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SOME ADJUNCTION-THEORETIC PROPERTIES OF CODIMENSION TWO NON-SINGULAR SUBVARITIES OF QUADRICS

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ABSTRACT. We make precise the structure of the first two reduction morphisms associated with codimension two non-singular subvarieties of non-singular quadrics Q^n, $n \geq 5$. We give a coarse classification of the same class of subvarieties when they are assumed not to be of log-general-type.

0. **Introduction.** The study of low codimension subvarieties of projective space has been a very active area of research in recent years. The papers [15], [6] and their bibliographies may serve the reader as a diving board towards a vast sea of general structure results, classification in low degree and conjectures concerning surfaces in $\mathbb{P}^4$ and threefolds in $\mathbb{P}^5$, respectively.

Low codimensional embeddings in projective space are special in many respects because, for example, of results such as the Barth-Larsen theorem and the double-point formulæ.

The Barth-Larsen theorem asserts that, given an embedding $\iota: X \hookrightarrow \mathbb{P}^N$, the group homomorphisms $\iota_m^*: H^m(\mathbb{P}^N, \mathbb{C}) \longrightarrow H^m(X, \mathbb{C})$ are isomorphisms in a certain range of dimensions which depends on the codimension of $\iota(X)$ in $\mathbb{P}^N$; see [3]. The reader can consult [7], section 2.3.10 for a precise statement and a set of references concerning this result and its generalization, due to Sommese, to homogeneous varieties, such as non-singular quadrics Q^n. In this paper we will use frequently these results in the form summarized by Remark 2.2.

Given an embedding as above, the double-point formulæ provide expressions for the Chern classes of the normal bundle of $\iota(X)$ in terms of the Chern classes of the ambient space $\mathbb{P}^N$ and of the embedding line bundle $L := \iota^* O_{\mathbb{P}^N}(1)$. Other information about the Chern classes of the normal bundle of the embedding can be used in conjunction with these formulæ to obtain numerical restrictions on X. See [19], Example 4.1.3, where the double point formulæ are used in conjunction with the self-intersection formulæ to find a numerical identity between the basic invariants of embedded surfaces in $\mathbb{P}^4$. A classical application is that abelian surfaces in $\mathbb{P}^4$ must have degree $d = 10$.

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The double-point formulæ hold in a wider context, *e.g.* $\iota\colon X \longrightarrow Y$ could be any morphism of non-singular varieties, with X complete; see [17], section 9.3. In this wider context, their utility is clear only when one has precise information about the cohomology ring of the target space Y.

In particular, the Barth-Larsen theorem and the double-point formulæ hold in the more general context of low codimensional embeddings in homogeneous spaces. It seems natural to the author to explore their consequences in this context. The first homogeneous space that should come to mind, after $\mathbb{P}^N$, is the non-singular quadric and, since hypersurfaces and complete intersections are considered trivial in this context, the first value for the codimension to be considered should be two.

In his dissertation, [11], the author concerned himself with the case of codimension two embeddings in non-singular quadrics $\iota\colon X^{n-2} \hookrightarrow \mathcal{Q}^n$, $n \geq 5$. This paper is part of a series of papers stemming from the results of his dissertation; the other papers are [12], [13] and [14]. Arrondo and Sols had previously studied surfaces in $\mathcal{Q}^4$, which is isomorphic to the Grassmannian of lines in $\mathbb{P}^3$; see [2].

In this paper we address the following question: *what are the special adjunction-theoretic properties of* $(n-2)$*-folds embedded in quadrics* $\mathcal{Q}^n$? Given a projective manifold X and an ample line bundle L on X, the *Adjunction Theory* of Fujita *et al*. and Sommese *et al*. studies these polarized varieties (X, L); see [16] and [7]. Roughly speaking, this theory studies the positivity properties of the $\mathbb{Q}$-divisors $K_X + tL$, $t \in \mathbb{Q}^+$. In general, the lack of this positivity is a detector of the presence of special projective morphisms defined on the variety. There are two cases. In the former one, the morphism is birational and it is called *reduction*; it contracts some special subvarieties and provides a new birational model where the positivity of naturally associated divisors can be investigated further; one may think of this as a step of an inductive analysis. In the latter one, the morphism is a fibration onto a lower dimensional variety, with general fibers Fano manifolds; this is how the *special varieties* of Adjunction Theory arise. Adjunction Theory acts as a flowchart. We start with a pair (X, L) and with $t = \dim X - 1$ as above and take on and inductive analysis using the morphisms as above. At each step we either perform a reduction and reduce t or we have a special fibration. However, the theory is complete only for $t > \dim X - 3$.

The paper [6] consists of a thorough analysis of the adjunction-theoretic properties of threefolds X^3 in $\mathbb{P}^5$. Under this stringent restriction, many of the already precise results of Adjunction Theory become explicit.

Inspired by the results and by the techniques employed in [6], in this paper we establish that some of the adjunction-theoretic properties of threefolds in $\mathbb{P}^5$ proved in [6] also hold for codimension two non-singular subvarieties of quadrics $\mathcal{Q}^n$, $n \geq 5$.

The paper is organized as follows. Section 1 contains preliminary material such as a little background in Adjunction Theory and results from the papers [12]and [14] which will be used in the subsequent sections. Section 2 contains the precise description of the first two reduction morphisms of Adjunction Theory for codimension two subvarieties of quadrics; as it turns out, by analogy with [6], Theorem 1.4, the reduction morphisms associated with these varieties are almost always isomorphisms; see Theorem 2.3. In Section 3 we give a coarse classification theorem for the varieties for which the second reduction morphism is not defined, the so-called varieties *not of log-general-type*; see Theorem 3.1, Theorem 3.2 and Theorem 3.4. In order to prove the third one, we need to analyze the case of Del Pezzo fibrations and, in the same way as in the paper [10], the case of conic bundles in Q^5. These two analyses are carried out in Sections 4 and 5, respectively.

Threefolds not of log-general-type in $\mathbb{P}^5$ are completely classified (*cf*. [10]) by the efforts of many authors and the complete classification of threefolds in $\mathbb{P}^5$ up to degree 11 (see [6] and its references) was instrumental in achieving that goal. The paper [14] completes the classification of varieties as in its title. In particular the classification is complete only for degree $d \leq 10$. In order to make Theorem 3.4 complete, we would need the above classification for degree $d \leq 20$.

NOTATION AND CONVENTIONS. Our basic reference is [19]. We work over any algebraically closed field of characteristic zero. A quadric Q^n, here, is a non-singular hypersurface of degree two in the projective space $\mathbb{P}^{n+1}$. Little or no distinction is made between line bundles, associated sheaves of sections and Cartier divisors.
By a *scroll* we mean a variety $X \subseteq \mathbb{P}^N$, for which $(X, \mathcal{O}_{\mathbb{P}^N}(1)_{|X}) \simeq (\mathbb{P}_Y(E), \xi_E)$, where E is a vector bundle on a non-singular variety Y. An adjunction-theoretic scroll (see [5]) is not, in general, a scroll; we denote them by *a. t. scrolls*.

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1. **Preliminary Material.**

1.1 *Background in Adjunction Theory.* For a self-contained introduction to Adjunction Theory we refer the reader to [5]. The book [7] summarizes the state of the art in Adjunction Theory.

Let us recall the notions of first and second reduction morphism.

Let X be a non-singular projective variety of dimension n and L be an ample line bundle on X which is spanned at all points by its global sections. We say that a pair (X', L'), consisting of a non-singular, projective variety X' and an ample line bundle L', is the *first reduction* of (X, L) if:

(1) *there exists a morphism, the first reduction morphism, $\phi: X \longrightarrow X'$ expressing X as X' blown-up at a finite set F of non-singular distinct points,* and

(2) $L = \phi^* L' - [\phi^{-1}(F)]$;

this last relation 2) is equivalent to 2′) $K_X + (n-1)L = \phi^*(K_{X'} + (n-1)L')$.

Moreover, ϕ induces a bijection between the smooth elements of $|L|$ and those of $|L'|$ passing through F; $K_X + (n-1)L$ is nef and big if and only if there exists the first reduction

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(X', L') of (X, L), which is unique up to isomorphism. The positive dimensional fibers of ϕ are exactly the integral divisors D with $D \simeq \mathbb{P}^{n-1}$, $L_{|D} \simeq \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ and with normal bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. Finally, $K_{X'} + (n-1)L'$ is ample on X'

If $K_X + (n-1)L$ is not nef and big, then (X, L) is on an explicit list of special polarized varieties; see [8], page 381.

Assume that (X, L) admits the first reduction (X', L') and that $K_{X'} + (n-2)L'$ is nef and big. Then a suitable positive power of it defines a birational morphism $\varphi: X' \longrightarrow X''$ onto a normal variety X''. φ will be referred to as the *second reduction morphism*. Its positive dimensional fibers are well understood; see [5], (0.2.1) and (0.2.2). The Weil divisor $(\varphi_* L')^{**}$ is actually $\mathbb{Q}$-Cartier. The pair (X'', L'') is called the *second reduction* of (X, L).

If (X, L) admits a first reduction (X', L'), but $K_{X'} + (n-2)L'$ fails to be nef and big then the pair (X', L') is on an explicit list of polarized varieties; see [8], pages 381–2.

Pairs for which the two reduction morphisms are not defined are called *not of log-general-type*.

1.2 *Codimension two subvarieties of quadrics.* Let $\iota: X \hookrightarrow \mathcal{Q}^n$ be the embedding of a degree d, non-singular subvariety of codimension two of $\mathcal{Q}^n$; let L denote the line bundle $\iota^* \mathcal{O}_{\mathcal{Q}^n}(1)$, g the genus of the curve C obtained by intersecting $(n-3)$ general elements of $|L|$. Denote by x_i the Chern classes of the tangent bundle of X and by n_i the ones of the normal bundle $\mathcal{N}_{X, \mathcal{Q}^n}$; by adjunction $K_X = -nL + n_1$ and by the self-intersection formula $n_2 = (1/2)dL^2$.

The following formulæ, which hold in the Chow ring of X for $n \geq 5$, are obtained using the double-point formulæ (see [17]) for ι.

$$n_2 = \frac{1}{2}(n^2 - n + 2)L^2 - nx_1 \cdot L + x_1^2 - x_2; \tag{1}$$

$$\frac{1}{6}(n^3 - 3n^2 + 8n - 12)L^3 + \frac{1}{2}(-n^2 + n - 2)x_1 L^2 + n(x_1^2 - x_2)L + 2x_1x_2 - x_1^3 - x_3 = 0. \tag{2}$$

The following formulæ for surfaces X in $\mathcal{Q}^4$ with balanced cohomology class can be found in [2].

$$2K_X^2 = \frac{1}{2}d^2 - 3d - 8(g-1) + 12\chi(\mathcal{O}_X). \tag{3}$$

In the case of $n = 5$, using the formulæ (1) and (2), we can express $K_X \cdot L^2$, $K_X^2 \cdot L$, K_X^3, $x_2 \cdot L$ and x_3 as functions of d, g, $\chi(\mathcal{O}_X)$, $\chi(\mathcal{O}_S)$; for example, omitting the dots from now on:

$$K_X L^2 = 2(g-1) - 2d, \tag{4}$$

$$K_X^2 L = \frac{1}{4}d^2 + \frac{3}{2}d - 8(g-1) + 6\chi(\mathcal{O}_S), \tag{5}$$

$$K_X^3 = -\frac{9}{4}d^2 + \frac{27}{2}d + gd + 18(g-1) - 30\chi(\mathcal{O}_S) - 24\chi(\mathcal{O}_X). \tag{6}$$

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PROPOSITION 1.1. *Let X be a non-singular threefold in $\mathcal{Q}^5$. Then*

$$60\chi(\mathcal{O}_S) \geq \frac{3}{2}d^2 - 12d + (d-48)(g-1) + 24\chi(\mathcal{O}_X)$$

and

$$\chi(\mathcal{O}_S) \leq \frac{2}{3}\frac{(g-1)^2}{d} - \frac{1}{24}d^2 + \frac{5}{12}d.$$

PROOF. Denote by s_i and n_i the Segre and Chern classes respectively of the normal bundle $\mathcal{N}$ of X in $\mathcal{Q}^5$. Since $\mathcal{N}$ is generated by global sections, we have $s_3 \geq 0$. Since $s_3 = n_1^3 - 2n_1n_2$, we get

$$0 \leq (K_X+5L)^3 - 2(K_X+5L)\frac{1}{2}dL^2 = K^3 + 15K_X^2L + 75K_XL^2 + 125d - d(K_X+5L)L^2.$$

The first inequality follows from (6), (5) and (4).
We use the Generalized Hodge Index theorem of [4]:

$$d(K_X^2L) \leq (K_XL^2)^2$$

and we make explicit the left hand side using (5) and the right hand side using (4). The second inequality follows. ∎

In what follows:

- $((a,b,c),\mathcal{O}(1))$ denotes the polarized pair given by a complete intersection of type (a,b,c) in $\mathbb{P}^{n+1}$ and the restriction of the hyperplane bundle to it;
- (X,L) denotes the polarized pair given by a variety $X \subseteq \mathcal{Q}^n$ and $L := \mathcal{O}_{\mathcal{Q}^n}(1)_{|X}$;
- g, q and p_g denote the sectional genus of the embedding line bundle, the irregularity and geometric genus of a surface section, respectively.

REMARK 1.2. Let $X \subseteq \mathcal{Q}^n$, $n \geq 5$, be any subvariety. Then the degree d of X is even. This follows from the fact that the cohomology class of $[X]$ equals the class $(1/2)\,d\,[\mathcal{Q}^{n-2}]$ in $H^4(\mathcal{Q}^n,\mathbb{Z})$.

PROPOSITION 1.3 (CF. [14]). *Let $X \subseteq \mathcal{Q}^n$, $n \geq 5$, a codimension two non-singular subvariety of degree $d \leq 10$. Then the pair (X,L) is one of the types below.*

Type A) $d=2$, $\big((1,1,2),\mathcal{O}(1)\big)$; $g=q=p_g=0$.
Type B) $d=4$, $\big((1,2,2),\mathcal{O}(1)\big)$; $g=1$, $q=p_g=0$.
Type C) $d=4$, $n=6$, $\big(\mathbb{P}^1\times\mathbb{P}^3,\mathcal{O}(1,1)\big)$; $g=q=p_g=0$.
Type D) $d=4$, $n=5$, $\Big(\mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1}(1)^2\oplus\mathcal{O}_{\mathbb{P}^1}(2)\big),\xi\Big)$; $g=q=p_g=0$.
Type E) $d=6$, $\big((1,2,3),\mathcal{O}(1)\big)$; $g=4$, $q=0$, $p_g=1$.
Type F) $d=6$, $n=5$, $\big(\mathbb{P}(T_{\mathbb{P}^2}),\xi\big)$, *embedded using a general codimension one linear system* $\mathfrak{l} \subseteq |\xi_{T_{\mathbb{P}^2}}|$; $g=1$, $q=p_g=0$.
Type G) $d=6$, $n=5$, $f\colon X \longrightarrow \mathbb{P}^1\times\mathbb{P}^2 =: Y$ *a double cover, branched along a divisor of type* $\mathcal{O}_Y(2,2)$, $L \simeq p^*\mathcal{O}_Y(1,1)$; $g=2$, $q=p_g=0$.

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Type H) $d = 8$, $\big((1, 2, 4), \mathcal{O}(1)\big)$; $g = 9$, $q = 0$, $p_g = 5$.
Type I) $d = 8$, $\big((2, 2, 2), \mathcal{O}(1)\big)$; $g = 5$, $q = 0$, $p_g = 1$.
Type L) $d = 8$, $n = 5$, $\big(\mathbb{P}(E), \xi\big)$, *$E$ a rank two vector bundle over $\mathcal{Q}^2$ as in[20];* $g = 4$, $q = p_g = 0$.
Type M) $d = 10$, $\big((1, 2, 5), \mathcal{O}(1)\big)$; $g = 16$, $q = 0$, $p_g = 14$.
Type N) $d = 10$, $n = 5$, $f_{|K_X+L|} \colon X \longrightarrow \mathbb{P}^1$ *is a fibration with general fiber a Del Pezzo surface* F, $K_F^2 = 4$, $K_X = -L + f^*\mathcal{O}_{\mathbb{P}^1}(1)$; $g = 8$, $q = 0$, $p_g = 2$.

We say that a non-singular threefold X on $\mathcal{Q}^5$ is of Type O), if it has degree $d = 12$ and it is a scroll over a minimal $K3$ surface. Such a threefold exists. See [14].

PROPOSITION 1.4 (CF. [14]). *The following is the complete list of non-singular codimension two subvarieties of quadrics $\mathcal{Q}^n$, $n \geq 5$, which are scrolls.*
Type C) $n = 6$, $d = 4$, *scroll over* $\mathbb{P}^1$ *and over* $\mathbb{P}^3$;
Type D) $n = 5$, $d = 4$, *scroll over* $\mathbb{P}^1$;
Type F) $n = 5$, $d = 6$, *scroll over* $\mathbb{P}^2$;
Type L) $n = 5$, $d = 8$, *scroll over* $\mathcal{Q}^2$;
Type O) $n = 5$, $d = 12$, *scroll over a minimal $K3$ surface.*

PROPOSITION 1.5 (CF. [12], OR [2] FOR THE CASE $d > 2k(k-1)$). *Let $C \subseteq \mathcal{Q}^3$ be an integral curve of degree d and geometric genus g. Assume that C is contained in a surface of $\mathcal{Q}^3$ of degree $2k$. Then*

$$g - 1 \leq \frac{d^2}{4k} + \frac{1}{2}(k-3)d.$$

PROPOSITION 1.6 (CF. [2], PROPOSITION 6.4). *Let C be an integral curve in $\mathcal{Q}^3$, not contained in any surface of $\mathcal{Q}^3$ of degree strictly less than $2k$. Then*

$$g - 1 \leq \frac{d^2}{2k} + \frac{1}{2}(k-4)d.$$

Let S be a non-singular surface in $\mathcal{Q}^4$, $\mathcal{N}$ its normal bundle, σ its postulation, C a non-singular hyperplane section of S, g its genus, d its degree. Let s be a positive integer, $V_s \in |\mathcal{I}_{S,\mathcal{Q}^4}(s)|$ be integral and $\mu_l := c_2\big(\mathcal{N}(-l)\big) = (1/2)d^2 + l(l-3)d - 2l(g-1)$, $\forall l \in \mathbb{Z}$.

LEMMA 1.7. *In the above situation:* $0 \leq \mu_s \leq s^2 d$.

PROOF. The left hand side inequality is just Proposition 1.5 above. To prove the right hand side we first assume $s = \sigma$. Using [2], Lemma 6.8 we conclude (from here on, the hypothesis $d > 2\sigma^2$ was not used there) in the case at hand. Now, for the general case, let $s = \sigma + t$, where t is a non-negative integer. Then, as it is easily checked, $\mu_s = \mu_\sigma + \sigma t d + t(\sigma + t - 3)d - 2t(g-1)$. We conclude by what was proved for μ_σ and by the obvious $g \geq 0$. ∎

2. **The structure of the reduction morphisms.** In this section we give, by a systematic use of the double-point formulæ, a precise description of the first two reduction morphisms of Adjunction Theory associated with codimension two subvarieties of quadrics Q^n, $n \geq 5$. We apply these formulæ also to the case of divisorial contractions of extremal rays on threefolds in Q^5.

From now on we shall make free use of the language of Adjunction Theory; we shall give a reference, almost never the original one, for the result used in the sequel.

Let $\nu := n - 2$.

LEMMA 2.1. *Let X be a codimension two non-singular subvariety of Q^n, $n \geq 5$. Let D be a divisor on X with $\big(D, O_D(D)\big) \simeq \big(\mathbb{P}^{\nu-1}, O_{\mathbb{P}^{\nu-1}}(-1)\big)$ and $(K_X + (\nu-1)L)_{|D} \simeq O_D$; then $n = 5, 6$ and $d = 10$.*
Let $n = 5$. Then we have the following list of possible degrees according to whether X contains a divisor of the given form $\big(D, O_D(D)\big)$ with $\big(K_X + (\nu-2)L\big)_{|D} \simeq O_D$:

(2.1.1) if $\big(D, O_D(D)\big) \simeq \big(\mathbb{P}^2, O_{\mathbb{P}^2}(-2)\big)$, then $d = 20$;
(2.1.2) if $\big(D, O_D(D)\big) \simeq \big(\mathbb{P}^2, O_{\mathbb{P}^2}(-1)\big)$, then $d = 14$;
(2.1.3) if $\big(D, O_D(D)\big) \simeq (\tilde{\mathbb{F}}_2, G)$, where $2G = K_D$, then $d = 14$;
(2.1.4) $\big(D, O_D(D)\big) \simeq (\mathbb{F}_0, G)$, where $2G = K_D$, then $d = 14$;
(2.1.5) the case in which D has two components as in [5], Theorem 0.2.1, case b5), cannot occur;
(2.1.6) the case $\big(D, O_D(D)\big) \simeq (\mathbb{F}_1, -E - f)$ cannot occur.
(2.1.7) the cases in which D is as in either a), or b) of [8] Theorem 2.3 cannot occur.

Let $n = 6$. Assume X contains a surface S such that $S \simeq \mathbb{P}^2$, $L_{|S} \simeq O_{\mathbb{P}^2}(1)$ and such that the normal bundle $N_{S,X} \simeq T^*_{\mathbb{P}^2}(1)$. Then $d = 14$.

PROOF. For $n = 5$ the proof is the same as the one of [6], Proposition 1.1, using (1) in the place of (0.8) of the quoted paper. For $n = 6$ we compute all the relevant Chern classes by using (1), the Euler sequence for $S \simeq \mathbb{P}^2$ and the exact sequence:

$$0 \longrightarrow T_S \longrightarrow T_{X|S} \longrightarrow N_{S,X} \longrightarrow 0. \qquad \blacksquare$$

The following remark will be used several times in the sequel of this paper.

REMARK 2.2. Let X be a non-singular codimension two subvariety of Q^n. As a consequence of the Barth-Larsen Theorem (see [3]), we have that: if $n \geq 6$, then the fundamental group $\pi_1(X)$ is trivial; if $n \geq 7$, then $\mathrm{Pic}(X) \simeq \mathbb{Z}$, generated by the hyperplane bundle, so that any projective morphism $f \colon X \longrightarrow Y$ with connected fibers onto a normal variety Y is either trivial or an isomorphism.

THEOREM 2.3 (STRUCTURE OF THE REDUCTION MORPHISMS). *Let X be a non-singular codimension two subvariety of Q^n, $n \geq 5$.*
Assume that (X, L) admits a first reduction (X', L'). Then the first reduction morphism is an isomorphism: $(X, L) \simeq (X', L')$.
Assume that (X, L) admits, in addition, a second reduction (X'', L''). We have:

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if $n = 5$ *and* $d \neq 14, 20$*, then* $(X, L) = (X', L')$ *and the second reduction map* $\varphi: X' \longrightarrow X''$ *is the blowing up on a non-singular* X'' *of a disjoint union of non-singular integral curves;*

if $n = 6$ *and* $d \neq 14$*, then* $(X, L) = (X', L')$ *and the second reduction map* $\varphi: X' \longrightarrow X''$ *is the blowing up on a non-singular* X'' *of a disjoint union of non-singular integral curves. If in addition* $d \neq 16, 22$*, then the second reduction morphism is an isomorphism:* $(X, L) \simeq (X', L') \simeq (X'', L'')$*;*

if $n \geq 7$*, then* $(X, L) \simeq (X', L') \simeq (X'', L'')$.

PROOF. Since (X, L) admits a first reduction, $K_X + (n-1)L$ is nef and big (*i.e.*, out of the lists of Theorems 3.1 and 3.2 below). Hence $K_X + (n-1)L$ fails to be ample only if the first reduction is not an isomorphism; in turn, that happens if and only if X contains some exceptional divisors of the first kind. By Lemma 2.1 this happens only if $d = 10$. By Proposition 1.3 the type is either M) or N); neither of them contains an exceptional divisor of the first kind. It follows that if the first reduction exists, then $(X, L) \simeq (X', L')$.
The statements concerning the second reduction morphism can be proved as follows. For $n = 5$, we use Theorem 0.2.1 of [5] coupled with Lemma 2.1.
For $n = 6$ we use Theorem 0.2.2 of [5] and then we take a general hyperplane section and reduce to the case $n = 5$, with the difference that now case b2) of Theorem 0.2.1 of [5] does not occur. The case of the blowing up of curves yields $d = 16, 22$, as we now show. Since $X \simeq X'$ we cut (1) with $F \simeq \mathbb{P}^2$, a general fiber of the blowing up. Define a to be the positive integer such that $L_{|F} \simeq \mathcal{O}_{\mathbb{P}^2}(a)$. Since $N_{F,X} \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ and $K_{X|F} \simeq \mathcal{O}_{\mathbb{P}^2}(-2)$ we get

$$(16 - d/2)a^2 = 12a - 4.$$

Since $a > 0$ we see that $d \leq 30$. The only integer solutions to the relation above are $(d, a) = (16, 1)$ and $(22, 2)$. This concludes the proof for $n = 6$.
Finally, for $n \geq 7$ we use Remark 2.2. ∎

Lemma 2.1 can also be used to describe Mori contractions for threefolds in Q^5. See [6], Corollary 1.2 for the analogous result on $\mathbb{P}^5$.

PROPOSITION 2.4. *Let* X *be a non-singular threefold in* Q^5*. Let* D *be an integral divisor on* X*. We have:*

(2.4.1) *if* $\big(D, \mathcal{O}_D(D)\big) \simeq \big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1)\big)$*, then either* $d = 10$ *and* $L_{|D} \simeq \mathcal{O}_{\mathbb{P}^2}(1)$*, or* $d = 14$ *and* $L_{|D} \simeq \mathcal{O}_{\mathbb{P}^2}(2)$*;*

(2.4.2) *if* $\big(D, \mathcal{O}_D(D)\big) \simeq \big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-2)\big)$*, then either* $d = 8$ *and* $L_{|D} \simeq \mathcal{O}_{\mathbb{P}^2}(1)$*, or* $d = 16$ *and* $L_{|D} \simeq \mathcal{O}_{\mathbb{P}^2}(2)$*;*

(2.4.3) *if* $\big(D, \mathcal{O}_D(D)\big) \simeq (\mathbb{F}_0, G)$*, then* $d \leq 20$*;*

(2.4.4) *if* $\big(D, \mathcal{O}_D(D)\big) \simeq (\tilde{\mathbb{F}}_2, G)$*, then* $d = 14$ *and* $L_D = -G$*.*

PROOF. The proof is the same as that of [6], Proposition 1.1, using (1) in the place of (0.8) of the quoted paper. ∎

PROPOSITION 2.5 (STRUCTURE OF MORI CONTRACTIONS). *Let X be a non-singular threefold embedded in Q^5 with $d \geq 22$ and K_X not nef. Let $\rho\colon X \longrightarrow Y$ be the contraction of any extremal ray on X. Then Y is non-singular and either ρ is birational and the blowing up of an integral non-singular curve on Y or ρ is a conic bundle in the sense of Mori Theory. In particular, if $d \gg 0$, then only the former case can occur.*

PROOF. The proof is the same as the one of [6], Corollary 1.2, using (1) in the place of (0.8) of the quoted paper. As for the last statement, if $\dim Y \leq 2$, then X is not of general type. We refer to the result of our paper [13] that there are only finitely many components of the Hilbert scheme corresponding to non-singular $(n-2)$-folds not of general type. ■

The following conjecture is due, in the case of 3-folds in $\mathbb{P}^5$, to Beltrametti, Schneider and Sommese. The idea is that, by virtue of Proposition 2.5, the failure of being a minimal model is detected, for $d \gg 0$, by the presence of special $\mathbb{P}^1$-bundles contained in X. Pushing the methods of Adjunction Theory, it may be possible to show that the degrees of these $\mathbb{P}^1$-bundles are bounded from above and this may, in turn, imply that the degrees of threefolds X in Q^5 which are not minimal models are bounded from above.

CONJECTURE 2.6. *There is an integer d_0 such that every non-singular threefold in Q^5 of degree $d \geq d_0$ is a minimal model.*

3. **Varieties not of log-general-type.** In this section we give a coarse classification of varieties as in the title. We still make free use of the language of Adjunction Theory.

Let $\nu := n-2$ and (X, L) be a degree d, ν-dimensional non-singular subvariety of Q^n endowed with its embedding line bundle L. The "Types" we shall consider correspond to the ones of Propositions 1.3 and 1.4.

We start by observing that $K_X + (\nu - 1)L$ is spanned by its global sections (spanned for short) except for three special pairs.

THEOREM 3.1. *Let (X, L) be as above. Then $K_X + (\nu - 1)L$ is spanned unless (X, L) is one of the three pairs* A), C) *or* D). *In particular, $d \leq 4$.*

PROOF. By the list on [8] page 381, and by the fact that there are no codimension two linear subspaces in Q^n, $\forall n \geq 5$, we need to analyze the a. t. scroll over a curve case only. By flatness an a. t. scroll over a curve is a scroll. The result follows from Theorem 1.4. ■

Now we classify those pairs for which $K_X + (\nu - 1)L$ is spanned, but for which $\kappa\big(K_X + (\nu - 1)L\big) < \nu$.

THEOREM 3.2. *Let (X, L) be as above. Assume that $K_X + (\nu - 1)L$ is spanned, i.e., (X, L) is not as in Theorem 3.1, but that it is not big. Then (X, L) is one of the following pairs:*

(3.2.1) (Del Pezzo variety): Type B); Type F);

(3.2.2) (Quadric Bundle over a curve): Type G);

(3.2.3) (A. t. scroll over a surface): Type L); Type O).

In particular, $d \leq 12$.

PROOF. Let $K_X + (\nu - 1)L$ be as in the theorem. Then, by [8], page 381, (X, L) is either a Del Pezzo variety, a quadric bundle or an a. t. scroll over a surface.

Let us assume that (X, L) is a Del Pezzo variety. By slicing with $(\dim X - 2)$ general hyperplanes we get a surface in $\mathcal{Q}^4$ with $K_S = -L_{|S}$. Since S is Del Pezzo we get $\chi(\mathcal{O}_S) = g(L) = 1$. We plug these values in (3) and get:

$$d^2 - 10d + 24 = 0.$$

It follows that either $d = 4$ or $d = 6$. The conclusion follows from Proposition 1.3.
Let us assume that (X, L) is a quadric bundle. Let $F \simeq \mathcal{Q}^{n-3}$ be a general fiber of the quadric fibration. Dotting (1) with F we get $d = 6$. We conclude using Proposition 1.3.
Let us assume that (X, L) is an a. t. scroll over a surface. By [7], Proposition 14.1.3 (X, L) is an ordinary scroll with $\kappa\big(K_X + (n-1)L\big) = 2$. We conclude by comparing with Proposition 1.4. ∎

Now we deal with the line bundle $K_X + (\nu - 2)L$. First we exclude the presence of some special pairs.

LEMMA 3.3. *Let* (X, L) *be as above. Then* (X, L) *cannot be isomorphic to any of the three pairs* $\big(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)\big)$, $\big(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)\big)$ *and* $\big(\mathcal{Q}^3, \mathcal{O}_{\mathcal{Q}^3}(2)\big)$. *Moreover, there are no Veronese bundles* (X', L') *associated with a pair* (X, L) *in* $\mathcal{Q}^5$.

PROOF. By contradiction assume that $(X, L) \simeq \big(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)\big)$. We intersect two general members of $|L|$ and get a non-singular surface section $(S, L_{|S})$ which is embedded in $\mathcal{Q}^4$ with $d = 16$, $g = 1$ and $\chi(\mathcal{O}_S) = 1$. This contradicts (3). We exclude the case in which $(X, L) \simeq \big(\mathcal{Q}^3, \mathcal{O}_{\mathcal{Q}^3}(2)\big)$ in a similar way.
The possibility $(X, L) \simeq \big(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)\big)$ is ruled out by Remark 1.2.
Let us assume that (X, L) is a pair for which (X', L') exists and is a Veronese bundle with associated morphism $p: X \longrightarrow Y$; in particular $n = 5$. By Theorem 2.3 $(X, L) \simeq (X', L')$. Dotting (1) with a general fiber F we get $d = 10$. Since for some ample line bundle $\mathcal{L}$ on Y $2K_X + 3L = p^*\mathcal{L}$, we have the following relation on a general surface section S of X:

$$L_{|S} = -2K_S + \mathcal{L}_{|S},$$

which "squared" gives $d = 10 \equiv 0 \bmod(4)$, a contradiction. ∎

THEOREM 3.4. *Assume that we are out of the lists of Theorems 3.1 and 3.2 so that* $(X, L) \simeq (X', L')$. *If* $K_X + (\nu - 2)L$ *is not nef and big then* (X, L) *is one of the following pairs:*

(3.4.1) (Mukai variety): Type E); Type I);

(3.4.2) (Del Pezzo fibration over a curve): either Type N), $d = 10$ *or as in (4.5.1),* $d = 12$*;*

(3.4.3) (Quadric bundle over a surface): $n = 5, 6$, *a flat quadric bundle over a non-singular surface: if* $n = 6$, *then* $d = 12$ *and if* $n = 5$, *then either* $d \leq 18$ *or* $d = 44$.

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(3.4.4) *(A. t. scroll over a threefold): $n = 6$, the scroll map is not flat and d is either* 14 *or* 20.

PROOF. Let $K_X + (\nu - 1)L$ be as in the hypothesis. Then by [8] page 381–2 and Lemma 3.3, (X, L) is either a Mukai variety, a Del Pezzo fibration over a curve, a quadric bundle over a surface or an a. t. scroll of dimension $\nu \geq 4$ over a normal threefold.

Let us assume that (X, L) is a Mukai variety. By slicing to a surface section S we find that $K_S = \mathcal{O}_S$ and, since X is simply connected, it follows that $\pi_1(S)$ is trivial as well; S is thus a $K3$ surface. Using (3) we get, using $\chi(\mathcal{O}_S) = 2$ and $2(g - 1) = d$, that either $d = 6$ or $d = 8$; accordingly $g = 4, 5$, respectively. The conclusion, in this case, follows from Proposition 1.3.

We deal with the case of Del Pezzo fibrations over a curve in Lemma 4.1 and Proposition 4.5.

We now deal with quadric bundles over surfaces. Again, $n = 5, 6$, by Remark 2.2.

Let $n = 5$ and assume, by contradiction, that there is a divisorial fiber F of the quadric bundle map $p: X \longrightarrow Y$. Then F is as in [8], Theorem 2.3. This contradicts case (2.1.7) of Lemma 2.1. It follows that all the fibers of p are equi-dimensional. By Lemma 5.6 it follows that p is a quadric fibration in the sense of Section 5. The statement follows from Proposition 5.4 and Remark 5.5.

Let $n = 6$. Since (X, L) is a quadric bundle over a surface, $p: X \longrightarrow Y$, so is its general hyperplane section. By what was proved for the case $n = 5$, the base surface Y is non-singular and by Corollary 5.7 we deduce that p is flat. If we cut (1) with a general fiber of p we get $d = 12$. Case (3.4.3) follows.

Finally Case (3.4.4) follows from Proposition 1.4 which ensures us of the absence, in $\mathcal{Q}^6$, of adjunction theoretic scrolls over threefolds for which the map p is flat: for if p were flat then Y would be non-singular by [21] Theorem 23.7 and then X would be a projective bundle, a contradiction. If one of these scrolls occurs, since p is not flat and $-K_X$ is p-ample, Lemma 5.6 and [21], Theorem 23. 1 ensures there must be a fiber F such that either F contains a divisor or, by [7], 14.1.4, F is a surface $\mathcal{S}$ as in Lemma 2.1. In the latter case we get $d = 14$. In the former, by slicing with a general hyperplane section, we get a threefold $\tilde{X}$ together with the morphism $\tilde{p} := p_{|\tilde{X}}: \tilde{X} \longrightarrow Y$, where Y is the base of the scroll. $\tilde{p}$ is the second reduction morphism for $(\tilde{X}, L_{|\tilde{X}})$. In particular, by slicing the fiber F with the same general hyperplane section, we obtain a scheme $F' \subseteq \tilde{X}$ which contains a divisor D contracted by $\tilde{p}$ to a point. This divisor D fits the assumptions of Lemma 2.1, by virtue of the structure theorem of the second reduction morphism (*cf.* [5], Theorem 0.2.1). This concludes the proof. ∎

4. **Fibrations over curves with general fiber a Del Pezzo manifold.** In order to prove Theorem 3.4, we need to analyze adjunction-theoretic Del Pezzo fibrations over curves. In this section we study a class of fibrations which includes the ones above. The main result is Proposition 4.5.

We now study codimension two non-singular subvarieties of $\mathcal{Q}^n$, $n \geq 5$, which admit a morphism $f: X \longrightarrow Y$, with connected fibers, onto a non-singular curve Y, such that the line bundle $K_X + (n-4)L$ is trivial on the general fiber. The general fiber will thus be a non-singular (adjunction-theoretic) Del Pezzo variety of the appropriate dimension $n-3$. By Remark 2.2 we have $n = 5, 6$.

A priori, not all such varieties are adjunction-theoretic Del Pezzo fibrations over a curve. We study these *a priori* more general objects for completeness. Section 4.1 contains some upper bounds for the degree of other special classes.

The following lemma ensures that these fibrations coincide with the Del Pezzo fibrations over curves of Adjunction Theory. Let S be a surface section of X.

LEMMA 4.1. *Let X be a fibration as above. Then $K_X + (n-1)L$ is ample and* $\kappa\big(K_X + (n-2)L\big) = \kappa(S) = 1$.

PROOF. By the above discussion, either $n = 5$ or $n = 6$. Without loss of generality we may assume that $n = 5$, for otherwise we could consider a general three-dimensional hyperplane section of X and it is easy to show that if the statements we want to prove hold for the threefold hyperplane section of X, then they also hold for X.

The generic fiber of f is a non-singular Del Pezzo surface F. Since $K_X + L$ is trivial on the fibers we define

$$\Delta := L^2 \cdot F = L^2_{|F} = K^2_F.$$

Cut (1) with F, using the facts that $K_{X|F} = K_F$ and that $x_2 \cdot F = 12 - \Delta$. We get

$$\Delta = \frac{24}{16-d}.$$

Since F is a Del Pezzo surface and L is very ample, we get $3 \leq \Delta \leq 9$. Since Δ is an integer we have only the following possibilities:

$$(\Delta, d) = (3, 8),\ (4, 10),\ (6, 12). \tag{7}$$

Using the above invariants, and the lists of Adjunction Theory, it is easy to show that $K_X + (n-1)L$ is ample and that $\kappa\big(K_X + (n-2)L\big) = 0, 1$. By Theorem 3.4 the case $K_X = -(n-2)L$ cannot occur, since these manifolds do not carry any nontrivial fibration. It follows that $K_X + 2L$ is ample, $\kappa(K_X + L) = 1$ and, by adjunction, $\kappa(S) = 1$. ∎

We need the following facts.

FACT 4.2. Let $f : X \longrightarrow Y$ be as above. By relative vanishing we have $h^i(\mathcal{O}_X) = h^i(\mathcal{O}_Y)$, $\forall i$.

FACT 4.3. $g(Y) = q(S)$, $2g - 2 - d = \big(p_g(S) + q(S) - 1\big)\Delta$; *moreover the elliptic fibration $S \longrightarrow Y$ has no multiple fibers*.

The assertion about $g(Y) = q(S)$ follows from the Lefschetz Theorem on hyperplane sections, $q(S) = h^1(\mathcal{O}_X)$, and from Fact 4.2; the other assertion follows from [24], 0.5.1.

FACT 4.4. $S \not\subseteq \mathbb{P}^4$.

To prove this, assume that $S \subseteq \mathbb{P}^4$. We use jointly the double-point formula for surfaces in $\mathbb{P}^4$, see [19], page 434, and (3) to compute the values of g and $\chi\big(\mathcal{O}(S)\big)$ to conclude that, $d = 8, 10$ would yield non-integer values, a contradiction, and that if $d = 12$ then $g = 25$, and $\chi(\mathcal{O}_S) = 13$; this system of invariants is inconsistent by Fact 4.3. This proves the assertion.

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PROPOSITION 4.5. *Let $X \subseteq Q^n$, $n \geq 5$, be a non-singular, codimension two subvariety which admits a fibration $f: X \longrightarrow Y$ in Del Pezzo manifolds onto a non-singular curve Y; in particular $\big(K_X + (\dim X - 2)L\big)_{|F} \simeq O_F$, F a general fiber.*
Then $Y \simeq \mathbb{P}^1$ and either (X, L) is of Type N) of Proposition 1.4 or only the following systems of invariants is possible:

(4.5.1) $n = 5$, $d = 12$, $K_F^2 = 6$, $g = 10$, $p_g(S) = 2$, $q(S) = 0$, $h^i(O_X) = 0$, $\forall i > 0$.

PROOF. By the proof of Lemma 4.1 and by the information of the cases of degree $d = 8, 10$ varieties stemming from Proposition 1.3, we only need to rule out the case $d = 8$ and make precise the invariants in the case $d = 12$. Moreover, by the same lemma, $\kappa(S) = 1$.
First let $n = 5$.
Now we determine the invariants in the case $d = 12$.
We apply formula (3) in the case $d = 12$. We get

$$2(g-1) - 3\chi(O_S) = 9. \tag{8}$$

By Fact 4.4 and by [13], Proposition 1.4 we are in the position to apply the Castelnuovo bound for curves in $\mathbb{P}^4$, which gives $g \leq 13$.
(8) implies that $\chi(O_S)$ is not a non-negative integer, unless $\big(g, \chi(O_S)\big) = (7, 1)$, $(10, 3)$, $(13, 5)$. We can rule out the cases: $d = 12$ and $\big(g, \chi(O_S)\big) = (7, 1)$, $(13, 5)$ using Fact 4.3 which gives $g - 7 = 3(p_g + q - 1)$; this last equality together with the given values of $\chi(O_S)$ and g gives a non-integer value for q, a contradiction. It follows that if $d = 12$, then $\big(g, \chi(O_S)\big) = (10, 3)$. To compute the values of p_g and q we use again Fact 4.3 which gives the number $p_g + q$. Since we know $\chi(O_S)$ we get the values of p_g and q.
Since $g = q$ we see that $Y \simeq \mathbb{P}^1$. The assertions about $h^i(O_X)$ follow from Fact 4.2.
The proposition is thus proved for $n = 5$.
Let $n = 6$, the only remaining case, by virtue of the Barth-Larsen theorem. By slicing with a general hyperplane we get a threefold with a fibration onto a curve whose general fiber is a Del Pezzo manifold so that the above analysis applies. The only difference is that the case $d = 10$ does not occur by Proposition 1.3.
Now we prove that the case $d = 12$ also does not occur.
The general fiber of f is a Del Pezzo threefold with $K_F = -2L_{|F}$ and $L_{|F}^3 = 6$. By explicit classification, see [16], page 72, either $F \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ or $F \simeq \mathbb{P}(T_{\mathbb{P}^2})$. In both cases formula (2) dotted with F gives $x_3 \cdot F = x_3(F) = 24$. But in the former case $x_3(F) = 8$, in the latter $x_3(F) = 6$. ∎

4.1 *More upper bonds.* This section is not needed for Theorem 3.4.

We now prove an upper bound for the degree of codimension two, non-singular subvarieties of Q^n, $n \geq 5$, which admit a morphism onto a curve such that the general fiber is a Fano variety. Again, by virtue of Remark 2.2, we need to worry only about the cases $n = 5, 6$.

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PROPOSITION 4.6. *Let $X \subseteq Q^n$ a non-singular subvariety of codimension two and degree d which admits a morphism onto a curve such that the general fiber is a Fano variety.*

If $n = 5$, then $d \leq 20$.
If $n = 6$, then $d \leq 30$.

PROOF. Let $n = 5$ and $\mathcal{L} := L_{|F}$. Assume that $d \geq 22$.
We cut (1) with a fiber, F, and obtain, on F:

$$(11 - d/2)\mathcal{L}^2 + 5K_F\mathcal{L} + K_F^2 - c_2(F) = 0.$$

Since $c_2(F) = 12 - K_F^2$, we get:

$$(d/2 - 11)\mathcal{L}^2 + 2K_F^2 - 12 + 5K_F\mathcal{L} = 0. \tag{9}$$

Now we use $K_F^2 \leq 9$ to get

$$(d/2 - 11)\mathcal{L}^2 \leq 6 + 5K_F\mathcal{L}. \tag{10}$$

Since $K_F\mathcal{L} \leq -1$, we see that either $d = 22$, or $d = 24$ and $\mathcal{L}^2 = -K_F\mathcal{L} = 1$. In the latter case $F \simeq \mathbb{P}^2$ and the Hodge Index Theorem, applied to the surface F, says that $K_{\mathbb{P}^2}^2 = 1$, a contradiction. In the former case we use (9):

$$2K_F^2 - 12 + 5K_F\mathcal{L} = 0,$$

which gives a contradiction for each value $K_F^2 = 1, \ldots, 9$. It follows that $d \leq 20$.
The proof of the statement for $n = 6$ is analogous to the proof of Proposition 4.7, where we use (1) with $n = 6$ cut with the cycle $K_X \cdot F$. ∎

In the same spirit we prove an upper bound on the degree of Fano threefolds in Q^5.

PROPOSITION 4.7. *Let $X \subseteq Q^5$ be a non-singular Fano threefold. Then $d \leq 20$.*

PROOF (CF. [6], COROLLARY 1.2). We cut (1) with K_X and get, using the fact that $x_1x_2 = 24\chi(\mathcal{O}_X) = 24$:

$$(11 - d/2)L^2K_X + 5LK_X^2 + K_X^3 + 24 = 0.$$

Let

$$\lambda := LK_X^2, \qquad 2\mu := -L^2K_X = -2g + 2 + 2d;$$

clearly λ and μ are positive integers and the above becomes:

$$(d - 22)\mu + 5\mu\lambda + 24 = -K_X^3. \tag{11}$$

By the Generalized Hodge Index Theorem, see [4], we get $(-K_X^3)(-K_XL^2) \leq (K_X^2L)^2$, or

$$(-K_X)^3(2\mu) \leq \lambda^2. \tag{12}$$

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By combining (11) and (12) we get

$$\lambda^2 - 10\mu\lambda - [2(d-22)\mu^2 + 48\mu] \geq 0. \tag{13}$$

If we solve the above in λ we get either $\lambda < 0$, a contradiction, or $\lambda > 10\mu$. This implies, in turn, that $\lambda \geq 11$. Since, by the classification of Fano threefolds, $-K_X^3 \leq 64$, (11) becomes

$$(d-22)\mu + 55 + 24 \leq 64,$$

a contradiction for $d \geq 22$. ■

5. **Quadric Fibrations.** In order to prove Theorem 3.4, we need to analyze adjunction-theoretic quadric bundles. In this section we study a class of fibrations which includes the ones above.

The main results are Proposition 5.4 and Remark 5.5.

The term *quadric bundle* is to be intended in the sense of Adjunction Theory. The term *quadric fibration* is introduced below. *A priori*, not all quadric fibrations are quadric bundles. We study these, *a priori*, more general objects for completeness. Section 5.1 ensures us that a quadric fibration with one-dimensional fibers is a conic bundle with a non-singular base.

By *quadric fibration* we mean a non-singular projective variety $X \subseteq \mathbb{P}$, of dimension x, together with a fibration $p: X \longrightarrow Y$ onto a (*a fortiori*) non-singular variety Y of positive dimension y, *all* of which fibers are quadrics, not necessarily integral, of the appropriate dimension $(x-y)$. One has non integral fibers only if the relative dimension is one.

The case $\dim Y = 0$ is trivial. By virtue of Remark 2.2 we have:

FACT 5.1. There are no codimension two quadric fibrations in $\mathcal{Q}^n$, for $n \geq 7$ and, for $n = 6$, any such is simply connected.

We restrict ourselves to the case of $n \geq 5$.

We begin by fixing some notation and establishing some simple facts.

Let L denote the restriction to X of the hyperplane bundle. The sheaf $\mathcal{E} := p_* L$ is locally free on Y of rank $(x-y+2)$. It is easy to check that $\mathcal{E}$ is generated by its global sections. The surjection $p^* p_*: L \longrightarrow L$ defines an embedding $: X \hookrightarrow \mathbb{P}(\mathcal{E})$, where $L = \xi_{\mathcal{E}|X}$ and X is defined by a nonzero section of the line bundle $2\xi - \pi^* \mathcal{M}$, for some $\mathcal{M} \in \mathrm{Pic}(Y)$, where $\pi: \mathbb{P}(\mathcal{E}) \longrightarrow Y$ is the bundle projection.

The following gives a sufficient condition for a general hyperplane section of X to be a quadric fibration over Y. It is a well known "counting dimensions" argument.

LEMMA 5.2. *Let $X \longrightarrow Y$ be a quadric fibration as above. Assume $2y < x+2$. Then a general hyperplane section X' of X is a quadric fibration over Y via $p_{|X'}: X' \longrightarrow Y$.*

PROOF. Since $\mathcal{E}$ is generated by global sections and, by assumption rank$(\mathcal{E}) > y$, a general section of it does not vanish on Y. Such a section will define, *for every* $z \in Y$, a hyperplane Λ_z of the corresponding fiber $\pi^{-1}(z) \subseteq \mathbb{P}(\mathcal{E})$. In the case in which the quadrics $p^{-1}(z)$ were integral $\forall z \in Y$, we would be done. This is, in general, not true. However, the singular quadrics of the fibration are parameterized by a proper closed subset D of Y with $\dim D \leq (y-1)$. The hyperplanes of $\mathbb{P}$ which contain the reduced part, $\Sigma \simeq \mathbb{P}^{x-y}$, of one of the components of one non-integral quadric of the fibration form a linear space of dimension $(\dim \mathbb{P} - x + y - 1)$ contained in $\mathbb{P}^{\vee}$. The space of these bad hyperplanes is of dimension at most $(\dim D + \dim \mathbb{P} - x + y - 1) \leq \dim \mathbb{P} - x + 2y - 2 < \dim \mathbb{P}^{\vee}$. It follows that the general section of E gives a hyperplane section of X which cuts *every* quadric of the fibration in a quadric of dimension one less. ∎

PROPOSITION 5.3. *There are no quadric fibrations over curves in Q^6. The only quadric fibrations over curves in Q^5 are of Type G) of Proposition 1.4. If there is a quadric fibration over a surface in Q^6, then it has degree $d = 12$.*

PROOF. As to quadric fibrations over curves, we cut (1) with a non-singular fiber $F \simeq Q^{n-3}$, we get $d = 6$. We conclude by comparing with Proposition 1.3.
As to quadric fibrations over a surface we cut (1) with a non-singular fiber $F \simeq Q^{n-4}$ and get $d = 12$. ∎

The following proposition and remark describe our knowledge of the situation for threefolds in Q^5 which quadric bundles over surfaces.

PROPOSITION 5.4. *Let $X \subseteq Q^5$ be a threefold quadric fibration (*conic bundle*) over a surface Y. Then either $d \leq 98$ or X is contained in a hypersurface $V \in |\mathcal{O}_{Q^5}(3)|$ and $d \leq 276$.*

PROOF. We denote the Chern classes of X and Y by x_i and b_i, respectively. We omit the symbol "p^*" for ease of notation. We follow closely the paper [10]. First we introduce the following entities and we report from [10], for the reader's convenience, the relations among them which are essential to the computations below (one warning: some of the equalities are only numerical equalities):

$\mathcal{M}$ was defined at the beginning of the section;

$D \in |2e_1 - 3\mathcal{M}|$, it is called the *discriminant divisor*; its points correspond to the singular fibers of p;

$2R \subseteq Y$ the *branching divisor* associated with a general hyperplane section, S, of X, which, in view of Lemma 5.2, is a cyclic double cover of Y;

$e_1 = 3R - D$;

$\mathcal{M} = 2R - D$;

$x_1 = L + b_1 - R$;

$x_2 = L^2 + L \cdot (b_1 - 2R + D) + (-2R^2 - R \cdot b_1 + D \cdot R + b_2 + e_2)$;

$x_3 = 2b_2 - D^2 + Db_1$;

$L \cdot W \cdot W' = 2W \cdot W'$, for every pair of divisors W and W' on Y;

$L^2 \cdot W = (4R - D) \cdot W$;

$e_2 = \frac{1}{2}(12R^2 + D^2 - 7DR - d)$.

Now we plug in the above values of x_1 and x_2 for x_1 and x_2 in (1):

$$(14) \qquad (6 - \frac{d}{2})L^2 - 4Lb_1 + 5LR + b_1^2 - b_1R - LD + 3R^2 - DR - b_2 - e_2 = 0.$$

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Next we equate the expression above for x_3 to the one of (2), using again the above expressions for x_1 and x_2:

$$(15)\quad -(2d+10)b_1R+2dR^2+(\frac{d}{2}+4)Db_1+D^2-10b_2+2b_1^2-(\frac{d}{2}+5)DR-d(\frac{d}{2}-13)=0.$$

Now we set

$$x := b_1^2 \quad \text{and} \quad y := DR,$$

we cut (14) with R, $-b_1$, D and L, respectively, so that we obtain four linear equations to which we add (15), after having substituted in x and y. The result is the following linear system of equations:

$$(16)\qquad Mv^t = c^t,$$

where

$$M := \begin{pmatrix} -8 & 34-2d & 0 & 0 & 0 \\ -2d-34 & 0 & -\frac{d}{2}+8 & 0 & 0 \\ 0 & 0 & -8 & \frac{d}{2}-8 & 0 \\ -18 & 14 & +4 & 0 & -2 \\ -2d-10 & 2d & \frac{d}{2}+4 & 1 & -10 \end{pmatrix},$$

$$v := (\, b_1R,\ R^2,\ Db_1,\ D^2,\ b_2 \,)$$

and

$$c := \Big((8-\frac{d}{2})y,\ -8x,\ (2d-34)y,\ 2x+4y+d(\frac{d}{2}-7),\ -2x+(\frac{d}{2}+5)y+d(\frac{d}{2}-13)\Big).$$

Since $P := -\frac{1}{2}\det M = 3d^3-27d^2-1520d+18976 > 0$, $\forall d > 0$, we can solve the above system (16) and obtain the unique solution:

$$\begin{aligned} b_1R &= -\frac{1}{2}[(-128d^2+4480d-39168)x+(2d^3-111d^2+2020d-12096)y \\ &\quad +(2d^5-120d^4+2678d^3-26304d^2+95744d)]/P, \\ R^2 &= \frac{1}{4}[(-1024d+18432)x+(3d^3-8d^2-2112d+23552)y \\ &\quad +(16d^4-688d^3+9728d^2-45056d)]/P, \\ b_1D &= -2[(-152d^2+4440d-32128)x+(2d^3-113d^2+2099d-12852)y \\ &\quad +(2d^5-122d^4+2766d^3-27574d^2+101728d)]/P, \\ D^2 &= -4[(-1216d+16064)x+(-3d^3+46d^2+893d-13736)y \\ &\quad +(16d^4-720d^3+10608d^2-50864d)]/P, \\ b_2 &= \frac{1}{4}[(12d^3+20d^2-3648d+13952)x+(d^3-30d^2+152d+960)y \\ &\quad +(d^5-27d^4+274d^3-4448d^2+46016d)]/P. \end{aligned}$$

Since E is generated by global sections and D is effective we see that $e_2 \geq 0$, $e_1 D \geq 0$. Also, [10], Lemma 2.9 gives $y = DR \geq 0$. We can make explicit e_2 and e_1 by the formulæ given at the beginning of this proof and deduce:

$$DR = y \geq 0,$$

$$e_2 \cdot P = (896d - 4480)x - (\frac{19}{2}d^2 - 366d + 3616)y - (\frac{19}{2}d^4 - \frac{843}{2}d^3 + 5864d^2 - 24656d) \geq 0,$$

$$\begin{aligned} e_1 D \cdot P = &-(4864d - 64256)x - (3d^3 - 103d^2 + 988d - 1984)y \\ &+ (64d^4 - 2880d^3 + 42432d^2 - 203456d) \geq 0. \end{aligned}$$

These three inequalities define a region of the plane (x, y). It is straightforward to check that the two lines $e_2 = 0$ and $e_1 D = 0$ have slopes a and b whose signs do not change with d if $d \geq 20$. One can check easily that $a > 0$ and $b < 0$. The intersection of the first line above with the x-axis is

$$(x_1, 0)_{e_2} = \Big(\frac{(19/2)d^4 - (843/2)d^3 + 5864d^2 - 24656d}{896d - 4480}, 0 \Big);$$

the intersection of the second line with the x-axis is

$$(x_2, 0)_{e_1 D} = \Big(\frac{64d^4 - 2880d^3 + 42432d^2 - 203456d}{4864d - 64256}, 0 \Big).$$

One can check, that, since $d \geq 20$, $x_1 < x_2$. The region we are interested in is a triangle with vertices $(x_1, 0)_{e_2}$, $(x_2, 0)_{e_1 D}$ and $(x_3, y_3)_{(e_2 = 0) \cap (e_1 D = 0)}$.
Now we compute the genus of a general curve section, C, of X. By adjunction $x_1 \cdot L^2 = 2d + 2 - 2g$, so that by what above:

$$\begin{aligned} g - 1 &= \frac{d}{2} - 2b_1 R + \frac{Db_1}{2} + 2R^2 - \frac{DR}{2} \\ &= -2b_1 R + \frac{Db_1}{2} + 2R^2 - \frac{y}{2} + \frac{d}{2} \\ &= \Big[(24d^2 - 472d + 2176)x + \big((23/2)d^2 - 375d + 3044 \big) y \\ &\quad + \big((23/2)d^4 - (891/2)d^3 + 5374d^2 - 19024d \big) \Big] / P. \end{aligned}$$

Again it is not difficult to check that the absolute value of the slope of the above line is bigger than $|b|$. It follows easily that the maximum possible value for $g - 1$ in our region is achieved at $(x_2, 0)_{e_1 D}$, while the minimum is at $(x_1, 0)_{e_2}$. We thus get

$$\frac{19d^3 - 187d^2 + 416d}{224d - 1120} \leq g - 1 \leq \frac{4d^3 - 77d^2 + 321d}{38d - 502}. \tag{17}$$

Assume that C is not contained in any surface of Q^3 of degree strictly less than $2 \cdot 11$. Then by (1.6) and by the left hand side inequality of (17), we get

$$\frac{19d^3 - 187d^2 + 416d}{224d - 1120} \leq \frac{d^2}{22} + \frac{7}{2}d,$$

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which, remembering that d is even and that we are assuming $d \geq 20$, implies $d \leq 98$.
Assume that C is contained in a surface of degree $2k$, with $k = 10, 9, \ldots, 3$. By Proposition 1.5 we infer:

$$\frac{19d^3 - 187d^2 + 416d}{224d - 1120} \leq \frac{d^2}{4k} + \frac{k-3}{2}d,$$

which implies, as above, that for $k = 10, 9, \ldots, 3$, $d \leq 64$, 58 54, 48, 44, 40, 40 and 276, respectively.
Finally, assume that C is contained in a surface of degree four or two. Using the right hand side inequality of (17) and Lemma 1.7 we get $d \leq 42$ and $d \leq 16$, respectively. Actually in the last case we get a contradiction, since we are assuming $d \geq 20$.
Finally if C is in a surface of degree six, then X is in a hypersurface of degree six in Q^5, provided, $d > 18$ (*cf.* [13], Proposition 1.4). ∎

REMARK 5.5. We have checked with a Maple routine what are the possible degrees of a threefold in Q^5 which is a quadric fibration over a surface. For $d \geq 20$, the results of this paper impose the following restrictions on the triples (d, x, y):

(1) $20 \leq d \leq 276$;
(2) for every fixed d as above (x, y) must belong to the triangle of the proof of Proposition 5.4;
(3) b_1R, R^2, b_1D, D^2, b_2, $g-1$, $\chi\big(\mathcal{O}(Y)\big)$ and $\chi\big(\mathcal{O}(S)\big)$ must be integers;
(4) $(g-1)$ must satisfy inequality (17) and the bound of Theorem 2.3 in [18];
(5) $\chi\big(\mathcal{O}(S)\big)$ must satisfy the two inequalities of Proposition 1.1;
(6) various inequalities stemming from the Hodge Index Theorem applied to Y as, for example, $(K_YR)^2 \geq K_Y^2R^2$;
(7) if $d > 98$ then $g - 1 \leq (1/12)d^2$, see Proposition 1.5;

The result is that the only possible degree, for $d \geq 20$ is $d = 44$.
By taking double covers of the four scrolls of [23], we see that there are flat conic bundles over surfaces for $d = 6, 12, 14, 18$. We do not know whether the case $d = 44$ occurs.

5.1 *Digression.*

In the course of the proof of Theorem 3.4 we used the fact, due to Besana [9], that the base of an adjunction theoretic quadric bundle over a surface is non-singular. The following lemma is a result with a similar flavor. It is probably well known. The first corollary is used in the proof of Theorem 3.4. The second one is, in a sense, a converse to the lemma.

LEMMA 5.6. *Let X a non-singular projective variety of dimension n, $p: X \longrightarrow Y$ a morphism onto a normal projective variety Y of dimension $n-1$ such that all fibers have the same dimension, the general scheme theoretic fiber over a closed point is isomorphic to a conic and $-K_X$ is p-ample. Then all the scheme-theoretic fibers are isomorphic to conics, p is flat and Y is non-singular.*

PROOF. The proof is the same as the one of [22] Lemma 3.25. The only necessary changes are the following: a) replace the line bundle H of [22], by a pull-back p^*A of

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any ample line bundle A on Y and use Kleiman's criterion of ampleness to obtain the result analogous to the last assertion of [22] Lemma; b) replace [22] Lemma 3.12 by [1] Lemma 1.5. ∎

COROLLARY 5.7. *Let X be a non-singular projective variety together with a morphism $p: X \longrightarrow Y$, where Y is a normal variety of dimension m. Let D_i, $i = 1, \ldots, n-m-1$ be divisors on X such that they intersect transversally; denote by X' their intersection. Assume that $p_{|X'}: X' \longrightarrow Y$ satisfies the hypothesis of Lemma 5.6. Then p is flat and Y is non-singular.*

PROOF. By the lemma, $p_{|X'}$ is flat. We can "lift" this flatness to p by virtue of [21], Corollary to Theorem 22.5. As above the flatness of $p_{|X'}$ (or of p) implies the non-singularity of Y. ∎

COROLLARY 5.8. *Let X a non-singular projective variety of dimension n, $p: X \longrightarrow Y$ a morphism onto a normal projective variety Y of dimension $n-1$ such that all fibers have the same dimension. If the general fiber of p is actually embeddable as conics with respect to an embedding of X, then all scheme theoretic fibers are actually embedded conics, p is flat, Y is non-singular and $-K_X$ is p-ample.*

PROOF. We argue as in the proof of the lemma with the simplifications due to the fact that a flat deformation of a conic in projective space is still a conic. The assertion about $-K_X$ follows by observing that, if L denotes the line bundle with which we embed X, $K_X + L$ is a pull-back from Y. ∎

REMARK 5.9. The assumption $-K_X$ is p-ample is essential in the lemma, as the blow up of a $\mathbb{P}^1$ bundle over a curve at two distinct points on a fiber shows. Moreover, the above Lemma does not follow directly from [22] or [1], since there are conic bundles for which the structural morphism is not a Mori contraction. Finally, the above theorem is certainly false if one has $\dim X = \dim Y$. It is a purely local question: consider the quotient of $\mathbb{A}^2$ by the involution $(x, y) \longrightarrow (-x, -y)$.

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