with $r = 7/20$ and $\rho = p_i = 1/4$ for $i = 1, 2, 3, 4$.

6. Comments and questions

The spectral dimension of certain infinite IFSs has been computed in [21]. The method in this paper can be applied to those IFSs to obtain $\tau(q)$.

It is interesting to compute $\tau(q)$ for $q < 0$ and see whether there is any phase transition. Our method cannot be applied to this case. We do not know whether the condition in Theorem 1.1 can be removed. Finally, we are not sure whether the method in this paper can be applied, after modifications if necessary, to infinite Bernoulli convolutions associated with Pisot numbers other than the golden ratio.

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