ON A FUNCTIONAL EQUATION FOR THE EXPONENTIAL FUNCTION OF A COMPLEX VARIABLE

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1. Introduction. The following result is well known in the theory of analytic functions; see **[1]**.

THEOREM A. *Suppose that $f(z)$ is an entire function of a complex variable z. Then $f(z)$ satisfies the functional equation*

$$|f(z)| = |f(x)|, \tag{1}$$

where $z = x+iy$ (x, y real), if and only if $f(z) = a\exp(sz)$, where a is an arbitrary complex constant and s is an arbitrary real constant.

We consider the following functional equation which is an extension of (1).

$$|f(z)|^2+|g(z)|^2 = |f(x)|^2+|g(x)|^2, \tag{2}$$

where $z = x+iy$ (x, y real) and $f(z)$, $g(z)$ are entire functions of a complex variable z.

The purpose of this note is to solve (2), i.e., to prove the following

THEOREM. *Let $f(z)$, $g(z)$ be entire functions of a complex variable z. Then $f(z)$ and $g(z)$ satisfy* (2) *if and only if $f(z) = \alpha A\exp(pz)+\beta B\exp(qz)$, $g(z) = \gamma A\exp(pz)+\delta B\exp(qz)$, where $A, B, \alpha, \beta, \gamma, \delta$ are complex constants, $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ is a unitary matrix and p, q are real constants.*

To this end we shall use the following two lemmas:

LEMMA 1. *If $f(z)$, $g(z)$ are regular in a domain D and $|f(z)|^2+|g(z)|^2 > 0$ in D, then we have in D*

$$\Delta\log(|f(z)|^2+|g(z)|^2) = \frac{4|f'(z)g(z)-f(z)g'(z)|^2}{(|f(z)|^2+|g(z)|^2)^2},$$

where Δ stands for the Laplacian $(\partial^2/\partial x^2)+(\partial^2/\partial y^2)$ ($z = x+iy$, x, y real).

Proof. See [3].

LEMMA 2. *If $f(z)$, $g(z)$, $h(z)$, $k(z)$ are entire functions of a complex variable z and satisfy $|f(z)|^2+|g(z)|^2 = |h(z)|^2+|k(z)|^2$ in $|z| < +\infty$, then there exists a unitary matrix $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ ($\alpha, \beta, \gamma, \delta$ are complex constants) such that*

$$h(z) = \alpha f(z)+\beta g(z) \quad \text{and} \quad k(z) = \gamma f(z)+\delta g(z)$$

in $|z| < +\infty$.

Proof. See [2, 4].

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2. Proof of the theorem. We may assume that $|f(z)|^2+|g(z)|^2 \not\equiv 0$ in $|z|<+\infty$. If we put

$$\varphi(x)=|f(x)|^2+|g(x)|^2(\geqq 0), \tag{3}$$

we have in $|z|<+\infty$, by (2),

$$|f(z)|^2+|g(z)|^2=\varphi(x). \tag{4}$$

We shall prove that

$$\varphi(x)>0 \tag{5}$$

in $-\infty<x<+\infty$.

If there exists a real number x_0 such that $\varphi(x_0)=0$, then, by (4), we have in $-\infty<y<+\infty$

$$|f(x_0+iy)|^2+|g(x_0+iy)|^2=0. \tag{6}$$

By (6) and by the identity theorem we have $f(z)\equiv 0$, $g(z)\equiv 0$ and hence $|f(z)|^2+|g(z)|^2\equiv 0$. This is contrary to our assumption that $|f(z)|^2+|g(z)|^2\not\equiv 0$. Hence (5) holds in $-\infty<x<+\infty$.

Taking the logarithms of both sides of (4) and taking the Laplacians $(\partial^2/\partial x^2)+(\partial^2/\partial y^2)$ of both sides of the resulting equality, by (5) we have in $|z|<+\infty$

$$\Delta\log(|f(z)|^2+|g(z)|^2)=\frac{\varphi(x)\varphi''(x)-\varphi'(x)^2}{\varphi(x)^2}. \tag{7}$$

By (7) and by Lemma 1 we have in $|z|<+\infty$

$$\frac{4|f'(z)g(z)-f(z)g'(z)|^2}{(|f(z)|^2+|g(z)|^2)^2}=\frac{\varphi(x)\varphi''(x)-\varphi'(x)^2}{\varphi(x)^2}. \tag{8}$$

By (4), (8) we have in $|z|<+\infty$

$$|f'(z)g(z)-f(z)g'(z)|^2=\tfrac{1}{4}(\varphi(x)\varphi''(x)-\varphi'(x)^2). \tag{9}$$

Putting $y=0$ in (8) and using (9), we have in $|z|<+\infty$

$$|f'(z)g(z)-f(z)g'(z)|=|f'(x)g(x)-f(x)g'(x)|. \tag{10}$$

By (10) and by Theorem A, we have in $|z|<+\infty$

$$f'(z)g(z)-f(z)g'(z)=a\exp(sz), \tag{11}$$

where a is a complex constant and s is a real constant.

By (9), (11) we have in $-\infty<x<+\infty$

$$\varphi(x)\varphi''(x)-\varphi'(x)^2=4|a|^2\exp(2sx). \tag{12}$$

Putting

$$\psi(x)=\exp(-sx)\varphi(x), \tag{13}$$

by (12) we have in $-\infty<x<+\infty$

$$\psi(x)\psi''(x)-\psi'(x)^2=4|a|^2. \tag{14}$$

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Differentiating both sides of (14) with respect to x, we have in $-\infty < x < +\infty$

$$\psi(x)\psi'''(x) - \psi'(x)\psi''(x) = 0. \tag{15}$$

By (5), (13) we have in $-\infty < x < +\infty$

$$\psi(x) > 0. \tag{16}$$

By (15), (16) we have in $-\infty < x < +\infty$

$$\left(\frac{\psi''(x)}{\psi(x)}\right)' = 0. \tag{17}$$

Hence, by (17) we have in $-\infty < x < +\infty$

$$\psi''(x) = K\psi(x), \tag{18}$$

where K is a real constant.

Taking (16) into account and solving (18), we have in $-\infty < x < +\infty$

$$\psi(x) = A_1 \exp(hx) + B_1 \exp(kx), \tag{19}$$

where A_1, B_1, h, k are real constants with $A_1 \geqq 0$, $B_1 \geqq 0$, $A_1 + B_1 > 0$. By (13), (19) we have in $-\infty < x < +\infty$

$$\varphi(x) = A_1 \exp((h+s)x) + B_1 \exp((k+s)x). \tag{20}$$

Since $\exp(x) = |\exp(z)|$ for every complex number $z = x+iy$ (x, y real), by (20) we have in $|z| < +\infty$

$$\varphi(x) = |A \exp(pz)|^2 + |B \exp(qz)|^2, \tag{21}$$

where A, B are complex constants with $|A|^2 = A_1$, $|B|^2 = B_1$ and p, q are real constants with $2p = h+s$, $2q = k+s$. By (4), (21) we have in $|z| < +\infty$

$$|f(z)|^2 + |g(z)|^2 = |A \exp(pz)|^2 + |B \exp(qz)|^2. \tag{22}$$

By (22) and by Lemma 2 we have in $|z| < +\infty$

$$f(z) = \alpha A \exp(pz) + \beta B \exp(qz), \qquad g(z) = \gamma A \exp(pz) + \delta B \exp(qz), \tag{23}$$

where A, B, α, β, γ, δ are complex constants, $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is a unitary matrix and p, q are real constants.

Conversely, since $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is a unitary matrix and $|\exp(z)| = \exp(x)$ for every complex number z, direct substitution shows that (23) satisfies our original functional equation (2).

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