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EXTREMAL PROBLEMS FOR A CLASS OF FUNCTIONS OF POSITIVE REAL PART AND APPLICATIONS

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Abstract

In this paper we determine the lower bound on $|z| = r < 1$ for the functional $\mathrm{Re}\{\alpha p(z) + \beta zp'(z)/p(z)\}$, $\alpha \geqslant 0$, $\beta \geqslant 0$, over the class $\mathbf{P}_k(A, B)$. By means of this result, sharp bounds for $|f(z)|$, $|f'(z)|$ in the family $\mathbf{S}_k^*(A, B)$ and the radius of convexity for $\mathbf{S}_k^*(A, B)$ are obtained. Furthermore, we establish the radius of starlikeness of order β, $0 \leqslant \beta < 1$, for the functions $F(z) = \lambda f(z) + (1 - \lambda)zf'(z)$, $|z| < 1$, where $-\infty < \lambda < 1$, and $f(z) \in \mathbf{S}_k^*(A, B)$.

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1. Introduction

Let $\mathbf{B}$ be the class of functions $w(z)$ regular in the unit disc $\Delta = \{z;\ |z| < 1\}$ and satisfying the conditions $w(0) = 0$, $|w(z)| < 1$ there. We denote by $\mathbf{P}(A, B)$, $-1 \leqslant B < A \leqslant 1$, the class of functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ defined by

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, \qquad z \in \Delta,$$

for some $w(z) \in \mathbf{B}$. We note that $\mathbf{P}(1, -1) \equiv \mathbf{P}$, the class of functions of positive real part in the unit disc.

Recently, Janowski [7] introduced the following general class of starlike functions: $\mathbf{S}^*(A, B) = \{f(z) = z + a_2 z^2 + \cdots :\ zf'(z)/f(z) \in \mathbf{P}(A, B),\ z \in \Delta\}$. This class reduces to well-known subclasses of starlike functions by special choices of A, B; for example, $\mathbf{S}^*(1 - 2\alpha, -1) = \{f(z) = z + a_2 z^2 + \cdots :$

$\operatorname{Re}\{zf'(z)/f(z)\} > \alpha,\ 0 \leqslant \alpha < 1,\ z \in \Delta\}$, $\mathbf{S}^*(1, 1/M - 1) = \{f(z) = z + a_2 z^2 + \cdots : |zf'(z)/f(z) - M| \langle M,\ M \rangle \frac{1}{2}, z \in \Delta\}$, $\mathbf{S}^*(\alpha, 0) = \{f(z) = z + a_2 z^2 + \cdots : |zf'(z)/f(z) - 1| < \alpha,\ 0 < \alpha \leqslant 1, z \in \Delta\}$, $\mathbf{S}^*(\alpha, -\alpha) = \{f(z) = z + a_2 z^2 + \cdots : |zf'(z)/f(z) - 1|/|zf'(z)/f(z) + 1| < \alpha, 0 \leqslant \alpha < 1,\ z \in \Delta\}$.

Problems associated with $\mathbf{S}^*(A, B)$ may be transformed into those of minimising or maximising on $|z| = r < 1$ functionals of the form $\operatorname{Re}\{F(p(z), zp'(z))\}$, where $F(u, v)$ is a given function regular in the v-plane and in the half-plane $\operatorname{Re} u > 0$, and where $p(z)$ varies in $\mathbf{P}(A, B)$.

By means of a result due to Robertson, Janowski [7] found the lower bounds for the functionals $\operatorname{Re}\{p(z) + zp'(z)/p(z)\}$ and $\operatorname{Re}\{zp'(z)/p(z)\}$ on $|z| = r < 1$, where $p(z) \in \mathbf{P}(A, B)$. However, the analysis is lengthy and rather involved. In this paper, we give an elementary solution to the more general problem

$$\text{(1.1)} \qquad \min_{|z| = r < 1} \operatorname{Re}\{\alpha p(z) + \beta zp'(z)/p(z)\}, \qquad \alpha \geqslant 0, \qquad \beta \geqslant 0,$$

where $p(z)$ varies in the class

$$\mathbf{P}_k(A, B) = \left\{p(z) = 1 + \sum_{n=k}^{\infty} p_n z^n \in \mathbf{P}(A, B) \colon k \geqslant 1,\ z \in \Delta\right\}.$$

Janowski's results correspond to the cases $\alpha = \beta = k = 1$ and $\alpha = 0$, $\beta = k = 1$, respectively.

For some applications of (1.1) we shall consider the following problems.

(i) Distortion, covering, radius of convexity for functions in $\mathbf{S}^*(A, B)$ with missing coefficients, that is, for the class

$$\mathbf{S}_k^*(A, B) = \left\{f(z) = z + \sum_{n=k}^{\infty} a_{n+1} z^{n+1} \colon zf'(z)/f(z) \in \mathbf{P}_k(A, B),\ z \in \Delta\right\}.$$

(ii) Radius of starlikeness of order β, $0 \leqslant \beta < 1$, for the functions

$$F(z) = \lambda f(z) + (1 - \lambda) zf'(z), \qquad z \in \Delta,$$

where $-\infty < \lambda < 1$, $f(z) \in \mathbf{S}_k^*(A, B)$.

The consideration of problem (ii) is motivated by recent investigations of Livingston [10], Bernardi [3], Goel and Singh [6]. Results of part (i) refine those given by Janowski [7] on functions of the class $\mathbf{S}^*(A, B)$.

2. The functional $\operatorname{Re}\{\alpha p(z) + \beta zp'(z)/p(z)\}$, $\alpha \geqslant 0$, $\beta \geqslant 0$, over $\mathbf{P}_k(A, B)$

Let $\mathbf{B}_k$ denote the class of regular functions of the form

$$w(z) = b_k z^k + b_{k+1} z^{k+1} + \cdots$$

such that $|w(z)| < 1$ in Δ. In view of the general Schwarz lemma, we have $|w(z)| \leqslant |z|^k$; therefore, we may write

$$w(z) = z^k \psi(z), \qquad z \in \Delta,$$

where $\psi(z)$ is regular and $|\psi(z)| \leqslant 1$ in Δ. An application of Carathéodory's inequality (see Carathéodory [4, page 18]) that

$$|\psi'(z)| \leqslant \frac{1-|\psi(z)|^2}{1-|z|^2}, \quad z \in \Delta,$$

now yields

$$\text{(2.1)} \qquad |zw'(z) - kw(z)| \leqslant \frac{|z|^{2k} - |w(z)|^2}{|z|^{k-1}(1-|z|^2)}, \qquad w(z) \in \mathbf{B}_k,\ z \in \Delta.$$

Equality in (2.1) occurs for functions of the form $z^k(z-c)/(1-cz)$, $|c| \leqslant 1$.

For every $p(z) \in \mathbf{P}_k(A, B)$, we have

$$\text{(2.2)} \qquad p(z) = H(w(z)), \qquad z \in \Delta,$$

for some $w(z) \in \mathbf{B}_k$, where $H(z) = (1 + Az)/(1 + Bz)$. Consequently, an application of the Subordination Principle (see Duren [5, pages 190–191]) yields that the image of $|z| \leqslant r$ under every $p(z) \in \mathbf{P}_k(A, B)$ is contained in the disc

$$\text{(2.3)} \qquad |p(z) - a_k| \leqslant d_k,$$

where

$$\text{(2.4)} \qquad a_k = \frac{1 - ABr^{2k}}{1 - B^2r^{2k}}, \qquad d_k = \frac{(A-B)r^k}{1 - B^2r^{2k}}.$$

It follows immediately from (2.3) and (2.4) that if $p(z) \in \mathbf{P}_k(A, B)$, then on $|z| = r < 1$, we have

$$\text{(2.5)} \qquad \frac{1 - Ar^k}{1 - Br^k} \leqslant \operatorname{Re}\{p(z)\} \leqslant |p(z)| \leqslant \frac{1 + Ar^k}{1 + Br^k}.$$

The inequalities are sharp for $p(z) = (1 + Az^k)/(1 + Bz^k)$.

We are now ready to prove our main theorem.

2.1. THEOREM. *If $p(z) \in \mathbf{P}_k(A, B)$, $\alpha \geqslant 0$, $\beta \geqslant 0$, then on $|z| = r < 1$, we have*

$$\operatorname{Re}\left\{\alpha p(z) + \beta \frac{zp'(z)}{p(z)}\right\} \geqslant \begin{cases} \dfrac{\alpha - [\beta k(A-B) + 2\alpha A]r^k + \alpha A^2 r^{2k}}{(1 - Ar^k)(1 - Br^k)}, & R_{k1} \leqslant R_{k2}, \\[2ex] \beta k \dfrac{A+B}{A-B} + 2\dfrac{(MN)^{1/2} - \beta(1 - ABr^{2k})}{(A-B)r^{k-1}(1-r^2)}, & R_{k2} \leqslant R_{k1}, \end{cases}$$

where $R_{k1} = (M/N)^{1/2}$, $R_{k2} = (1 - Ar^k)/(1 - Br^k)$, $M = \beta(1 - kAr^{k-1} + kAr^{k+1} - A^2r^{2k})$, and $N = \beta + [\alpha(A-B) - \beta kB]r^{k-1} - [\alpha(A-B) - \beta kB]r^{k+1} - \beta B^2 r^{2k}$. The result is sharp.

PROOF. From the representation formula (2.2) we may write

$$\alpha p(z)+\beta\frac{zp'(z)}{p(z)}=\alpha\frac{1+Aw(z)}{1+Bw(z)}+\beta\frac{(A-B)zw'(z)}{[1+Aw(z)][1+Bw(z)]},\qquad w(z)\in\mathbf{B}_k.$$

Applying (2.1) to the second term of the right-hand side, we find

$$\operatorname{Re}\left\{p(z)+\frac{zp'(z)}{p(z)}\right\}\geqslant\operatorname{Re}\left\{\alpha\frac{1+A\omega(z)}{1+B\omega(z)}+\frac{\beta(A-B)k\omega(z)}{(1+A\omega(z))(1+B\omega(z))}\right\}-\frac{\beta(A-B)\left(|z|^{2k}-|\omega(z)|^2\right)}{|z|^{k-1}\left(1-|z|^2\right)|1+A\omega(z)|\,|1+B\omega(z)|}.$$

From (2.2), we also have, for $\omega(z)\in\mathbf{B}_k$, that

$$\omega(z)=\frac{p(z)-1}{A-Bp(z)},\qquad p(z)\in\mathbf{P}_k(A,B).$$

Hence, in terms of $p(z)$, the above inequality becomes

$$\text{(2.6)}\quad \operatorname{Re}\left\{\alpha p(z)+\beta\frac{zp'(z)}{p(z)}\right\}\geqslant\beta k\frac{A+B}{A-B}+\frac{1}{A-B}\operatorname{Re}\left\{[\alpha(A-B)-\beta kB]\,p(z)-\frac{\beta kA}{p(z)}\right\}-\beta\frac{r^{2k}|A-Bp(z)|^2-|p(z)-1|^2}{(A-B)r^{k-1}(1-r^2)|p(z)|}.$$

Put $p(z)=a_k+u+iv$, let $|p(z)|=R$, and denote the right-hand side of (2.6) by $S(u,v)$. Then

$$S(u,v)=\beta k\frac{A+B}{A-B}+\frac{1}{A-B}\left\{[\alpha(A-B)-\beta kB](a_k+u)-\frac{\beta kA(a_k+u)}{R^2}-\beta\frac{1-B^2r^{2k}}{r^{k-1}(1-r^2)}\cdot\frac{d_k^2-u^2-v^2}{R}\right\}.$$

Now,

$$\text{(2.7)}\qquad \frac{\partial S}{\partial v}=\frac{\beta}{A-B}\cdot\frac{v}{R^4}T(u,v),$$

where

$$T(u,v)=2kA(a_k+u)+\frac{1-B^2r^{2k}}{r^{k-1}(1-r^2)}\left[2R^3+\left(d_k^2-u^2-v^2\right)R\right]\geqslant 2(a_k+u)\left[kA+\frac{1-B^2r^{2k}}{r^{k-1}(1-r^2)}(a_k-d_k)^2\right]$$

(as $d_k^2 - u^2 - v^2 \geqslant 0$, and $R^3 \geqslant (a_k + u)(a_k - d_k)^2$). Therefore

$$(2.8) \qquad T(u,v) \geqslant 2(a_k + u)\left[kA + \frac{1 - B^2r^{2k}}{r^{k-1}(1-r^2)} \cdot \left(\frac{1 - Ar^k}{1 - Br^k}\right)^2\right].$$

We want to show now that

$$(2.9) \qquad \frac{1 - B^2r^{2k}}{r^{k-1}(1-r^2)} \geqslant k.$$

In fact,

$$\frac{1 - B^2r^{2k}}{r^{k-1}(1-r^2)} \geqslant \frac{1 - r^{2k}}{r^{k-1}(1-r^2)} \geqslant k$$

if and only if $1 - r^{2k} \geqslant kr^{k-1}(1 - r^2)$, that is, if and only if

$$F(k,r) \equiv 1 + r^2 + r^4 + \cdots + r^{2(k-1)} - kr^{k-1} \geqslant 0.$$

If the following expressions are written out completely, it is seen that for k even, $F(k,r) = (1 - r^{k-1})^2 + r^2(1 - r^{k-3})^2 + \cdots + r^{k-2}(1 - r)^2 > 0$, and for k odd, $F(k,r) = (1 - r^{k-1})^2 + r^2(1 - r^{k-3})^2 + \cdots + r^{k-3}(1 - r^2)^2 > 0$. Hence, inequality (2.9) always holds. This inequality together with (2.8) imply that

$$T(u,v) \geqslant 2k(a_k + u)\left[A + \left(\frac{1 - Ar^k}{1 - Br^k}\right)^2\right].$$

Now $A(1 - Br^k)^2 + (1 - Ar^k)^2 = (1 + B)(1 - Ar^k)^2 + (A - B)(1 - ABr^{2k}) > 0$. Thus $T(u,v) > 0$, and it follows from (2.7) that the minimum of $S(u,v)$ on the disc $|p(z) - a_k| \leqslant d_k$ is attained when $v = 0$, $u \in [-d_k, d_k]$. Setting $v = 0$ in the expression for $S(u,v)$, we get

$$S(u,0) = \beta k\frac{A+B}{A-B} + \frac{1}{A-B}\left\{\left[\alpha(A-B) - \beta kB + \beta\frac{1 - B^2r^{2k}}{r^{k-1}(1-r^2)}\right](a_k + u) - \beta\left[kA - \frac{1 - A^2r^{2k}}{r^{k-1}(1-r^2)}\right]\frac{1}{a_k + u} - 2\beta\frac{1 - ABr^{2k}}{r^{k-1}(1-r^2)}\right\},$$

which yields

$$\frac{dS(u,0)}{du} = \frac{1}{A-B}\left\{\alpha(A-B) - \beta kB + \beta\frac{1 - B^2r^{2k}}{r^{k-1}(1-r^2)} + \beta\left[kA - \frac{1 - A^2r^{2k}}{r^{k-1}(1-r^2)}\right]\frac{1}{(a_k+u)^2}\right\}.$$

We see that the absolute minimum of $S(u,0)$ occurs at the point $u_0 = (M/N)^{1/2} - a_k$ if u_0 lies inside $[-d_k, d_k]$. Its value is

$$S(u_0,0) = \beta k\frac{A+B}{A-B} + 2\frac{(MN)^{1/2} - \beta(1 - ABr^{2k})}{(A-B)r^{k-1}(1-r^2)}.$$

We next want to show that $u_0 < d_k$. Indeed, it is seen from (2.9) that

$$\frac{1 - B^2 r^{2k}}{r^{k-1}(1 - r^2)} - kB \geqslant k(1 - B) \geqslant 0,$$

and similarly,

$$\frac{1 - A^2 r^{2k}}{r^{k-1}(1 - r^2)} - kA \geqslant k(1 - A) \geqslant 0.$$

Also,

$$\begin{aligned} N &= \beta(1 - B^2 r^{2k}) + (\alpha(A - B) - \beta k B) r^{k-1}(1 - r^2) \\ &\geqslant \beta(1 - kBr^{k-1} + kBr^{k+1} - B^2 r^{2k}), \end{aligned}$$

as $\alpha(A - B) \geqslant 0$. Thus,

$$\begin{aligned} (a_k + u_0)^2 &< \frac{1 - kAr^{k-1} + kAr^{k+1} - A^2 r^{2k}}{1 - kBr^{k-1} + kBr^{k+1} - B^2 r^{2k}} \\ &= \frac{k - Ar^{k-1}}{k - Br^{k-1}} \left(\frac{1 - kAr^{k-1}}{k - Ar^{k-1}} + Ar^{k+1} \right) \left(\frac{1 - kBr^{k-1}}{k - Br^{k-1}} + Br^{k+1} \right)^{-1}. \end{aligned}$$

Since $0 < (k - Ar^{k-1})/(k - Br^{k-1}) < 1$, and since the second and third factors are positive, the above inequality reduces to

$$(2.10) \quad (a_k + u_0)^2 < \left(\frac{1 - kAr^{k-1}}{k - Ar^{k-1}} + Ar^{k+1} \right) \left(\frac{1 - kBr^{k-1}}{k - Br^{k-1}} + Br^{k+1} \right)^{-1}.$$

The right-hand side of (2.10) is less than or equal to $(1 + Ar^{k+1})/(1 + Br^{k+1})$ if and only if

$$\begin{aligned} &\frac{1 - kAr^{k-1}}{k - Ar^{k-1}} + (1 - k)\left(\frac{1 + Ar^{k-1}}{k - Ar^{k-1}} \right) Br^{k+1} \\ &\qquad \leqslant \frac{1 - kBr^{k-1}}{k - Br^{k-1}} + (1 - k)\left(\frac{1 + Br^{k-1}}{k - Br^{k-1}} \right) Ar^{k+1}, \end{aligned}$$

that is, if and only if

$$\begin{aligned} &\left[(1 - k)Br^{k+1} + (1 - k)ABr^{2k} + 1 - kAr^{k-1}\right](k - Br^{k-1}) \\ &\qquad \leqslant \left[(1 - k)Ar^{k+1} + (1 - k)ABr^{2k} + 1 - kBr^{k-1}\right](k - Ar^{k-1}). \end{aligned}$$

This inequality is equivalent to

$$(2.11) \qquad (k - 1)\left[1 + (A + B)r^{k+1} + ABr^{2k} + k(1 - r^2)\right] \geqslant 0.$$

Put $G(A, B, r) = 1 + (A + B)r^{k+1} + ABr^{2k}$. Then

$$\frac{\partial G}{\partial B} = r^{k+1}(1 + Ar^{k-1}) > 0.$$

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Thus,

$$G(A,B,r) \geqslant G(A,-1,r) = 1 - r^{k+1} + Ar^{k+1}(1 - r^{k-1}) \geqslant (1 - r^{k-1})(1 + Ar^{k+1}) > 0.$$

This implies that condition (2.11) is always satisfied. Consequently, in view of (2.10) and these intermediate steps, we have that

$$(a_k + u_0)^2 < \frac{1 + Ar^{k+1}}{1 + Br^{k+1}}.$$

Furthermore, it is clear that

$$\frac{1 + Ar^{k+1}}{1 + Br^{k+1}} < \frac{1 + Ar^k}{1 + Br^k} < \left(\frac{1 + Ar^k}{1 + Br^k}\right)^2 = (a_k + d_k)^2.$$

Hence, $u_0 < d_k$. However, u_0 is not always greater than $-d_k$. For the case $u_0 \leqslant -d_k$, that is, if $R_{k1} \leqslant R_{k2}$, the absolute minimum of $S(u,0)$ occurs at the end-point $u = -d_k$, the value of which is

$$S(-d_k, 0) = \frac{\alpha - [\beta k(A - B) + 2\alpha A] r^k + \alpha A^2 r^{2k}}{(1 - Ar^k)(1 - Br^k)}.$$

To see that the result is sharp, we consider the functions

$$p(z) = \frac{1 + Az^k}{1 + Bz^k}, \quad \text{for } R_{k1} \leqslant R_{k2},$$

$$p(z) = \frac{1 + Aw_k(z)}{1 + Bw_k(z)}, \quad \text{for } R_{k2} \leqslant R_{k1},$$

where $w_k(z) = z^k(z - c_k)/(1 - c_k z)$, with c_k such that $\operatorname{Re}\{[1 + Aw_k(z)]/[1 + Bw_k(z)]\} = R_{k1}$ at $z = re^{i\pi/k}$.

3. Some geometric properties of the class $S_k^*(A, B)$

In this section we derive the sharp bounds for $|f(z)|$ and $|f'(z)|$ in the family $S_k^*(A, B)$ and the radius of convexity for $S_k^*(A, B)$. Letting $r \to 1$ in the lower bound for $|f(z)|$, we obtain the disc which is covered by the image of the unit disc under every $f(z)$ in $S_k^*(A, B)$.

3.1. **Theorem.** *Let $f(z) \in S_k^*(A, B)$. Then on $|z| = r < 1$, we have*

(i)
$$r(1 - Br^k)^{(A-B)/kB} \leqslant |f(z)| \leqslant r(1 + Br^k)^{(A-B)/kB}, \quad \text{if } B \neq 0,$$
$$r\exp\left(-\frac{Ar^k}{k}\right) \leqslant |f(z)| \leqslant r\exp\left(\frac{Ar^k}{k}\right), \quad \text{if } B = 0;$$

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$$\text{(ii)}\quad \begin{aligned} &(1 - Ar^k)(1 - Br^k)^{[A-(1+k)B]/B} \\ &\qquad \leqslant |f'(z)| \leqslant (1 + Ar^k)(1 + Br^k)^{[A-(1+k)B]/B}, \quad \text{if } B \neq 0, \\ &(1 - Ar^k)\exp\left(-\frac{Ar^k}{k}\right) \leqslant |f'(z)| \leqslant (1 + Ar^k)\exp\left(\frac{Ar^k}{k}\right), \quad \text{if } B = 0. \end{aligned}$$

PROOF. Write $zf'(z)/f'(z) = p(z)$, $p(z) \in \mathbf{P}_k(A, B)$. Then

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{1}{z}[p(z) - 1].$$

Hence, on integrating both sides, we get

$$\log \frac{f(z)}{z} = \int_0^z [p(\xi) - 1]\frac{d\xi}{\xi},$$

that is,

$$\frac{f(z)}{z} = \exp \int_0^z \frac{p(\xi) - 1}{\xi}\, d\xi, \qquad p(z) \in \mathbf{P}_k(A, B).$$

Therefore,

$$\left|\frac{f(z)}{z}\right| = \exp\left[\operatorname{Re}\left\{\int_0^z \frac{p(\xi) - 1}{\xi}\, d\xi\right\}\right].$$

Substituting ξ by zt in the integral we have

$$\left|\frac{f(z)}{z}\right| = \exp \int_0^1 \operatorname{Re}\left\{\frac{p(zt) - 1}{t}\right\} dt.$$

It follows from (2.5) that, on $|zt| = rt$, we have

$$\operatorname{Re}\left\{\frac{p(zt) - 1}{t}\right\} \leqslant \frac{(A - B)r^k t^{k-1}}{1 + Br^k t^k}.$$

Hence, for $B \neq 0$,

$$\left|\frac{f(z)}{z}\right| \leqslant \exp \int_0^1 \frac{(A - B)r^k t^{k-1}}{1 + Br^k t^k}\, dt = (1 + Br^k)^{(A-B)/kB}.$$

The lower bound may be obtained similarly. The case $B = 0$ is trivial. To prove (ii), we note that

$$|f'(z)| = \left|\frac{f(z)}{z}\right| |p(z)|, \qquad p(z) \in \mathbf{P}_k(A, B).$$

Hence, by applying the above results and (2.5), the assertions follow.

All the bounds are sharp for the function

$$f(z) = z(1 + Bz^k)^{(A-B)/kB}, \quad \text{if } B \neq 0,$$

$$f(z) = z\exp\left(\frac{Az^k}{k}\right), \quad \text{if } B = 0.$$

The corollary of Theorem 1 of Zawadzki [13] corresponds to the special case $A = 1 - 2\alpha$, $B = -1$.

Letting $r \to 1$ in the lower bound for $|f(z)|$ we obtain the covering theorem for $\mathbf{S}_k^*(A, B)$.

3.2. COROLLARY. *The image of the unit disc under a function* $f(z) \in \mathbf{S}_k^*(A, B)$ *contains the disc of centre* 0 *and radius* $(1 - B)^{(A-B)/kB}$ *if* $B \neq 0$, *and radius* $\exp(-A/k)$ *if* $B = 0$.

We now derive the radius of convexity of $\mathbf{S}_k^*(A, B)$. This radius is given by the smallest root in $(0, 1]$ of the equation $\Omega(r) = 0$, where

$$\Omega(r) = \min\left\{\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}: |z| = r < 1,\ f(z) \in \mathbf{S}_k^*(A, B)\right\}$$
$$= \min\left\{\operatorname{Re}\left\{p(z) + \frac{zp'(z)}{p(z)}\right\}: |z| = r < 1,\ p(z) \in \mathbf{P}_k(A, B)\right\}.$$

An application of Theorem 2.1 with $\alpha = 1$, $\beta = 1$ gives $\Omega(r)$, and, upon solving the equation $\Omega(r) = 0$, we obtain

3,3. COROLLARY. *The radius of convexity of* $\mathbf{S}_k^*(A, B)$ *is given by the smallest root in* $(0, 1]$ *of*

(i) $$1 - [(2 + k)A - kB]r^k + A^2r^{2k} = 0, \quad \text{if } R_{k1} \leqslant R_{k2},$$

(ii) $$-4 + 4r^2 + k[k(A - B) + 4A]r^{k-1} - 2[(k^2 - 2)(A - B) + 4kA]r^{k+1}$$
$$+k[k(A - B) + 4A]r^{k+3} + 4A^2r^{2k} - 4A^2r^{2k+2} = 0, \quad \text{if } R_{k2} \leqslant R_{k1},$$

where R_{k1}, R_{k2} *are as given in Theorem* 2.1.

The result previously obtained by Zmorovič [14] and Singh and Goel [12] corresponds to the case in which $k = 1$, $A = 1 - 2\alpha$, and $B = -1$.

4. On Livingston's problem

Libera [8] showed that if $f(z) \in \mathbf{S}^* \equiv \mathbf{S}^*(1, -1)$ then the function

$$g(z) = \frac{2}{z}\int_0^z f(\xi)\, d\xi \tag{4.1}$$

is also in $\mathbf{S}^*$. Livingston [10] studied the converse problem: namely, if $g(z) \in \mathbf{S}^*$, what is the radius of starlikeness of the function

$$f(z) = \tfrac{1}{2}[g(z) + zg'(z)]? \tag{4.2}$$

Livingston showed that $f(z)$ is starlike in $|z| < \frac{1}{2}$. This result has been refined and generalised in different ways by many authors. Padmanabhan [11] proved that if $g(z) \in \mathbf{S}_\alpha^*$, $0 \leqslant \alpha \leqslant \frac{1}{2}$, then $f(z)$, as defined by (4.2), is starlike of the same order in $|z| < [\alpha - 2 + (\alpha^2 + 4)^{1/2}]/2\alpha$. Libera and Livingston [9] extended Padmanabhan's result to include the range $\frac{1}{2} < \alpha < 1$. These authors obtained the radius of the disc in which $f(z)$ is starlike of order β, where $f(z)$ is given by (4.2) with $g(z) \in \mathbf{S}_\alpha^*$, $0 \leqslant \alpha < 1$ and $\beta \geqslant \alpha$. The complementary case $0 \leqslant \beta < \alpha$ was studied by Al-Amiri [1] and Bajpai and Singh [2].

In another direction, Bernardi [11] found the radius of starlikeness of the functions $f(z)$ defined by

$$f(z) = \frac{1}{1+c}[cg(z) + zg'(z)], \tag{4.3}$$

where $c = 1, 2, 3, \ldots$ and $g(z) \in \mathbf{S}^*$. Goel and Singh [6] extended and generalised Bernardi's result to the case in which c is any real number such that $c + \beta > 0$ and $g(z)$ belongs to a more restricted family characterised by the condition

$$\left|\left(\frac{zg'(z)}{g(z)} - \beta\right)/(1-\beta) - \alpha\right| < \alpha, \qquad 0 \leqslant \beta < 1, \alpha > \tfrac{1}{2}.$$

We note that this class is a special case of $\mathbf{S}^*(A, B)$ with $A = [\alpha(1-\beta) + (1-\alpha)\beta]/\alpha$, $B = (1-\alpha)/\alpha$. We further remark that for $1 + c > 0$, and for $\lambda = c/(1+c)$, equation (4.3) is equivalent to

$$f(z) = \lambda g(z) + (1-\lambda) zg'(z), \qquad -\infty < \lambda < 1. \tag{4.4}$$

The restriction $c + \beta > 0$ in Goel and Singh's analysis corresponds to $\beta/(\beta - 1) < \lambda < 1$.

In the following, as another direct application of Theorem 2.1, we determine the sharp radius of the disc in which every $f(z)$, as given by (4.4) with $g(z) \in \mathbf{S}_k^*(A, B)$, is starlike of order γ, $0 \leqslant \gamma < 1$. All the above-mentioned results are special cases of this result, with $k = 1$, and with appropriately chosen values of A, B, γ.

4.1. Theorem. *Let* $f(z) = \lambda g(z) + (1-\lambda)zg'(z)$, *where* $(A-1)/(A-B) \leqslant \lambda < 1$, *and let* $g(z) \in \mathbf{S}_k^*(A, B)$. *Let* r_{k1} *be the smallest root in* $(0,1]$ *of the equation*

$$(1-\lambda)(1-\gamma) + [(\lambda + \gamma(1-\lambda))(B+C) - k(1-\lambda)(C-B) - 2C]r^k + [C^2 - (\lambda + \gamma(1-\lambda))BC]r^{2k} = 0,$$

and let r_{k2} *be the smallest root in* $(0,1]$ *of the equation*

$$\begin{aligned}4(1-\lambda)[D-E+(1-\lambda)kC] &- 4(1-\lambda)[D-E+(1-\lambda)kC]r^2\\&+[D^2+4(1-\lambda)kCE]r^{k-1}\\&+\left[4(1-\lambda)^2(C-B)^2-2D^2-8(1-\lambda)kCE\right]r^{k+1}\\&+[D^2+4(1-\lambda)kCE]r^{k+3}\\&+4(1-\lambda)[C^2E-CBD-(1-\lambda)kCB^2]r^{2k}\\&-4(1-\lambda)[C^2E-CBD-(1-\lambda)kCB^2]r^{2k+2}=0,\end{aligned}$$

where $C=(1-\lambda)A+\lambda B$, $D=[\lambda+\gamma(1-\lambda)](C-B)-k(1-\lambda)(C+B)$, *and* $E=C-B-k(1-\lambda)B$. *Then* $f(z)$ *is starlike of order* γ, $0\leqslant\gamma<1$, *in*

$$|z|<\begin{cases}r_{k1}, & \text{for } R_{k1}\leqslant R_{k2},\\ r_{k2}, & \text{for } R_{k2}\leqslant R_{k1},\end{cases}$$

where R_{k1}, R_{k2} *are as given in Theorem* 2.1 *with* A *replaced by* C, *with* $\alpha=1$, $\beta=1-\lambda$.

Proof. Since $g(z)\in \mathbf{S}_k^*(A,B)$, we may write

$$\frac{zg'(z)}{g(z)}=p(z),\qquad p(z)\in\mathbf{P}_k(A,B).$$

Then, from the definition of $f(z)$, we have

$$\text{(4.5)}\qquad \frac{zf'(z)}{f(z)}=p(z)+\frac{zp'(z)}{p(z)+\mu},\qquad \mu=\frac{\lambda}{1-\lambda},\ -1<\mu<\infty.$$

Put $q(z)=[p(z)+\mu]/(1+\mu)$. Then, in terms of functions of $\mathbf{B}_k$, we have

$$q(z)=\frac{1+[(1-\lambda)A+\lambda B]w(z)}{1+Bw(z)},\qquad w(z)\in\mathbf{B}_k.$$

Hence $q(z)\in\mathbf{P}_k(C,B)$ and

$$\text{(4.6)}\quad \frac{zq'(z)}{q(z)}=\frac{zp'(z)}{p(z)+\mu}=\frac{(1-\lambda)(A-B)zw'(z)}{[1+Bw(z)][1+((1-\lambda)A+\lambda B)w(z)]}.$$

It is clear from (4.6) that the function $zp'(z)/[p(z)+\mu]$ may not be regular in Δ if $(1-\lambda)A+\lambda B>1$, that is, if $\lambda<(A-1)/(A-B)$. Hence we confine λ to the range $(A-1)/(A-B)\leqslant\lambda<1$ so that $zp'(z)/[p(z)+\mu]$ is regular in the entire unit disc. Equation (4.5) may be rewritten as

(4.7)

$$\frac{zf'(z)}{f(z)}=-\frac{\lambda}{1-\lambda}+\frac{1}{1-\lambda}\left[q(z)+(1-\lambda)\frac{zq'(z)}{q(z)}\right],\qquad q(z)\in\mathbf{P}_k(C,B).$$

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Now, the radius of starlikeness of order γ of $f(z)$ is determined by the equation

$$\min_{f(z)\in S_k^*(A,B)} \min_{|z|=r<1} \operatorname{Re}\left\{\frac{zf'(z)}{f(z)} - \gamma\right\} = 0,$$

or equivalently, from (4.7),

(4.8)

$$\min_{q(z)\in P_k(C,B)} \min_{|z|=r<1} \operatorname{Re}\left\{-\gamma - \frac{\lambda}{1-\lambda} + \frac{1}{1-\lambda}\left[q(z) + (1-\lambda)\frac{zq'(z)}{q(z)}\right]\right\} = 0.$$

Hence an application of Theorem 2.1 (with A replaced by C, $\alpha = 1$, and $\beta = 1 - \lambda$) to (4.8) will yield the equations which give the starlikeness of $f(z)$. The sharpness of the result follows from that of Theorem 2.1.

Theorem 1 of Bernardi [3] is recovered when we put $k = 1$, $\gamma = 0$, $A = 1$, $B = -1$, and $C = 1 - 2\lambda$ in the above theorem. Theorem 3.2 of Singh and Goel [12] corresponds to the case in which $k = 1$, $\lambda = 1/2$, $\gamma = 0$, $A = 1 - 2\beta$, $B = -1$, and $C = -\beta$.

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References

[1] H. S. Al-Amiri, 'On the radius of β-convexity of starlike functions of order α', *Proc. Amer. Math. Soc.* **39** (1973), 101–109.
[2] P. L. Bajpai and P. Singh, 'The radius of starlikeness of certain analytic functions', *Proc. Amer. Math. Soc.* **44** (1974), 395–402.
[3] S. D. Bernardi, 'The radius of univalence of certain analytic functions', *Proc. Amer. Math. Soc.* **24** (1970), 312–318.
[4] C. Carathéodory, *Theory of functions of a complex variable*, Vol. 2 (Chelsea Publishing Company, New York, 1960).
[5] P. L. Duren, *Univalent Functions* (Springer Verlag, New York, 1983).
[6] R. M. Goel and V. Singh, 'On the radii of univalence of certain analytic functions', *Indian J. Pure Appl. Math.* **4** (1973), 402–421.
[7] W. Janowski, 'Some extremal problems for certain families of analytic functions I', *Ann. Polon. Math.* **28** (1973), 298–326.
[8] R. J. Libera, 'Some classes of regular univalent functions', *Proc. Amer. Math. Soc.* **16** (1965), 755–758.

[9] R. J. Libera and A. E. Livingston, 'On the univalence of some classes of regular functions', *Proc. Amer. Math. Soc.* **30** (1971), 327–336.

[10] A. E. Livingston, 'On the radius of univalence of certain analytic functions', *Proc. Amer. Math. Soc.* **17** (1966), 352–357.

[11] K. S. Padmanabhan, 'On the radius of univalence of certain classes of analytic functions', *J. London Math. Soc.* (2) **1** (1969), 225–231.

[12] V. Singh and R. M. Goel, 'On radii of convexity and starlikeness of some classes of functions', *J. Math. Soc. Japan* **23** (1971), 323–339.

[13] R. Zawadzki, 'On some theorems on distortion and rotation in the class of k-symmetric starlike functions of order α', *Ann. Polon. Math.* **24** (1971), 169–185.

[14] V. A. Zmorovič, 'On bounds of convexity for starlike functions of order α in the circle $|z| < 1$ and in the circular region $0 < |z| < 1$', *Mat. Sb.* **68**(110) (1965), 518–526, English transl., *Amer. Math. Soc. Transl.* (2) **80** (1969), 203–213.

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