COUNTABLY QUASI-SUPRABARRELLED SPACES

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In this paper we obtain some permanence properties of a class of locally convex spaces located between quasi-suprabarrelled spaces and quasi-totally barrelled spaces, for which a closed graph theorem is given.

1. INTRODUCTION

Throughout this paper the word "space" will stand for "Hausdorff locally convex topological vector space defined over the field $\mathbb{K}$ of real or complex numbers". Let us recall a space $E$ is quasi-suprabarrelled [1] if, given an increasing sequence of subspaces of $E$ covering $E$, there is one which is barrelled; $E$ satisfies condition (G) [4] if, given a sequence of subspaces of $E$ covering $E$, there is one which is barrelled; $E$ is quasi-totally barrelled [2] if, given a sequence of subspaces of $E$ covering $E$, there is one which is barrelled and its closure has countable codimension in $E$; $E$ is totally barrelled [12] if, given a sequence of subspaces of $E$ covering $E$, there is one which is barrelled and its closure has finite codimension in $E$; $E$ is unordered Baire-like [6] if, given a sequence of closed absolutely convex subsets of $E$ covering $E$, there is one which is a neighbourhood of the origin; and $E$ is suprabarrelled [9] ((bd) in [5]) if, given an increasing sequence of subspaces of $E$ covering $E$, there is one which is barrelled and dense in $E$. The relationship among these classes of spaces is the following:

unordered Baire-like $\Rightarrow$ totally barrelled $\Rightarrow$ suprabarrelled $\Rightarrow$ barrelled.

and

totally barrelled $\Rightarrow$ quasi-totally barrelled $\Rightarrow$ (G) $\Rightarrow$ quasi-suprabarrelled

$\Rightarrow$ barrelled.

In this paper we shall introduce a class of spaces located between quasi-totally barrelled spaces and quasi-suprabarrelled spaces, which enjoys good permanence properties, and satisfies a closed graph theorem.

Our notation is standard, so if $A$ is a subset of a linear space, $\langle A \rangle$ will denote its linear span and if $\{E_i : i \in I\}$ is a family of spaces, $E = \Pi\{E_i : i \in I\}$ and $J$ is a subset of $I$, $E(J)$ will denote the subspace of $E$ consisting of those elements of support $J$.

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2. COUNTABLY QUASI-SUPRABARRELLED SPACES

**Definition**: We shall say a space $E$ is countably quasi-suprabarrelled if, given an increasing sequence of subspaces $\{E_n: n \in \mathbb{N}\}$ covering $E$, there is one of them, say $E_p$, which is barrelled and its closure, $\overline{E_p}$, has countable codimension in $E$.

Clearly, quasi-totally barrelled $\Rightarrow$ countably quasi-suprabarrelled $\Rightarrow$ quasi-suprabarrelled, and suprabarrelled $\Rightarrow$ countably quasi-suprabarrelled.

It is easy to check that if $\mathfrak{c}$ is the cardinal of the continuum, $\mathfrak{c}$ verifies condition (G) and, consequently, is quasi-suprabarrelled but not countably quasi-suprabarrelled, [2, Example 2]. On the other hand, $\varphi$ is a non-suprabarrelled countably quasi-suprabarrelled space since every linear subspace of it has countable codimension. Moreover, if $I$ is any index set, it follows from Theorem 1 below that $\varphi^I$ is also a non-suprabarrelled countably quasi-suprabarrelled space, and, in general, each countably quasi-suprabarrelled space containing a complemented copy of $\varphi$ is not suprabarrelled.

**Examples**. Countably quasi-suprabarrelled spaces which are not quasi-totally barrelled.

1. Let $E$ be a locally convex space and $m_0(E)$ the space of the $2^\mathbb{N}$-simple functions defined over $\mathbb{N}$ with values in $E$ endowed with the uniform convergence topology. From [3] it follows that if $E$ is nuclear and unordered Baire-like, then $m_0(E)$ is suprabarrelled and, consequently, countably quasi-suprabarrelled. If $\{A_n: n \in \mathbb{N}\}$ denotes the sequence of all the subsets of two different positive integers of $\mathbb{N}$ and $L_n$ is the linear subspace of $m_0(E)$ of all the $f \in m_0(E)$ which are constant on $A_n$, it is clear that each $L_n$ is closed in $m_0(E)$, that $\{L_n: n \in \mathbb{N}\}$ covers $m_0(E)$ and that if $\dim E$ is uncountable, then each $L_n$ has uncountable codimension in $m_0(E)$. Hence, if $E$ is a nuclear unordered Baire-like space of uncountable dimension, $m_0(E)$ is a countably quasi-suprabarrelled space which is not quasi-totally barrelled.

2. Let $E$ be a Banach space containing a sequence of closed linear subspaces $\{X_n: n \in \mathbb{N}\}$ of infinite dimension such that for each $n \in \mathbb{N}$, the closed linear hull of $\{X_m: m > n\}$ is a topological complement of $X_1 + \ldots + X_n$ and let $E_n$ be the closed linear hull of $\{X_m: m \in \mathbb{N} \setminus \{n\}\}$. If $\mathcal{U}$ is an ultrafilter in $\mathbb{N}$ which contains the filter of all the subsets of $\mathbb{N}$ whose complement has zero density, $L(\mathcal{U})$ the closure in $E$ of the linear hull of $\bigcup\{X_n: n \in \mathbb{N} \setminus U\}$ for each $U \in \mathcal{U}$, and $L = \bigcup\{L(U): U \in \mathcal{U}\}$, then $L$ is a suprabarrelled and dense subspace in $E$, [11, Proposition 12]. If each $X_n$ has infinite dimension, then $E_n \cap L$ is a subspace of uncountable codimension in $L$. Finally, as each $E_n$ is closed and $\{E_n \cap L: n \in \mathbb{N}\}$ covers $L$, we obtain that $L$ is a countably quasi-suprabarrelled space which is not quasi-totally barrelled.

Clearly, the topological product of $\varphi$ and any non-quasi-totally barrelled countably quasi-suprabarrelled space is an example of a countably quasi-suprabarrelled space.
which is neither suprabarrelled nor quasi-totally barrelled. On the other hand, a metrizable space $E$ is countably quasi-suprabarrelled if and only if $E$ is suprabarrelled. But, as we have mentioned above, there exist non-suprabarrelled countably quasi-suprabarrelled spaces. Next we shall show the following.

**Proposition 1.** Let $E$ be a countably quasi-suprabarrelled space. If $E$ is not suprabarrelled then $E$ is not Baire-like either.

**Proof:** If $E$ is not suprabarrelled, there exists an increasing sequence of linear subspaces $\{E_n: n \in \mathbb{N}\}$ of $E$ covering $E$, such that no $E_n$ is barrelled and dense at the same time. As $E$ is countably quasi-suprabarrelled, we may suppose that each $E_n$ is barrelled and its closure is of countable codimension. Hence $E$ cannot be Baire-like since it may be covered by an increasing sequence of closed linear subspaces of infinite countable codimension. 

3. **Properties of countably quasi-suprabarrelled spaces**

Next we shall obtain some permanence properties of countably quasi-suprabarrelled spaces.

**Proposition 2.** Let $E$ be a countably quasi-suprabarrelled space. If $F$ is a closed linear subspace of $E$ then $E/F$ is countably quasi-suprabarrelled.

**Proof:** Let $\{G_n: n \in \mathbb{N}\}$ be an increasing sequence of subspaces of $E/F$ covering $E/F$. Let $k$ be the canonical mapping of $E$ onto $E/F$. Then $\{k^{-1}(G_n): n \in \mathbb{N}\}$ is an increasing sequence of subspaces of $E$ covering $E$, so there must be some some $p \in \mathbb{N}$ such that $k^{-1}(G_p)$ is barrelled and $\text{cod}E_k^{-1}(G_p) \leq \aleph_0$. Now, $G_p = k(k^{-1}(G_p))$ is barrelled and if $L$ is an algebraic complement of $k^{-1}(G_p)$ in $E$, then $G_p + k(L) = k(k^{-1}(G_p)) + k(L) \supset k(k^{-1}(G_p) + L) = k(E) = E/F$. Hence $G_p$ has countable codimension in $E/F$.

**Proposition 3.** Let $F$ be a dense linear subspace of $E$. If $F$ is countably quasi-suprabarrelled then $E$ is countably quasi-suprabarrelled.

**Proof:** Let $\{E_n: n \in \mathbb{N}\}$ be an increasing sequence of subspaces of $E$ covering $E$. Since $F$ is countably quasi-suprabarrelled there is some $p \in \mathbb{N}$ such that each $F \cap E_p$ is barrelled and $\text{cod}F_k F \cap E_p^F \leq \aleph_0$. Let $L$ be a topological complement of $F \cap E_p^F$ in $F$. $F \cap E_p \oplus L$ coincides with $E$ since it is closed and $F \subset F \cap E_p \oplus L$, so $E_p + L = E$ and $\text{cod}E_k E_p \leq \aleph_0$. Besides, $F \cap E_p \oplus_L L$ is a barrelled dense subspace of $E_p + L$. Hence $E_p$ is barrelled.

**Proposition 4.** Let $F$ be a countable codimensional subspace of $E$. If $E$ is countably quasi-suprabarrelled then $F$ is countably quasi-suprabarrelled.
PROOF: Let \( \{ F_n : n \in \mathbb{N} \} \) be an increasing sequence of subspaces of \( F \) covering \( F \). Let \( G \) be an algebraic complement of \( F \) in \( E \). As \( \{ F_n + G : n \in \mathbb{N} \} \) is an increasing sequence of subspaces of \( E \) covering \( E \), we may assume that every \( F_n \) is barrelled. On the other hand, as \( \{ F_n + G : n \in \mathbb{N} \} \) is also an increasing sequence of subspaces of \( E \) covering \( E \), we may assume that every \( F_n + G \) is barrelled. So, if \( L_n \) is a topological complement of \( \overline{F_n} \) in \( \overline{F_n + G} \), \( L_n \) is a closed subspace of \( E \) for every \( n \in \mathbb{N} \) and there must be some \( p \in \mathbb{N} \) so that \( F_p + G \), and consequently \( \overline{F_p} \), has countable codimension in \( E \). Hence \( \overline{F_p} \) has countable codimension in \( F \). \( \Box \)

PROPOSITION 5. The topological product of finitely many countably quasi-suprabarrelled spaces is countably quasi-suprabarrelled.

PROOF: Assume \( E_1 \) and \( E_2 \) are countably quasi-suprabarrelled and \( E = E_1 \times E_2 \). Let \( \{ F_n : n \in \mathbb{N} \} \) be an increasing sequence of subspaces of \( E \) covering \( E \). Then there exists a subsequence \( \{ F_{n_p} : p \in \mathbb{N} \} \) such that \( \text{cod}_{E_i} (F_{n_p} \cap \overline{E_i}) \leq \aleph_0 \), \( i = 1, 2 \). For each \( p \in \mathbb{N} \) let \( A_{p,i} \) be a cobasis of \( F_{n_p} \cap \overline{E_i} \) in \( E_i \), \( i = 1, 2 \). Set \( A := \bigcup \{ A_{p,1} \cup A_{p,2} : p \in \mathbb{N} \} \) and, for each \( p \in \mathbb{N} \), let \( L_p := (F_{n_p} \cup A) \). If some \( L_p \) were barrelled, \( F_{n_p} \) would be barrelled and the proof would be finished since \( E = E_1 \times E_2 = (F_{n_p} \cap \overline{E_1}) \times (F_{n_p} \cap \overline{E_2}) + (A_{p,1} \cup A_{p,2}), \) that is \( \text{cod} E (F_{n_p}) \leq \aleph_0 \).

Let us suppose that none of the \( L_p \) is barrelled. Then for each \( p \in \mathbb{N} \) there is a barrel, say \( T_p \), in \( L_p \) which is not a neighbourhood of the origin in \( L_p \). Now, since \( \{ L_p \cap E_i : p \in \mathbb{N} \} \) is an increasing sequence of subspaces of \( E_i \) covering \( E_i \), \( i = 1, 2 \), there must be some positive integer \( q \in \mathbb{N} \) such that \( L_q \cap E_i \) is barrelled. Therefore, setting \( V_q := \overline{T_q} \cap \overline{E_i} \cap \overline{L_q} \cap \overline{E_i} \) is a neighbourhood of the origin in \( L_q \cap E_i \).

On the other hand, \( L_q \cap E_i \) is dense in \( E_i \) since \( L_q \cap E_i \supset (F_{n_p} \cap \overline{E_i}) \cup (A_{q,i}) = E_i \). Therefore, \( V_q \cap L_q \cap \overline{E_i} \) is a neighbourhood of the origin in \( E_i \), \( i = 1, 2 \), and \( V_q \) is a neighbourhood of the origin in \( E \) since \( V_q \cap L_q \cap \overline{E_1} \times V_q \cap L_q \cap \overline{E_2} \supset V_q + V_q = 2V_q \). Hence \( T_q \) is a neighbourhood of the origin in \( L_q \), which is not possible. \( \Box \)

In order to show that this result is true for arbitrarily many spaces we shall need [1, Theorem 2] and [2, Proposition 4]:

**Lemma 1.** Let \( \{ E_i : i \in I \} \) be a family of spaces such that for every finite subset \( H \subset I \), \( E(H) \) is quasi-suprabarrelled. Then \( E = \prod \{ E_i : i \in I \} \) is quasi-suprabarrelled.

**Lemma 2.** Let \( \{ E_i : i \in I \} \) be a family of spaces and \( B \) a countable family of closed absolutely convex subsets of \( E = \prod \{ E_i : i \in I \} \) such that \( \text{cod} E(B) > \aleph_0 \) for each \( B \in B \). Suppose that \( \mathcal{F} := \{ (B) : B \in B \} \) covers \( E \) and let \( \mathcal{F}_i := \{ F \in \mathcal{F} : \text{cod}_{E_i}(F) \cap E(\{i\}) > \aleph_0 \} \). If for each \( F \in \mathcal{F} \) there is a finite subset \( J(F) \) of \( I \)
such that \( F \supset E(I \setminus J(F)) \), then there exists some \( j \in I \) such that \( F_j \) covers \( E(\{j\}) \).

**Theorem 1**. If \( \{E_i : i \in I\} \) is a family of countably quasi-suprabarrelled spaces, then \( E = \Pi \{E_i : i \in I\} \) is countably quasi-suprabarrelled.

**Proof**: By Lemma 1, \( E \) is quasi-suprabarrelled. So, if \( E \) is not countably quasi-suprabarrelled, there exists an increasing sequence of barrelled subspaces of \( E \) covering \( E \), \( \{F_n : n \in \mathbb{N}\} \), such that \( \text{cod}_{E} F_n^E > \aleph_0 \) for every \( n \in \mathbb{N} \). Then \( \{F_n : n \in \mathbb{N} \text{ and } F_n \supset E(I \setminus J_n) \) with \( J_n \) a finite subset of \( I \) \) is also an increasing sequence of barrelled subspaces of \( E \) covering \( E \), [12, Proposition 4].

Now, by Lemma 2, there exists some \( j \in I \) such that \( \{F_n : n \in \mathbb{N} \text{ and } \text{cod}_{E} \{j\} F_n^E > \aleph_0 \} \) covers \( E(\{j\}) \), which is not possible since \( E(\{j\}) \) is countably quasi-suprabarrelled. \( \square \)

Finally let us recall that a locally convex space \( E \) is a \( \Gamma_r \)-space if given any quasi-complete subspace \( G \) of \( E^*(\sigma(E^*, E)) \) such that \( G \cap E' \) is dense in \( E^*(\sigma(E', E)) \), then \( G \) contains \( E' \), and that \( \Gamma_r \)-spaces are the maximal class of locally convex spaces satisfying the closed graph theorem when barrelled spaces are considered as the domain, (see [8] and [10, Chapter 1, Section 6.2]). Moreover [8, Corolario 1.14] provides:

**Lemma 3**. Let \( f \) be a continuous linear mapping from a barrelled space \( E \) into \( F \). If \( F \) is a \( \Gamma_r \)-space then \( f \) has a continuous extension from the completion of \( E \) into \( F \).

**Theorem 2**. Let \( E \) be a countably quasi-suprabarrelled space and suppose \( \{F_n : n \in \mathbb{N}\} \) is an increasing sequence of subspaces of \( F \) such that on each \( F_n \) there exists a topology, \( \tau_n \), finer than the original one so that \( F_n(\tau_n) \) is a \( \Gamma_r \)-space. If \( f \) is a linear mapping from \( E \) into \( F \) with closed graph then either there is some \( p \in \mathbb{N} \) such that \( f(E) \subset F_p \) and \( f \) is continuous or there is a topological complement \( H \) of \( \varphi \) in \( E \) such that \( f(H) \subset F_p \), \( f \) being continuous.

**Proof**: The sequence of subspaces \( \{f^{-1}(F_n) : n \in \mathbb{N}\} \) of \( E \) is increasing and covers \( E \), so there must to some \( p \in \mathbb{N} \) such that \( f^{-1}(F_p) \) is barrelled and its closure, \( H \), has countable codimension in \( E \). Let \( L \) be a topological complement of \( H \) in \( E \). If \( \dim L < \aleph_0 \), then the restriction of \( f \) to \( L \) is continuous. If \( \dim L = \aleph_0 \), then \( L \cong \varphi \) and the restriction of \( f \) to \( L \) is continuous, too. Thus in order to see that \( f \) is continuous it is enough to show that the restriction \( f |_H \) of \( f \) to \( H \) is continuous. The restriction \( g \) of \( f \) to \( f^{-1}(F_p) \) has closed graph in \( f^{-1}(F_p) \times F_p(\tau_p) \) and thus is continuous. Now Lemma 3 allows us to extend \( g \) to a continuous linear mapping \( h : H \to F_p(\tau_p) \). Let us show that \( h = f|_H \). Given \( x \in H \), let \( \{x_i : i \in I\} \) be a net in \( f^{-1}(F_p) \) converging to \( x \) in \( H \). Then the net \( \{f(x_i) : i \in I\} \) converges to \( h(x) \) in \( F_p(\tau_p) \) and, consequently, in \( F \). Hence \( f(x) = h(x) \) since \( f|_H \) has closed graph in \( H \times F \).
As \( f(H) \subseteq F_p \), the proof is complete if \( \dim L = \aleph_0 \). If \( \dim L < \aleph_0 \), it is clear that there is some \( q \in \mathbb{N} \) so that \( f(L) \subseteq F_q \), and therefore \( r = \max\{p, q\} \) gives \( f(E) \subseteq F_r \).

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