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A NOTE ON UPPER BOUNDS FOR THE EIGENVALUES OF $y''+\lambda py=0$

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The natural modes of a small planar transversal vibration of a fixed string of unit length and tension are determined by the eigenvalues and associated eigenfunctions of the differential equation

$$y''(x)+\lambda p(x)y(x) = 0 \tag{1}$$

subject to the boundary condition

$$y(0) = y(1) = 0 \tag{2}$$

where the non-negative function p describes the mass distribution of the string. That the distribution of mass on the string influences the modes of vibration, may be reflected by observing that the eigenvalues determined by the system (1-2) may be considered functions of the density p, $\lambda_n(p)$, where $\lambda_1(p)<\lambda_2(p)<\ldots$. It is thus a natural problem to investigate the restrictions imposed upon the eigenvalues $\lambda_n(p)$ when the density p is restricted to a specific class of measurable functions. The main result which we derive is the establishment of an upper bound for the nth eigenvalue $\lambda_n(p)$ when p is a class of functions $E(M, h, H)$ which we define in the following. Our result is a generalization of results of Krein [11, Theorem 4] and Schwarz [12, Theorem 3]. Krein investigated the extremum of the eigenvalues $\lambda_n(p)$ when p is restricted to the class of bounded measurable functions having total mass equal to a fixed constant, M, and Schwarz considered the restriction to piecewise continuous equimeasurable densities. Others [1, 2, 3, 4, 5] have examined bounds for the eigenvalue $\lambda_n(p)$ when the function p is restricted to proper subclasses of the class of functions considered by Krein and when the density p is not necessarily bounded but satisfies restrictions on its form such as being convex, increasing, or $\int_0^x p(t)\,dt$ being concave.

We consider a class of measurable functions defined on [0, 1] which is a natural extension of the class considered by Krein, $E(M, h, H)$, defined for each choice of measurable functions h and H and constant M satisfying

$$0 \le h(x) \le H(x) \quad \text{for} \quad x \in [0, 1],$$

and

$$\int_0^1 h(x)\,dx < M < \int_0^1 H(x)\,dx,$$

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by

$$E(M, h, H) = \left\{p \mid h(x) \le p(x) \le H(x), \int_0^1 p(x)\,dx = M\right\}.$$

For $p \in E(M, h, H)$ $\lambda_n(p)$ satisfies the inequalities $\lambda_n(H) \le \lambda_n(p) \le \lambda_n(h)$, when $h \not\equiv 0$ as the eigenvalues of (1) are known to vary inversely as the density function varies pointwise [8]. Thus $\lambda_n(p)$ is bounded below for any choice of H and h. However, as h approaches the zero function the nth eigenvalue $\lambda_n(h)$ approaches $+\infty$. Thus the least upper bound for $\lambda_n(p)$ as p varies over $E(M, h, H)$ conceivably could increase to infinity if the function h were decreased to the zero function. That this is not the case, and that in fact there exists an upper bound for $\lambda_n(p)$ entirely dependent on the function H and independent of h is given by the

THEOREM. *Let $\lambda_n(p)$ denote the nth eigenvalue of a vibrating string under unit tension, with fixed end points, and mass density given by the non-negative measurable function p. If p is in the class of functions defined on* $[0, 1]$ *having total mass M and bounded pointwise above by a measurable function H and below by a non-negative measurable function h, then $\lambda_n(p) \le \lambda_n^*$ where λ_n^* is a function of H and n only.*

Before proving the Theorem we observe that if the upper boundary function H is bounded with $\mathscr{H} = \sup H(x) < \infty$, $x \in [0, 1]$, then our result is contained in those of Krein [11, Th. 4] and Schwarz [12, Th. 3]. Explicitly Krein showed that in this case

$$\lambda_n(p) \le H(\pi n/M)^2.$$

Thus the principle significance of our result lies in its application in the situation where the function H is not bounded. In this case it seems appropriate to comment on the existence of eigenvalues associated with the functions in $E(M, h, H)$, the function H, and other functions derived from H in the following. Generally a solution of equation (1) is assumed to be a piecewise continuously differentiable function y which satisfies the equation (1) throughout the interval [0, 1]. However, since a measurable function p is determined only up to a set of measure zero, it does not make sense to ask that (1) hold for all $x \in [0, 1]$. Thus instead of requiring a solution to satisfy (1) we ask that $\theta = yp^{1/2}$ satisfy the equivalent integral equation

$$\theta(x) = \lambda \int_0^1 G(x, \xi)(p(x))^{1/2}(p(\xi))^{1/2}\theta(\xi)\,d\xi \tag{3}$$

where G is the Greens function of the system $-y''(x) = 0$, $y(0) = y(1) = 0$. In this sense, since the equation (3) is of Fredholm type with a symmetric L^2 kernel, the system (1-2) will have a sequence of eigenvalues and associated eigenfunctions possessing the same properties as those obtained in the usual sense when p is assumed continuous. (See Tricomi [13, section 3.13].) In particular, as noted by Krein [11, (1.1)], for any non-negative measurable density p defined on $[0, l]$,

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the first eigenvalue of (1) on $[0, l]$ subject to zero boundary conditions is given by

$$\lambda_1(p) = \min_{\psi\in C^1}\left(\int_0^l [\psi'(x)]^2\,dx \Big/ \int_0^l \psi^2(x)p(x)\,dx\right) \tag{4}$$

where C^1 is the class of continuously differentiable functions ψ defined on $[0, l]$ and satisfying $\psi(0)=\psi(l)=0$.

Proof. Let ρ be a function in $E(M, h, H)$ and denote by u_n the nth eigenfunction of the system (1-2) with $p=\rho$. The function u_n partitions the interval [0, 1] into n nodes determined by its zeros $0=x_0<x_1<\cdots<x_n=1$. There exists at least one choice of the index i such that the interval $I=(x_i, x_{i+i})$ satisfies the inequality $\int_I \rho(x)\,dx\geq M/n$. We then denote by J, the largest subinterval of I, (x_i, x_i+S) such that $\int_J \rho(x)\,dx=M/n$. The nth eigenvalue $\lambda_n(\rho)$ of the system (1-2) with $p=\rho$ is then the first eigenvalue of the equation (1) with the boundary conditions $y(0)=y(x_{i+1}-x_i)=0$ and $p(x)=\rho(x_i+x)$. By further constraining the problem we cannot decrease the eigenvalues and thus we have

$$\lambda_n(\rho) \leq \mu(\rho_0)$$

where $\mu(p)$ denotes the first eigenvalue of the equation (1) subject to the conditions $y(0)=y(S)=0$, and $\rho_0(x)\equiv\rho(x_i+x)$, Courant [8, p. 408].

Denote by ρ_0^+ the symmetrically increasing rearrangement of ρ_0 on the interval $[O, S]$. (See Beesack and Schwarz [7] and [10].) We next invoke a Theorem due to Beesack and Schwarz [7, Th. 2] to obtain the inequality

$$\mu(\rho_0) \leq \mu(\rho_0^+).$$

We note that Beesack and Schwarz stated their Theorem 2 for continuous densities, but utilizing the characterization (4) and noting that the Theorem 378 of [10] which they employed in their proof is applicable to measurable functions, their proof extends to measurable densities as considered here.

We next let H^+ denote the symmetrically increasing rearrangement of the function H over the interval [0, 1]. Let t_n denote the least value of t such that $\int_0^t H^+(x)\,dx=M/2n$. We then have

$$\frac{M}{n} = 2\int_0^{t_n} H^+(x)\,dx \geq \int_{x_i}^{x_i+2t_n} H(x)\,dx \geq \int_{x_i}^{x_i+2t_n} \rho(x)\,dx.$$

Consequently, if we had $2t_n>S$, it would then follow from the above that

$$\frac{M}{n} > \int_{x_i}^{x_i+S} \rho(x)\,dx = \frac{M}{n}.$$

Hence $2t_n\leq S$ and we can define the symmetric function H_0 over $[O, S]$ by

$$H_0(x) = \begin{cases} H^+(x), & \text{if } 0\leq x\leq t_n, \\ 0, & \text{if } t_n < x \leq S/2, \\ H_0(S-x), & \text{if } S/2 < x \leq S. \end{cases}$$

Since $\rho(x) \leq H(x)$ for $x \in J$ we have

$$\rho_0^+(x) \leq H_0(x) \quad \text{for} \quad x \in [O, t_n] \cup [S-t_n, S]$$

and clearly $\rho_0^+(x) \geq H_0(x)$ for $x \in (t_n, S-t_n)$. Consequently we may apply Theorem III of Beesack [6] to the functions H_0 and ρ_0^+ on the interval $[O, S]$ to obtain the inequality

$$\mu(\rho_0^+) \leq \mu(H_0).$$

Again, although Beesack stated his Theorem III for continuous densities, his proof extends to cover the case of measurable densities as considered here by utilizing the characterization (4) and noting that Beesacks Theorem II, utilized in his proof of Theorem III, applies to measurable functions.

Finally, we denote by λ_n^* the first eigenvalue of the system

$$u''(x)+\lambda_n^* H^+(x)u(x) = 0, \qquad x \in [0, t_n]$$
$$u(0) = u'(t_n) = 0. \tag{5}$$

Then, due to the symmetry of H_0, we have

$$\mu(H_0) = \lambda_n^*.$$

Combining the above we obtain our result

$$\lambda_n(\rho) \leq \lambda_n^*.$$

The number λ_n^* is not dependent on the choice of ρ and since t_n is a monotone decreasing function of n, we see that to increase n tends to further constrain the system (5) and thus λ_n^* is monotone increasing in n.

The bound λ_n^* given in the above theorem will not generally provide a sharp upper bound for $\lambda_n(p)$ as p varies over $E(M, h, H)$ unless $n=1$ and H is symmetrically decreasing. However the apriori existence of such a bound, valid for any choice of the function h, allows the application of Theorems 3 and 4 of Gentry and Banks [9]. These theorems then assert that for any differentiable function $f(x_1, x_2, \ldots, x_n)$, the functional $F(p)=f(\lambda_1(p), \lambda_2(p), \ldots, \lambda_n(p))$ will actually assume its supremum on the class $E(M, h, H)$ at a density ρ which is itself extremal in the sense that $\rho(x)$ is equal to either $H(x)$ or $h(x)$ for each $x \in [0, 1]$. This generalizes the results of Krein [11, Th. 5].

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