Can. J. Math., Vol. XXII, No. 6, 1970, pp. 1133–1155

ALGEBRAIC EXTENSIONS OF COMMUTATIVE REGULAR RINGS

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Dedicated to the memory of Peter Schulz

Introduction. In this paper we study algebraic closures for commutative semiprime rings. The main interest, however, is with rings which are regular in the sense of von Neumann. These play the same role with respect to semiprime rings as fields do with respect to integral domains. Two generally distinct notions are defined: "algebraic" and "weak-algebraic" extensions. Each has the transitivity property and yields a closure which is unique up to isomorphism and is "universal". Both coincide in fields.

The extensions here called "algebraic" were studied independently by Enochs [**5**] and myself. Our results on these extensions proceed from a different point of view, and allow us to answer a question posed by Enochs. Furthermore, these results are required (and were developed) in order to obtain the weak-algebraic closure, which was the original closure sought. The motivation for the weak-algebraic extensions is found in the work of Shoda [**14**, p. 134, no. 1].

As applications, algebraic closures are obtained for some regular rings which arise in the study of rings of continuous functions, and for some group rings.

Henceforth, all rings are commutative with 1, and all ring homomorphisms are 1-preserving. Terminology and notation not otherwise specified are used as in Lambek [**12**]. We also assume familiarity with the properties of regular rings, as found in that book.

Acknowledgements. I wish to express my appreciation to Professor J. Lambek for his advice, encouragement, and criticisms. As well, I am indebted to W. Burgess, I. Connell, and H. Storrer, for many discussions.

1. Essential extensions of rings.

1.1. Let R and S be rings, and let R be a subring of S. Then the following are easily seen to be equivalent:

Received January 14, 1970 and in revised form, August 10, 1970. This paper is derived from the author's doctoral dissertation, written at McGill University under the supervision of J. Lambek and the subvention of the National Research Council of Canada. The results presented incorporate announcements 68T-H29 and 69T-A10 in the Notices of the American Mathematical Society.

(1) every non-zero ideal of S intersects R in a non-zero ideal;
(2) every non-zero principal ideal of S intersects R in a non-zero ideal;
(3) for any $s \neq 0$ in S there is a $t \in S$ such that $st \in R$, and $st \neq 0$;
(4) a ring homomorphism with domain S is a monomorphism if and only if its restriction to R is a monomorphism.

If this holds we will say that S is an *essential* extension of R, or that S is essential over R. An embedding of rings $m: R \to S$ will be essential if S is essential over $m(R)$. If R is semiprime, then one checks that S is essential over R if every non-zero semiprime ideal of S has non-zero intersection with R. These extensions have been called "intrinsic" by Faith and Utumi [**6**] and "tight" by Enochs [**5**]. Our work is closer to that of Storrer [**15**] whose term we use.

The essential extensions of a field are precisely its overfields. Also any ring of quotients of a commutative ring R [**12**, p. 40, Proposition 6] is essential over R.

1.2. LEMMA (*Transitivity*). *Let R, S, and T be rings, $R \subset S \subset T$, let S be an essential extension of R, and let T be an essential extension of S. Then T is essential over R.*

Proof. Let I be an ideal of T. If $I \neq (0)$, then $I \cap S$ is a non-zero ideal of S. Now $I \cap R = (I \cap S) \cap R \neq (0)$.

1.3. LEMMA. *Let R be a semiprime ring and let S be essential over R. Then S is semiprime.*

Proof. $(\text{rad } S) \cap R = \text{rad } R = (0)$. Therefore $\text{rad } S = (0)$.

1.4. *Definition* [**10**, p. 3]. A ring is *Baer* if all of its annihilator ideals are direct summands (i.e. are principal ideals generated by idempotents). If R is Baer, then it must be semiprime. For if not, there exist an $x \in R$ and an integer $n > 1$ such that $x^n = 0$, and $x^{n-1} \neq 0$. Thus $x \in (x^{n-1})^* = eR$ say, where e is an idempotent of R. But then $x = ex$, and so

$$0 = ex^{n-1} = (ex)x^{n-2} = x^{n-1},$$

a contradiction.

On the other hand, any semiprime rationally complete ring is Baer [**12**, p. 44, Proposition 4], yielding the following result.

1.5. PROPOSITION. *If R is a semiprime ring, then R has an essential Baer extension.*

Proof. Embed R into its complete ring of quotients.

The following is motivated by the non-commutative work of Faith and Utumi [**6**, 2.4].

1.6. LEMMA. *Let R be a Baer ring and let S be essential over R. Then $S \setminus R$ contains no idempotents.*

https://doi.org/10.4153/CJM-1970-132-4 Published online by Cambridge University Press

Proof. By contradiction. Take $e = e^2 \in S \setminus R$. eS is the annihilator of $(1 - e)S$ in S. Therefore $(eS \cap R)[(1 - e)S \cap R] = (0)$ yielding

$$eS \cap R \subset [(1 - e)S \cap R]^*,$$

the annihilator being taken in the ring R.

Suppose that $x \in R$ and $x[(1 - e)S \cap R] = 0$. Then $x(1 - e) = 0$, for if $x(1 - e) \neq 0$ then by the essentiality of S over R there is a $y \in S$ such that $(1 - e)xy$ is in $(1 - e)S \cap R$ and is different from 0. But x annihilates $(1 - e)S \cap R$, and so $x(1 - e)xy = 0$. Therefore $[(1 - e)xy]^2 = 0$ yielding $(1 - e)xy = 0$ since R is semiprime, a contradiction. Thus $x(1 - e) = 0$ and $x = ex \in eS \cap R$. Therefore $[(1 - e)S \cap R]^* = eS \cap R$. But R is Baer, and so there is an idempotent f in R such that $fR = [(1 - e)S \cap R]^* = eS \cap R$. Clearly $ef = f$. If $e \neq ef$, then $0 \neq e(1 - f)$ and by essentiality there exist $r \in R$ and $t \in S$ such that $0 \neq r = e(1 - f)t$.

Now $r \in eS \cap R = fR$; hence $r = fr = fe(1 - f)t = ef(1 - f)t = 0$, a contradiction. Therefore $e = ef = f \in R$.

Recall that a ring R is regular in the sense of von Neumann if for each $r \in R$ there is at least one element $r' \in R$ such that $r = r^2r'$. r' is often called a quasi-inverse for r, after the case when R is a field. Clearly, a regular ring must be semiprime.

1.7. Lemma. *Let R be a semiprime ring and let T be an essential extension of R. If S is a between ring of R and T and if S is regular, then S is essential over R.*

Proof. Let x be a non-zero element of S. For some $y \in S$, $x = x^2y$, and so xy is a non-zero idempotent. By essentiality, there exists $t \in T$ such that $xyt \in R$ and is non-zero. But $xyt = (xy)^2t = x[y(xyt)]$. Since $y(xyt)$ is in S, the proof is complete.

1.8. Proposition [**2**, p. 183]. *A semiprime ring is regular if and only if each of its prime ideals is a maximal ideal.*

1.9. Lemma. *Let R be a regular ring and let S be an over-ring of R which is semiprime and is integrally dependent on R. Then S is a regular ring.*

Proof. Integral dependence of rings is defined and discussed in [**16**, p. 259], where it is shown that if S is integral over R, and if P is an arbitrary prime ideal of S, then P is a maximal ideal of S if and only if the ideal $P \cap R$ is maximal in R. Since by 1.8 all prime ideals of R are maximal, the same must be true for those of S. Since S is also semiprime, 1.8 guarantees that S is regular.

1.10. Corollary. *If R is a regular ring and S is an over-ring of R which is both integral and essential over R, then S is regular.*

Let R be a ring. We follow [**13**] in denoting the set of idempotents of R by $B(R)$. $B(R)$ is a Boolean ring in which multiplication coincides with that of R, but addition differs in general.

1.11. PROPOSITION. *Let R be a regular ring, and let S be a regular over-ring of R. Then S is an essential extension of R if and only if $B(S)$ is a ring of quotients of $B(R)$.*

Proof. Assume that S is essential over R, and let $e \in B(S)$. Then there exist $r \in R$, $s \in S$, $r \neq 0$, such that $r = es$. Let r' be a quasi-inverse for r, and let $f = rr'$. Clearly $f \neq 0$, and $f \in B(R)$. Thus $f = rr' = (es)r' = e(esr') = ef$. Since $ef = f \neq 0$, $B(S)$ is a ring of quotients of $B(R)$.

Assume that $B(S)$ is a ring of quotients of $B(R)$. Let s be a non-zero element in S, s' a quasi-inverse for s. Then as above, $ss' \neq 0$ and $ss' \in B(S)$. There exist $f \in B(R)$, $e \in B(S)$, $f \neq 0$, such that $f = ss'e$. Therefore S is essential over R.

1.12. LEMMA. *A regular ring is Baer if and only if its Boolean ring of idempotents is complete.*

Proof. In a regular ring, principal ideals are direct summands [**12**, p. 67]; thus the annihilators of individual elements are direct summands. Since the annihilator of a set is the intersection of the annihilators of its elements, the ring will be Baer if and only if any intersection of direct summands is again a direct summand, i.e. if and only if the Boolean ring of idempotents is complete.

1.13. COROLLARY. *Let R be a regular Baer ring and let S be a regular essential extension of R. Then S is Baer.*

Proof. By 1.6, $B(S) = B(R)$, a complete Boolean algebra.

Let R be a ring and let S be an over-ring of R. Recall [**16**, p. 218] that if J is an ideal of S, then the ideal $J \cap R$ is called its contraction to R. An ideal of R is contracted (with respect to S) if it is the contraction of an ideal of S. If I is an ideal of R, then IS is its extension to S and an ideal of S is extended if it is the extension of an ideal of R.

1.14. LEMMA. *Let R be a regular ring and let S be an over-ring of R. Then any ideal of R is the contraction of an ideal of S.*

Proof. In a regular ring the zero ideal is the intersection of all the prime (= maximal) ideals. Since any factor ring of a regular ring is regular, any ideal in a regular ring is the intersection of the maximal ideals containing it. Thus it suffices to establish the contraction result for maximal ideals.

Let M be a maximal ideal in R. Clearly MS is an ideal of S. Assume that $MS = S$. Then for suitable $m_i \in M$ and $s_i \in S$, $i = 1, 2, \ldots, k$, $1 = m_1s_1 + \ldots + m_ks_k$. Thus $S = 1S = (m_1R + \ldots + m_kR)S$. Since R is regular, by von Neumann's lemma [**12**, p. 68], there exists an $e \in M$ such that $m_1R + \ldots + m_kR = eR$. Furthermore, e can be taken to be an idempotent since principal ideals are generated by (single) idempotents. Thus

https://doi.org/10.4153/CJM-1970-132-4 Published online by Cambridge University Press

$S = (eR)S = eRS = eS$. Thus $1 = es$ for some $s \in S$. Therefore $1 - e = (1 - e)1 = (1 - e)es = es - es = 0$. Thus $1 = e \in M$, which contradicts the fact that M is a proper ideal of R.

Thus MS is proper in S. Now $M \subset MS \cap R$, a proper ideal in R. Since M is maximal, $M = MS \cap R$.

The following result is of independent interest.

1.15. PROPOSITION. *Let R be regular and let S be a regular Baer ring which is essential over R ($Q(R)$ for example). Then* (Spec S, f) *is the projective cover* [**9**, 2.1] *of* Spec R, *where f is defined by contracting the prime ideals of S to those of R.*

Proof. Since R and S are regular, both spaces in question are compact and Hausdorff. Furthermore, Spec S is extremally disconnected since S is Baer. f is well known to be continuous and 1.14 shows that it is onto. Thus all that remains is to show that f is an "irreducible" map, i.e. that if C is a proper closed subset of Spec S, then $f(C)$ is proper in Spec R. By the definition of the Stone topology, $C = \{M_i\}_{i \in I}$ where $\{M_i\}$ is the set of all maximal ideals of S containing the ideal $J = \bigcap_i M_i$. Since C is proper, $J \neq (0)$. Now $f(C) = \{M_i \cap C\}$, the closed set in Spec R defined by the ideal $J \cap R$. Since S is essential over R, $J \cap R \neq (0)$. Thus $J \cap R$ is not contained in all the maximal ideals of R and $f(C)$ is proper.

1.16. PROPOSITION. *Let R be a regular ring and let S be a regular essential extension of R. Then all ideals of S are extensions of ideals of R if and only if $B(R) = B(S)$.*

Proof. If $B(R) = B(S)$ and $x \in J$ (J an arbitrary ideal in S), then by regularity of S, $x = xyx$ for some $y \in S$. Hence $x = ex$, where $e = xy$ is an idempotent in the ideal J. Thus $x \in (J \cap R)S$. Therefore $J \subset (J \cap R)S$. The opposite inclusion however is trivial, and so J is the extension of $J \cap R$.

Suppose that $B(R) \neq B(S)$ but that the ideal generated by $e \in B(S) \setminus B(R)$ is the extension of an ideal, say A, of R, so that $eS = AS$. Then as in 1.14, we can replace A first by a finitely generated ideal of R and then by a direct summand of R, generated by the idempotent f say. Thus $eS = fRS = fS$. This implies that $e = f$, contradicting the fact that f is in R and e is not. Thus e generates an ideal that is not extended with respect to R.

1.17. *Remark.* If R is regular and Baer and S is a regular essential extension of R, we have (by 1.6, 1.14, and 1.16) that all ideals of R are contracted with respect to S and that all ideals of S are extended with respect to R; in fact, the operations of contraction and extension define an isomorphism between the lattice of ideals of R and the lattice of ideals of S. Thus under these stricter conditions, Storrer's result [**15**, 10.3] on annihilator ideals can be extended to all ideals.

1.18. PROPOSITION. *Let R be a regular Baer ring and let S be essential over R. Then the minimal prime ideals of S are precisely the ideals which are extensions of maximal ideals in R.*

Proof. We wedge S between R and a regular essential extension of R. Consider the embeddings $R \to S \to Q(S)$, where $Q(S)$ is the complete ring of quotients of S. By 1.2, $Q(S)$ is essential over R, and by 1.3 and [**12**, p. 42, Proposition 1], $Q(S)$ is regular. Let M be a maximal ideal in R. By 1.17, $MQ(S)$ is a maximal ideal in $Q(S)$, and it obviously extends M. Suppose that $x \in MQ(S) \cap S$; then $xQ(S) = eQ(S) \subset MQ(S)$ for some idempotent $e \in B(Q(S)) = B(R)$, since R is Baer. Thus $e \in MQ(S) \cap R = M$, and so $x = ex \in MS$. Therefore $MQ(S) \cap S \subset MS$ which implies that $MQ(S) \cap S = MS$, the second inclusion being trivial. MS is a prime ideal in S because it is the contraction of a prime ideal of $Q(S)$. Is it minimal? Suppose that P is a prime ideal in S contained in MS. Then $P \cap R \subset MS \cap R = M$. But $P \cap R$ is a prime ideal in R, a regular ring, and so it is a maximal ideal and $P \cap R = M$. Therefore $P = PS \supset (P \cap R)S = MS \supset P$. Therefore MS is a minimal prime ideal.

We have shown that an arbitrary maximal ideal M in R extends to a minimal prime ideal in S. Now let L be a minimal prime ideal in S. Then $L = LS \supset (L \cap R)S$, which is a minimal prime ideal in S, as was just shown. Therefore $L = (L \cap R)S$ and L is of the claimed form. It is interesting to note that S has the property that each prime ideal of S contains a unique minimal prime ideal.

Recall that a ring is Bézout if its finitely generated ideals are principal.

1.19. PROPOSITION. *Let R be a regular Baer ring and let S be an essential extension of R. Then the following are equivalent:*

(1) *S is regular;*

(2) *S is Bézout and all non-zero divisors in S are units.*

Proof. (1) $\Rightarrow$ (2). This is true by von Neumann's Lemma and [**12**, p. 33, Proposition 3(1)].

(2) $\Rightarrow$ (1). By 1.18, we can write an arbitrary minimal prime ideal of S in the form MS, where M is a maximal ideal of R. Let N be an ideal of S such that $N \supset MS$. Let $Q(S)$ be the complete ring of quotients of S. Then $NQ(S) \supset MSQ(S) = MQ(S)$. But by 1.17, $MQ(S)$ is a maximal ideal in $Q(S)$, since $Q(S)$ is regular. Therefore $NQ(S) = Q(S)$ or $NQ(S) = MQ(S)$.

Suppose that $NQ(S) = Q(S)$. Then for suitable $n_i \in N$, $q_i \in Q(S)$, $i = 1, 2, \ldots, k$, we have $n_1q_1 + \ldots + n_kq_k = 1$. Thus

$$Q(S) = (n_1S + \ldots + n_kS)Q(S).$$

Since S is Bézout, there exists $x \in S$ such that $xS = n_1S + \ldots + n_kS$.

Clearly $x \in N$, and $Q(S) = (xS)Q(S) = xQ(S)$. Thus x is a unit in the ring $Q(S)$. Clearly it is not at the same time a zero-divisor in S, and so it must be a unit in S. But $x \in N$; thus $N = S$.

Suppose that $NQ(S) = MQ(S)$. Then

$$N = N \cap S \subset NQ(S) \cap S = MQ(S) \cap S = MS$$

as in 1.18. Therefore $N = MS$.

Thus the minimal prime ideals of S are maximal ideals, and so all prime ideals of S are maximal. S is also semiprime, and hence it is regular by 1.8.

2. Algebraic extensions.

2.1. *Definition.* Let R be a ring and let S be an over-ring of R. We will call S an *algebraic extension* of R if it is both essential and integral over R. It should be noted that the two properties are independent; if F is a field and x is an indeterminate, then $F(x)$ is essential but not integral over F. On the other hand, the complete Boolean algebra on a two-element set is integral but not essential over the copy of the two-element field which it contains.

2.2. PROPOSITION. *Let R be a semiprime ring and let S be an over-ring of R. Then S is algebraic over R if and only if for each $s \in S$, $s \neq 0$, there exist $r_i \in R$, $i = 0, 1, \ldots, n-1$, $r_0 \neq 0$ such that $s^n + r_{n-1}s^{n-1} + \ldots + r_1 s + r_0 = 0$.*

Proof. Let S be algebraic over R and let $s \in S$, $s \neq 0$. Since S is integral over R, we have

$$s^n + r_{n-1}s^{n-1} + \ldots + r_1 s + r_0 = 0 \tag{1}$$

for some $r_i \in R$, $i = 0, 1, \ldots, n-1$. If $r_0 \neq 0$, then our proof is complete.

Suppose that $r_0 = 0$. Since S is essential over R, there exists $t \in S$ such that $st = a \in R$, $a \neq 0$. If $a^i r_i = 0$ for $i = 0, 1, \ldots, n-1$, then multiplication of (1) by t^n yields:

$$0 = t^n s^n + t r_{n-1} a^{n-1} + t^2 r_{n-2} a^{n-2} + \ldots + t^{n-1} r_1 a = a^n$$

which implies that $a = 0$, a contradiction. Thus there exists a positive integer $m < n$ such that $a^m r_i = 0$ for all $i < m$, and $a^m r_m \neq 0$.

Multiplication of (1) by t^m yields:

$$a^m s^{n-m} + a^m r_{n-1} s^{n-m-1} + \ldots + a^m r_{m+1} s + a^m r_m = 0. \tag{2}$$

It is easy to see that addition of equations (1) and (2) yields an equation of the desired form.

Conversely, suppose that the condition holds. Clearly S is integral over R. Let $s \in S$, $s \neq 0$. Then for appropriate $r_i \in R$, $i = 0, 1, \ldots, n-1$, one has

$$0 \neq r_0 = s(r_1 + r_2 s + \ldots + r_{n-1}s^{n-2} + s^{n-1}) \in sS \cap R,$$

showing that S is essential over R.

2.3. PROPOSITION. *Let R be a regular ring and let S be a regular essential extension of R. The following are equivalent:*

(1) *S is an algebraic extension of R;*
(2) *all between rings of R and S are regular;*
(3) *$R[s]$ is a regular ring, s any element of S;*
(4) *$R[u]$ is a regular ring, u any unit of S;*
(5) *all units of S are integral over R.*

Proof. Clearly (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

(1) $\Rightarrow$ (2). Since S is essential over R, it is semiprime and so are all the between rings of R and S. Moreover, the latter are integral over R, since S is. Now invoke 1.9 to yield (2).

(4) $\Rightarrow$ (5). Let u be a unit in S. $R[u]$ is regular, and so its non-zero divisors are units [**12**, p. 33, Proposition 3 (1)]. Therefore u is a unit in $R[u]$. Thus there exist $r_i \in R$, $i = 0, 1, \ldots, n$, such that $1 = u(r_n u^n + \ldots + r_1 u + r_0) = r_n u^{n+1} + \ldots + r_1 u^2 + r_0 u$. Transposition and multiplication of both sides of the equation by $(u^{-1})^{n+1}$ yield

$$(u^{-1})^{n+1} - r_0(u^{-1})^n - \ldots - r_{n-1}(u^{-1}) - r_n = 0,$$

which is an equation of integral dependence over R for the element u^{-1}. But every unit is the inverse of a unit.

(5) $\Rightarrow$ (1). It is pointed out [**12**, p. 36, Exercise 4] that in a commutative regular ring the quasi-inverse of an element may be chosen to be a unit. (If s' is a quasi-inverse for s, then so is $u = s'ss' + 1 - ss'$, and $u^{-1} = s + 1 - ss'$.) We shall make use of this result several times in this article. If $s \in S$ and u is both a quasi-inverse for s and a unit, then $su = e$, an idempotent, and $s = eu^{-1}$; i.e., every element of S is the product of an idempotent and a unit. Clearly the idempotents of S are integral over R; thus if the units are integral as well, S is integral over R.

2.4. LEMMA. *Let R, S, and T be rings, $R \subset S \subset T$. Suppose that S is an algebraic extension of R and that T is an algebraic extension of S. Then T is an algebraic extension of R.*

Proof. The transitivity of essentiality was pointed out in 1.2. The transitivity of integral dependence was established in [**16**].

2.5. LEMMA. *Let R be a semiprime ring. Then R has a Baer algebraic extension.*

Proof. We know that R can be embedded into $Q(R)$, a Baer ring. Let T be the integral closure of R in $Q(R)$. Then T is essential and integral over R, and $B(T) = B(Q(R))$. We show that T is Baer. Let t be a zero divisor in T. Since $Q(R)$ is Baer, there exists $e \in B(T)$ such that $eQ(R) = (t)^*$ in $Q(R)$. Therefore $eT = eQ(R) \cap T \subset (t)^*$ in T. Now if $st = 0$, for some $s \in T$, then $s \in eQ(R) \cap T = eT$. Therefore $eT = (t)^*$ in T. This shows that the annihilators of elements are direct summands. But $B(T)$ is complete, and so all annihilators in T are direct summands, and T is Baer.

2.6. *Definition.* Let R be a ring. Then R is *algebraically closed* if it has no proper algebraic extensions. In terms of maps this means that if $m: R \rightarrow S$ is a monomorphism such that S is algebraic over $m(R)$, then m is onto. For example, an algebraically closed field is an algebraically closed ring.

2.7. PROPOSITION. *Let R be a regular ring. Then the following are equivalent:*
(1) *R is algebraically closed;*
(2) *R is Baer and every monic polynomial equation over R has a root in R;*
(3) *R is Baer and all the factor fields of R are algebraically closed.*

Proof. (1) $\Rightarrow$ (2). If R is not Baer, then the embedding given in 2.5 is proper and algebraic; hence R is not algebraically closed.

Now suppose that $f(x)$ is a monic polynomial over R with the property that no element of R is a zero of it. One notes in passing that its absolute term must be non-zero. We "adjoin a root". Embed R into $R[x]$, where x is an indeterminate, and consider the ideal $I = (f(x))$. Then $I \cap R = (0)$. For a typical element of I is of the form $f(x) \cdot g(x)$, where $g(x)$ is a polynomial in x over R. If the highest power of x appearing in f is n, and if the highest non-zero coefficient appearing in g is b, as the coefficient of x^m say, then $f(x) \cdot g(x)$ is a polynomial with b as the coefficient of x^{m+n}. Since x is an indeterminate, such a polynomial is not in R.

In view of Zorn's lemma, there exists an ideal J, which is maximal in the family of ideals of $R[x]$ which contain I and have trivial intersection with R. Let p be the projection from $R[x]$ to $R[x]/J = S$, say. Thus we have

$$R \to R[x] \xrightarrow{p} S.$$

It is clear that S is essential over $p|R$ by the choice of J. Also since J contains I, it follows that $p(x)$ is integral over the image of R in S. But $p(x)$ and $p(R)$ generate the ring S. Therefore S is integral and essential over $p(R)$, i.e. $p|R$ is an algebraic embedding of R into S. Furthermore, the embedding is proper since $p(x)$ satisfies the equation $f(x) = 0$, as does no element of R. Thus we have a proper algebraic extension of R, a ring which is given by assumption to be algebraically closed; a contradiction. Therefore all monic polynomials over R have zeros in R.

(2) $\Rightarrow$ (3). This is straightforward. The field R/M will be algebraically closed if it has a root for every monic polynomial equation over itself. Take such a monic over R/M. Lift each of the coefficients back to one of its preimages under the canonical map from R to R/M, but insist that the preimage of the first coefficient be the 1 in R. By (2), the resulting monic over R has a root in R, say a. Then the image of a under the canonical map above is a root for the monic over R/M. Since the question of being Baer is not at issue, this part of the proof is complete.

(3) $\Rightarrow$ (1). Assume (3) and let $\lambda: R \to S$ be an algebraic embedding. Since our rings are commutative, λ is conformal in the sense of Pierce [**13**, p. 8, 2.1]. In [**13**, 6.6] it is shown that the category of rings and conformal maps is

equivalent to the category of reduced ring spaces and their morphisms. The mapping corresponding to λ is denoted $\lambda^0 = (\lambda^*, \bar{\lambda})$ by Pierce [**13**]. λ^* maps $\mathrm{Spec}(B(S))$ into $\mathrm{Spec}(B(R))$ by the obvious contraction. Since R is Baer, $B(R) = B(S)$ canonically, and so λ^* is a homeomorphism.

One claims that $\bar{\lambda}$ as defined in [**13**, p. 22] is also a homeomorphism. First note that if M is a maximal ideal of $B(R)$, R a regular ring, then $B(R)R$ is a maximal ideal of R. This follows by the same argument, now customary, as appears in 1.16 (by writing $x = ex$, *et cetera*). Thus pointwise $\bar{\lambda}$ is the field embedding given by the following (using some of Pierce's notation): $\lambda(M, *)$: $R/\lambda^{-1}(\lambda(R) \cap MS) \rightarrow \lambda(R)/\lambda(R) \cap MS \rightarrow S/MS$ at the maximal ideal M in $\mathrm{Spec}(B(R))$. The first mapping indicated is one-to-one and onto and is included simply because we are distinguishing between "extensions" and "embeddings". Since S is an algebraic extension of $\lambda(R)$, since the latter ring has algebraically closed factor fields (given), and since the property of being an algebraic extension is preserved under factoring, therefore the second containment map indicated is not proper; i.e., $\bar{\lambda}(M, *)$ is one-to-one and onto. Since this holds at each M in $\mathrm{Spec}(B(R))$, the map $\bar{\lambda}$ is a homeomorphism.

Thus λ^0 is both mono and epi in the category of reduced ring spaces and their maps. By the equivalence of categories established by Pierce, λ is mono and epi in the category of rings and conformal maps. Therefore λ is epi in the (smaller) category of commutative rings and ring homomorphisms. But Storrer [**15**, 3.2 and 6.1] has shown that a mono-epimorphism in this category whose domain is a regular ring must be onto. Thus λ is not a proper map between R and S, and we have shown that R is algebraically closed.

2.8. COROLLARY. *If R is regular and algebraically closed and I is an ideal of R, then R/I is algebraically closed if and only if it is Baer.*

Proof. R/I is regular. Its quotient fields lie among those of R, and so they are algebraically closed. The result now follows by the equivalence of (1) and (3) of 2.7.

2.9. PROPOSITION. *A product of algebraically closed regular rings is also algebraically closed.*

Proof. Let $R = \prod R_i$, where $\{R_i\}_{i \in I}$ is a family of algebraically closed regular rings. R is regular by common knowledge. Also $B(R) \cong \prod B(R_i)$, a product of complete Boolean algebras. The latter is complete by [**12**, p. 41, Proposition 8]. Thus R is Baer. Now if one must demonstrate a root for a monic polynomial, one notes that the polynomial gives a monic over each R_i under projection onto the ith component. Thus one can solve locally at each i, to arrive at the sought root.

At this point one could construct an algebraic closure for regular rings directly. Instead we turn to the parallel work of Enochs. It is clear from [**5**, p. 701 and Theorem 2] that algebraically closed rings coincide with the totally integrally closed rings of Enochs. Furthermore, his Theorem 2 translates into: given a semiprime ring R, there exists an algebraic extension $\Omega(R)$ of R, which is algebraically closed. Furthermore, $\Omega(R)$ is unique up to isomorphism over R, and contains a copy, over R, of every algebraic extension of R. We call $\Omega(R)$ the algebraic closure of R. It is clear from 1.10 that the algebraic closure of a regular ring is regular.

2.10. *Remark.* If one has available the algebraic closure in the regular case, one can realize the closure in the semiprime case as the integral closure of the ring in the algebraic closure of its complete ring of quotients.

2.11. PROPOSITION. *Ω commutes with finite Cartesian products.*

Proof. In view of [**5**, Proposition 1], one need only make the straightforward verification that finite products preserve the equational condition of 2.2.

2.12. PROPOSITION. *Let R be a semiprime ring and let $\Omega(R)$ be its algebraic closure; then $\Omega(R)$ is Baer.*

Proof. If $\Omega(R)$ were not Baer, then by 2.5 it would have an algebraic extension which, being Baer, would be a proper extension.

In closing his paper Enochs remarks that "it is an open question whether A totally integrally closed implies $S^{-1}A$ totally integrally closed for every multiplicative set $S \subset A$". We now take up this question and show the answer to be negative by means of some general considerations which also yield related results on rings of quotients. It will not be necessary to look beyond regular rings to resolve this question.

2.13. LEMMA. *Let R be a regular ring and let I be an ideal in R. Then idempotents can be lifted modulo I; i.e. an element of R/I is an idempotent if and only if it is the image under factoring by I, of an idempotent of R.*

Proof. The images of idempotents are again idempotents. Conversely, let $\bar{x} = \bar{x}^2$ in R/I. Then $x^2 - x \in I$. By regularity there exists $y \in R$ such that $x = x^2y = xe$, where $e = e^2 = xy$. Since $x^2 - x$ is in I, so is $y(x^2 - x) = x - e$. Thus the idempotent e is mapped onto $\bar{x}$ in R/I.

2.14. LEMMA. *Let R be a regular ring and let $B(R)$ be its ring of idempotents. Then:*

(1) *if I is any ideal of R, then the ring of idempotents of R/I is isomorphic to $B(R)/I \cap B(R)$,*

(2) *any ideal of $B(R)$ is extended by an ideal of R.*

Proof. (1) The multiplication in $B(R)$ coincides with that in R, and the addition is given in terms of the operations in R by $e \oplus f = e + f - 2ef$. If h is the projection from R onto R/I, it is easily checked that $h|B(R) \to B(R/I)$ is a ring homomorphism. Furthermore, its kernel is $I \cap B(R)$ and 2.13 ensures that it is onto.

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(2) Let J be any ideal in $B(R)$. Let $I = \{er: e \in J, r \in R\}$. Then I is an ideal of R [**13**, p. 17, Lemma 1.6]. If $er \in I \cap B(R)$, then $er = e(er)$ and since e is in the ideal J, so is er.

2.15. Lemma. *Let R be a regular Baer ring with ring of idempotents $B(R)$. Then R is completely reducible if and only if $B(R)$ is finite.*

Proof. If R is completely reducible, then it is the Cartesian product of finitely many fields and it is obvious that it contains only a finite number of idempotents.

The converse can be established by induction on the order of $B(R)$. If $|B(R)| = 2$, then R is a field. If $|B(R)| = n > 2$, let $e \in B(R)\setminus\{0, 1\}$. Then $R \cong eR \times (1 - e)R$ and $|B(eR)| < n$, $|B((1 - e)R)| < n$.

2.16. Theorem (Dwinger). *A Boolean algebra has the property that all of its quotient algebras are complete if and only if it is finite.*

Proof. [**4**, p. 456, Theorem 4.3].

2.17. Proposition. *If R is semiprime and rationally complete, then all quotient rings are rationally complete if and only if R is completely reducible.*

Proof. Suppose that R is not completely reducible. By 2.15, $B(R)$ is infinite. By 2.16, there exists an ideal J in $B(R)$ such that $B(R)/J$ is not complete. By 2.14 (2), J is extended by an ideal say I of R, and by 2.14 (1), the ring of idempotents of R/I is not complete. Thus R/I is not rationally complete. Thus if all quotient objects of R are to be complete, then $B(R)$ must be finite, and R must be completely reducible. The opposite implication is straightforward.

2.18. Proposition. *If R is algebraically closed, then all quotient objects of R are algebraically closed if and only if R is completely reducible.*

Proof. The quotient rings of R are always regular and their factor fields are always algebraically closed. Thus the only property at issue is that of being Baer, and this argument proceeds as in 2.17.

2.19. Lemma. *Let R be a regular ring and let I be an ideal of R. Then I is the kernel of a localization of R, and the localization is R/I.*

Proof. Let $I = \bigcap_i M_i$, where $\{M_i\}$ is the family of maximal ideals of R containing I. The set $S = \bigcap_i (R - M_i)$ is multiplicative and I is the kernel of the localization with respect to S. For if r is in the kernel, $rs = 0$, for some $s \in S$. Since s is in no M_i, r is in each M_i, and therefore in I. Conversely, if $r \in I$, then $rR = eR$, for some idempotent e of R. Since e is in each M_i, $1 - e$ is in each $R - M_i$, i.e. in S. Since $r(1 - e) = 0$, we conclude that r is in the kernel of the localization with respect to S. The localization R_S will be R/I because R/I consists only of zero divisors and units.

Enochs' question can now be answered as follows.

2.20. PROPOSITION. *Let R be an algebraically closed regular ring. Then every localization of R is algebraically closed if and only if R is completely reducible, i.e. if and only if R is a finite Cartesian product of algebraically closed fields.*

Proof. 2.18 and 2.19.

2.21. *Remark.* It is also now clear that the demands that the ring be Baer in 2.7 are not superfluous.

3. Weak-algebraic extensions. The motivation for our work is the following theorem [**1**, p. 84]: *If S is a field extension of the field R, then S is algebraic over R if and only if all the rings between R and S are fields.*

3.1. *Definition.* Let R be a semiprime ring and let S be an over-ring of R. We will say that S is a *weak-algebraic extension* of R if all rings between R and S are essential over R. It is easy to verify that this coincides with the (usual) field-theoretic notion. Furthermore, it is clear that this condition is strictly stronger than essentiality; the complex field, for example, has many over-fields but no proper weak-algebraic extensions. If R is a semiprime ring, then $Q(R)$ is a weak-algebraic extension of R.

3.2. LEMMA. *Let R be a semiprime ring and let S be an over-ring of R. Then S is weak-algebraic over R if and only if each non-zero element of S satisfies a polynomial equation with coefficients from R which has a non-zero absolute term.*

Proof. Assume that S is a weak-algebraic extension of R. If $s \in S \setminus R$, then $R[s]$ is essential over R, and so there exists a polynomial, say, $f = r_n s^n + \ldots + r_1 s + r_0$ in $R[s]$ and an $a \in R$, $a \neq 0$, such that $sf = a$. Transposition yields $r_n s^{n+1} + \ldots + r_1 s^2 + r_0 s - a = 0$, which is an equation of the required form. That the elements of R satisfy such an equation is a triviality.

Suppose, now, that all elements of $S \setminus R$ satisfy such equations, and let T be a between ring of R and S. If $t \in T$, then $T \supset R[t]$, and

$$tT \cap R \supset tR[t] \cap R \neq (0),$$

as is evident from the polynomial equation over R for the element t. Therefore T is essential over R, and the proof is complete. We will subsequently refer to such polynomials and polynomial equations as *weak-algebraic.*

3.3. COROLLARY. *Let R be a subring of T, and let T be a subring of S. Then if S is a weak-algebraic extension of R, it is also a weak-algebraic extension of T.*

3.4. PROPOSITION. *Let $\{R_i\}_{i \in I}$ be a family of semiprime rings and let $\{S_i\}_{i \in I}$ be a family of rings such that S_i is a weak-algebraic extension of R_i for each i. Then $S = \prod_i S_i$ is a weak-algebraic extension of $R = \prod_i R_i$.*

Proof. Let s be a non-zero element of S. Then there exists $i \in I$ such that $(s)_i \neq 0_i$. Since S_i is a weak-algebraic extension of R_i, there exist $r_i^{(k)}$, $k = 0, 1, 2, 3, \ldots, n$, in R_i, $r_i^{(0)} \neq 0_i$, such that

$$\sum_{k=0}^{n} r_i^{(k)} (s)_i^k = 0_i. \tag{3}$$

Define $f^{(k)}$, $k = 0, 1, 2, \ldots, n$, as follows.

$$(f^{(k)})_i = r_i^{(k)}, \quad (f^{(k)})_j = 0_j \quad \text{for all } j \in I, j \neq i.$$

Then $f^{(k)} \in R$ for all k, $f^{(0)} \neq 0$, and by (3), $\sum_k f^{(k)} s^k = 0$. Thus s satisfies a weak-algebraic polynomial equation over R.

Earlier we established the transitivity of essential extensions; we now present a partial result on transitivity for weak-algebraic extensions.

3.5. LEMMA. *Let R be a semiprime ring and let S be a weak-algebraic extension of R. Then $Q(S)$ is a weak-algebraic extension of R.*

Proof. Let q be a non-zero element of $Q(S)$. Then there exist $s_1, s_2 \in S$ such that $qs_1 = s_2 \neq 0$. Since $Q(S)$ is regular, q generates the same principal ideal in $Q(S)$ as does some (unique) idempotent, say e. Then $q = qe$ and $qes_1 = s_2$. Clearly es_1 is non-zero, and so there exist $s_3, s_4 \in S$ such that $es_1s_3 = s_4 \neq 0$, whence $qs_4 = qes_1s_3 = s_2s_3$. If $s_2s_3 = 0$, then $qs_4 = 0$, whence $0 = es_4 = s_4$, a contradiction. Thus $qs_4 = s_2s_3 \neq 0$. Since S is essential over R, there exist $s_5 \in S$, $r \in R$ such that $s_4s_5 = r \neq 0$. Therefore $qr = qs_4s_5 = s_2s_3s_5$. If $qs_4s_5 = 0$, then $0 = es_4s_5 = s_4s_5 = r$, a contradiction. Letting $s = s_2s_3s_5$ we have $qr = s$ a non-zero element of S. Recall that s satisfies a weak-algebraic polynomial equation over R. Substitution of qr in the latter yields a suitable equation for q.

3.6. LEMMA. *Let R be a semiprime ring and let S be a weak-algebraic extension of R. Let T be the integral closure of R in S; then T is an algebraic extension of R and S is a ring of quotients of T.*

Proof. T is essential over R, and it is given to be integral over R; hence it is algebraic over R. Take $x \in S$. Since S is weak-algebraic over R, we have $r_nx^n + \ldots + r_1x + r_0 = 0$, for some $r_i \in R$, $r_0 \neq 0$. Multiplication by r_0 yields $r_0r_nx^n + \ldots + r_0r_1x + r_0^2 = 0$. Now the absolute term is still non-zero, and so it is not possible that all other terms be zero. Let the first non-zero term be $r_0r_mx^m$. Now multiply the equation through by $(r_0r_m)^{m-1}$ to obtain:

$$(r_0r_mx)^m + r_{m-1}r_0(r_0r_mx)^{m-1} + r_{m-2}r_0^2r_m(r_0r_mx)^{m-2} + \ldots$$
$$+ r_1r_0^{m-1}r_m^{m-2}(r_0r_mx) + r_0^{m+1}r_m^{m-1} = 0.$$

Now if $r_0^{m+1}r_m^{m-1} = 0$, then $(r_0r_m)^{m+1} = 0$ whence $r_0r_m = 0$, since R is semiprime, and this would contradict the choice of m. Therefore the absolute term in the equation is not 0. Therefore $r_0r_mx \neq 0$. But the equation states that

$r_0 r_m x$ is integral over R, and so it must lie in T. Thus $0 \neq r_0 r_m x \in T$. Since such an equation is available for each x in S, we invoke [**12**, p. 46, Exercise 5] to conclude that S is a ring of quotients of T.

3.7. Corollary. *Let R be an algebraically closed semiprime ring and let S be a weak-algebraic extension of R. Then S is a ring of quotients of R.*

Proof. The integral closure of R in S is an algebraic extension of R, and must, therefore, coincide with R.

3.8. *Definition.* A ring is *weak-algebraically closed* if it has no proper weak-algebraic extensions. It is obvious that weak-algebraically closed rings are algebraically closed. As well, a weak-algebraically closed ring must be regular since the complete ring of quotients of a semiprime ring is regular.

3.9. Proposition. *Let R be regular; then the following are equivalent:*
(1) *R is weak-algebraically closed;*
(2) *R is self-injective and all monic equations over R have roots in R;*
(3) *R is self-injective and all of its factor fields are algebraically closed;*
(4) *R is self-injective and algebraically closed;*
(5) *R is the complete ring of quotients of an algebraically closed ring.*

Proof. (1) $\Rightarrow$ (2). If R is weak-algebraically closed, then $R = Q(R)$, a self-injective ring [**12**, p. 46, Exercise 6]. Also by 2.7 (2), monics over R have roots in R.

(2) $\Rightarrow$ (3) as in 2.7.

(3) $\Rightarrow$ (4) since self-injective regular rings are Baer.

(4) $\Rightarrow$ (1) by 3.7 and the fact that a semiprime self-injective ring coincides with its maximal ring of quotients.

(5) $\Rightarrow$ (4) by 3.7 and (4) $\Rightarrow$ (5) trivially.

3.10. Corollary. *A Cartesian product of weak-algebraically closed rings is a weak-algebraically closed ring.*

Proof. By [**12**, p. 41, Proposition 8], a product of self-injective regular rings is again regular self-injective. Now invoke 2.8 and 3.9 (4) above.

3.11. Corollary. *If R is weak-algebraically closed, then all quotient objects of R are weak-algebraically closed if and only if R is completely reducible.*

Proof. The quotient rings of R are regular and their factor fields are algebraically closed. Thus the only property at issue is self-injectivity and this proceeds as in 2.17.

We now state the following result due to Storrer.

3.12. Theorem [**15**, 10.1]. *Let R be semiprime and let $m\colon R \to S$ be an essential embedding of R into S. Then if $Q(R)$ and $Q(S)$ are complete rings of quotients of R and S, respectively, there exists an embedding $m'\colon Q(R) \to Q(S)$ such that $m'|R = m$.*

3.13. THEOREM. *Let R be a semiprime ring. Then R has a weak-algebraic closure $\Omega'(R)$; i.e. there exists a weak-algebraically closed ring $\Omega'(R)$ which is a weak-algebraic extension of R. Furthermore, $\Omega'(R)$ is "universal" with respect to weak-algebraic extensions of R.*

Proof. Let $\Omega'(R) = Q(\Omega(R))$. By 3.5, $\Omega'(R)$ is a weak-algebraic extension of R. By 3.9 (5), $\Omega'(R)$ is weak-algebraically closed. Now for universality; let A be a weak-algebraic extension of R, and B the integral closure of R in A. By the universality of the algebraic closure, B can be embedded into $\Omega(R)$. Furthermore, this embedding is essential since the image of B is a between ring of R and $\Omega(R)$. Thus by 3.12, this embedding can be extended to an embedding of A into $\Omega'(R)$ over R.

3.14. COROLLARY. *Any two weak-algebraic closures of R are isomorphic (over R).*

3.15. COROLLARY. *The weak-algebraic closure commutes with arbitrary Cartesian products.*

Proof. 3.4 and 3.10.

3.16. PROPOSITION (*Transitivity*). *Let R be a subring of S and let S be a subring of T; then if T is weak-algebraic over S, and if S is weak-algebraic over R, then T is weak-algebraic over R.*

Proof. Clearly $\Omega'(R) = \Omega'(S)$. Thus there is a copy over R, of T in $\Omega'(R)$, and every element of the copy satisfies a weak-algebraic polynomial equation over R. Since these equations will be preserved under isomorphism over R, every element of T satisfies a weak-algebraic polynomial equation over R.

The following result is now clear.

3.17. LEMMA. *Let R be regular; then the following are equivalent:*
(1) *the algebraic closure of R is self-injective;*
(2) *every weak-algebraic extension of R is algebraic;*
(3) *every weak-algebraic extension is regular.*

3.18. LEMMA. *Let R be regular and let $\Omega(R)$ be its algebraic closure. Then if $\Omega(R)$ is rationally complete, $Q(R)$ is integral over R.*

Proof. If $\Omega(R)$ is rationally complete, then, by 3.12, $\Omega(R)$ contains a copy of $Q(R)$ over R.

We shall see that the converse to this lemma is not true in general.

4. Some examples. Since the two closures Ω and Ω' agree on fields, they also agree on completely reducible rings since the latter are just finite products of fields. In this section we will show that the two closures do differ in general. We begin with an example from the study of rings of continuous functions; our notation is that of Gillman and Jerison [**8**].

4.1. Let X be a P-space without isolated points. These exist by [**8**, p. 193, 13 P]. Then $C(X)$ is a commutative regular ring with 1. If p is an arbitrary point of X, then p defines the maximal ideal $M_p = \{f \in C(X): f(p) = 0\}$ and $C(X)/M_p$ is isomorphic to the real field [**8**, p. 56, 4.6 (a)]. Let $Q(X)$ denote the complete ring of quotients of $C(X)$. Then since $C(X)$ is regular, there is at least one maximal ideal, say N, in $Q(X)$ lying over M_p: It is shown in [**9**, 3.1, 4.2] that if X is any space, then the (totally ordered) factor fields of $Q(X)$ which are isomorphic to the reals are in bijection with the isolated points of X. Therefore in our situation all such factor fields (in particular $Q(X)/N$) are hyper-real. Now if $G = Q(X)/N$ were algebraic over the reals, it would be isomorphic with the reals since G is totally ordered and the reals are real-closed (cf. [**8**, p. 172, (3)]). Since this fails, $Q(X)$ is not an algebraic extension of $C(X)$. But it is a weak-algebraic extension of $C(X)$. From this example we draw a number of conclusions.

4.2. A weak-algebraic extension need not be algebraic.

4.3. An essential extension of a regular ring need not be regular. This follows directly from 2.3 (2).

4.4. A regular Baer ring need not be self-injective. Such a ring is given, for example, by the integral closure of $C(X)$ in $Q(X)$ where X is the space of 4.1.

4.5. In [**7**, 4.3] the rings $Q_L(X)$ and $Q_F(X)$ are introduced and it is shown that the former is the complete ring of quotients of the latter. To obtain $Q_L(X)$ one considers the set of all locally constant continuous real-valued functions whose domains of definition are dense open subsets of the topological space X, and divides out by the equivalence relation which identifies two functions which agree on the intersection of their domains. $Q_F(X)$ is the subring determined by the functions with finite range. Both rings are regular and $Q_F(X)$ is Baer since it contains all the idempotents of $Q_L(X)$. It is not hard to see that the two rings differ if X is the real field in its order topology. Let $g \in Q_L(X)$ and suppose that $g^n + g^{n-1}f_{n-1} + \ldots + f_0 = 0$ for some $f_i \in Q_F(X)$. We may assume that all the functions are defined on the domain D given by the intersection of their individual domains. Each f_i is defined on a finite clopen partition Π_i of D, on the elements of which it is fixed. If Π is the common refinement of the Π_i, then Π is finite and each f_i is fixed on the elements of Π. Now g can only assume a finite number of different values on a given element of Π, since it must satisfy the polynomial above. Therefore g has finite range. Thus $Q_F(X)$ is integrally closed in $Q_L(X)$, therefore the conclusions 4.2 and 4.3 are valid for regular Baer rings. Also it should be noted that $Q_F(X)$ is a nice example of a regular Baer ring which is not self-injective. The algebraic closure of $Q_L(X)$ is discussed in § 6.

4.6. The following example is motivated by model-theoretic considerations. Let F be a finite field of order p^n, p a prime. Let $R = \prod_{\aleph_0} F$. Then R is regular,

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and self-injective and by 3.15, $\Omega'(R) = \prod_{\aleph_0} \Omega(F)$. Let M be a maximal ideal in $\Omega'(R)$ which is defined by a non-principal ultrafilter on the set indexing the Cartesian product of copies of $\Omega(F)$ (cf. [**11**, 8.1]). The field $\Omega'(R)/M$ is an ultrapower of $\Omega(F)$ and since $\Omega(F)$ is countable, $|\Omega'(R)/M| = 2^{\aleph_0}$ by [**11**, 5.6]. Now $R/M \cap R$ is an ultrapower of F and since F is *finite*, so are all of its ultrapowers, in particular so is $R/M \cap R$. Thus, considerations of cardinality show that $\Omega'(R)/M$ is not algebraic over $R/M \cap R$. Therefore $\Omega'(R)$ is not algebraic over R.

4.7. Ω need not commute with infinite Cartesian products.

Proof. 4.6.

4.8. The algebraic closure of a self-injective ring need not be self-injective.

Proof. 4.6.

5. Applications to group rings. We first state a number of results concerning group rings. All can be found in [**12**], except for 5.4, which was mentioned to me by I. Connell.

5.1. The group ring $R = AG$ is regular if and only if A is regular, G is locally finite, and the order of any finite subgroup of G is invertible in A.

5.2. If A is self-injective and G is finite, then R is self-injective.

5.3. If R is self-injective, then A is self-injective and G is locally finite.

5.4. If R is Baer, then A is Baer and the orders of the elements of G are invertible in A.

These results suggest an investigation of when group rings are algebraically closed, and when they are weak-algebraically closed.

5.5. Lemma. *Let $R = AG$, R, A both regular. Then the quotient fields of R are algebraically closed if and only if the quotient fields of A are algebraically closed.*

Proof. Necessity. $R/\Delta \cong A$, where Δ is the augmentation ideal of R.
Sufficiency. Let M be a maximal ideal in R, and consider the field embedding $\bar{A} = A/M \cap A \rightarrow R/M = \bar{R}$. Let g be any element of G. By 5.1, $g^n = 1$ for some integer n. Thus $\bar{g}$ in $\bar{R}$ is algebraic over $\bar{A}$. Thus the field embedding is algebraic, and since $\bar{A}$ is algebraically closed, the embedding is an isomorphism.

5.6. Proposition. *Let $R = AG$, let R and A be regular and let G be finite. Then R is weak-algebraically closed if and only if A is.*

Proof. 5.2, 5.3, 5.5, and 3.9 (3).

5.7. Proposition. *Let $R = AG$ be regular. Then if R is algebraically closed, so is A.*

Proof. 5.4, 5.5, and 2.7.

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5.8. LEMMA. *Let G be a finite group, and $R = AG$, a regular group ring, where A is regular, algebraically closed and of prime characteristic. Then R is algebraically closed.*

Proof. By 5.5 and 2.7, it suffices to show that R is Baer. $Q(R) = Q(A)G$ by [**3**, 3.6]. Let e be an idempotent in $Q(R)$, say

$$e = \sum_{i=1}^{n} q_i g_i, \qquad q_i \in Q(A), \quad g_i \in G, \quad i = 1, 2, \ldots, n.$$

Let p be the characteristic of A. By the binomial theorem, $e = e^{p^\alpha} = \sum_i q_i^{p^\alpha} g_i^{p^\alpha}$, α any positive integer. Thus exponentiation by p^α acts as a permutation on the support of e.

Let g_i be an element in the support of e. By the above remark the set $\{g_i^{p^j}: j = 0, 1, 2, \ldots\}$ lies in the (finite) support of e. Therefore there exist positive integers $m, n, m > n$, such that $g_i^{p^m} = g_i^{p^n}$. Reading off its coefficients in the equation $e^{p^m} = e^{p^n}$ yields the equation $q_i^{p^m} = q_i^{p^n}$. Thus q_i is integral over A. But since A is algebraically closed, it coincides with its integral closure in $Q(A)$. Thus AG contains all idempotents of $Q(A)G$ and is consequently Baer.

As a partial converse to 5.7 we have the following result.

5.9. PROPOSITION. *Let $R = AG$ be a regular group ring. Then R is algebraically closed provided that G is finite and A is algebraically closed and of non-zero characteristic.*

Proof. A is regular and so it has no non-trivial nilpotent elements. In particular, the characteristic of A must be square-free, say $n = \prod p_i$. If $A_i = \{a \in A: ap_i = 0\}$, then one verifies straightforwardly that A_i is a regular ring of characteristic p_i and that $A \cong \prod A_i$. Since A is Baer, each A_i is Baer, whence it is algebraically closed by 2.8. Now $R = AG \cong \prod A_i G$. By 5.8, each $A_i G$ is algebraically closed. By 2.9, R is algebraically closed.

6. Applications to rings of continuous functions. [**6**] is concerned with the study of $C(X)$, the ring of real-valued continuous functions defined on a topological space X. While $C(X)$ is in general far from regular, there are a number of regular rings related to $C(X)$ which arise in [**7**], and the question of determining their algebraic closures is posed. Any modifications made in the notation of [**7**] will be obvious in meaning.

Let $Q_R(X)$ denote the complete ring of quotients of $C(X)$. It can be realized as the set of all continuous real-valued functions defined on dense open subsets of X, modulo the relation which identifies functions which agree on the intersection of their domains. Addition and multiplication are pointwise. Let $Q_C(X)$ denote the ring of all complex-valued continuous functions defined by the same filter, modulo the same relation.

6.1. THEOREM. $\Omega(Q_R(X)) = Q_C(X)$.

Proof. The natural embedding is algebraic. Take $f \in Q_C(X)$, $f \neq 0$. Let a and b denote the purely real and purely complex parts of f; these are continuous real-valued functions on the domain of f because they are the compositions with f of the two projections from C to R. Now $f = a + ib$ and $f^2 - 2af + (a^2 + b^2) = 0$ identically on the domain of f. This is a monic equation in f with coefficients from $Q_R(X)$; furthermore, the absolute term is not zero because f is not the zero function.

$Q_C(X)$ is algebraically closed. The factor fields of $Q_R(X)$ are real closed. This can be seen by noticing that the proof of this theorem for the ring $C(X)$, given in [**8**, p. 175], goes over entirely; simply argue on the intersection of the domains of the finitely many functions discussed. The factor fields of $Q_C(X)$ are algebraic over these real-closed fields and they contain the image of the function which has constant value i. Thus they are algebraically closed [**8**, p. 172]. Since $Q_C(X)$ is Baer, we invoke 2.7 to complete the proof.

The ring $\bar{Q}_R(X)$, presented in [**7**, Chapter 4], is the Dedekind completion of $Q(X)$ and is isomorphic to the ring of continuous functions defined by the filter of dense G_δs of X, modulo the usual relation.

6.2. THEOREM. $\Omega(\bar{Q}_R(X)) = \bar{Q}_C(X)$.

Proof. The fact that $\bar{Q}_C(X)$ has algebraically closed factor fields and that the embedding is algebraic proceeds as in 6.1. $\bar{Q}_C(X)$ is Baer because $\bar{Q}_R(X)$ is rationally complete [**7**, 4.8] and therefore Baer.

The ring ${}_RQ_L(X)$, introduced in [**7**, 4.3], consists of those functions in $Q_R(X)$, which are locally constant on their domains of definition.

6.3. THEOREM. $\Omega[{}_RQ_L(X)] = {}_CQ_L(X)$.

Proof. The composition of a locally constant complex-valued function with either projection to the reals is still locally constant. Thus our embedding is algebraic as in 6.1. Also since ${}_RQ_L(X)$ is Baer [**7**, 4.3], so is ${}_CQ_L(X)$.

Now a monic over ${}_CQ_L(X)$ is one over $Q_C(X)$, so it has a root in $Q_C(X)$, since this ring is algebraically closed (6.1). In fact, the root must lie in ${}_CQ_L(X)$. Specifically, let $g \in Q_C(X)$ satisfy $g^n + f_{n-1}g^{n-1} + \ldots + f_1 g + f_0 = 0$, $f_i \in {}_CQ_L(X)$. Let D be a dense open set in X, common to the domains of definition of the $n + 1$ functions appearing in the equation. To show that g is locally constant, we assume that $g(d) = z$, z some complex number, and show that g equals z on some neighbourhood of d. z is a root of the equation

$$x^n + f_{n-1}(d)x^{n-1} + \ldots + f_1(d)x + f_0(d) = 0. \tag{4}$$

Since each f_i is locally constant, there exist U_i, $i = 0, 1, \ldots, n - 1$, open neighbourhoods of d in D such that f_i is fixed on U_i. Let $U = \bigcap_i U_i$. Then each f_i is fixed on U, and so we conclude that g assumes on U only values among the finitely many roots of (4). If one chooses (as is clearly possible since C is Hausdorff) W, an open set in C containing z and excluding all other roots of (4), then g is constant on the neighbourhood $U \cap g^{-1}(W)$ of p. Thus all monics over ${}_CQ_L(X)$ have roots in ${}_CQ_L(X)$ and by 2.7, ${}_CQ_L(X)$ is algebraically closed.

We turn to the case of continuous functions from a topological space X into a finite field F (topologically discrete) denoting the ring by $C(X, F)$. It is easy to see that compact Hausdorff totally disconnected spaces [**8**, p. 247] suffice for the study of functions to F just as completely regular spaces suffice for rings of real-valued functions.

6.4. PROPOSITION. *Let X be compact, Hausdorff, and totally disconnected. Then $Q(C(X, F)) \cong C(G(X), F)$, where $G(X)$ is the projective cover of X, due to Gleason.*

Proof. From the construction and properties of the projective cover we have a surjection t: $G(X) \to X$, with the property that t maps any proper closed subset of $G(X)$ onto a proper subset of X. As well, $G(X)$ is compact, T_2, and extremally disconnected. It is clear that t induces a ring monomorphism t^*: $C(X, F) \to C(G(X), F)$ under composition.

By [**13**, p. 104, 24.2], $C(G(X), F)$ is rationally complete. One claims that $C(G(X), F)$ is a ring of quotients of $C(X, F)$. Take $f \in C(G(X), F), f \neq 0$, then f defines a partition of $G(X)$ into disjoint clopen sets $A_1, A_2, \ldots, A_n$, where the A_i are the inverse images under f of the different elements of F. Now since f is non-zero, we assume, without loss of generality, that $f(A_1) = d \neq 0$ in F. The set $B = \bigcup_{i=2}^{n} A_i$ is a proper closed set in $G(X)$. Thus $t(B)$ is a proper closed set in X. Let $D = X \diagdown t(B)$. Then D is open and it contains a non-void clopen set, say C. Clearly $t^{-1}(C) \subset A_1$. Consider the function $h \in C(X, F)$ defined as follows: $h(C) = 1$, and $h(X \diagdown C) = 0$. Then $t^*(h) \in C(G(X), F)$ and is 1 on the clopen set $t^{-1}(C)$ in A_1 and zero elsewhere. Thus $ft^*(h)$ is the function which is d on $t^{-1}(C)$. But this is the function $t^*(hd)$. Thus we have $ft^*(h) = t^*(hd) \neq 0$.

6.5. *Remark.* Thus in the case of the ring of functions to a finite field, the complete ring of quotients is always a ring of functions itself. This contrasts strongly with Hager's result [**9**] for the ring of real-valued continuous functions.

6.6. *Remark.* $Q(C(X, F))$ is an algebraic extension of $C(X, F)$. For if n is the order of F, then $r^n = r$ for all $r \in C(X, F)$. If $q \in Q(C(X, F))$, then there is a dense ideal D of $C(X, F)$ such that $qD \subset C(X, F)$. If $d \in D$, then $q^n d = q^n d^n = (qd)^n = qd$. Therefore $(q^n - q)D = (0)$ and $q^n - q = 0$, an equation of integral dependence.

Thus one can restrict the study of the algebraic closure of $C(X, F)$ to the case where X is extremally disconnected.

6.7. THEOREM. *Let X be compact, Hausdorff, and extremally disconnected. Then $\Omega C(X, F) = C(X, \Omega F)$, where $\Omega(F)$ is given the discrete topology.*

Proof. It is clear that $C(X, \Omega F)$, under pointwise addition and multiplication, is a commutative ring with 1, which extends $C(X, F)$. Since X is compact, the elements of $C(X, \Omega F)$ have finite range, and so they are defined on a finite partition of X into clopen sets; from this it is easy to see the regularity of $C(X, \Omega F)$.

$C(X, \Omega F)$ is essential over $C(X, F)$. Take $f \in C(X, \Omega F)$, $f \neq 0$, defined on the partition $X = \bigcup_{i=1}^{n} A_i$, with $f(A_i) = x_i \in \Omega(F)$. Let g be defined on the same partition as follows; if f is 0 on A_i, then so is g; if $f(A_i) \neq 0$, then $g(A_i) = (f(A_i))^{-1}$. Clearly g is in $C(X, \Omega F)$, and fg is defined on a finite partition of X and is 0- or 1-valued. Since $f \neq 0$, it follows that

$$0 \neq fg \in C(X, F)$$

establishing essentiality.

$C(X, \Omega F)$ is integral over $C(X, F)$. Consider an arbitrary finite (clopen) partition of X, say $X = \bigcup_{i=1}^{n} A_i$. Let k_i be the function defined as follows; $k_i(A_i) = x_i$, an arbitrary element of $\Omega(F)$, $k_i(X \setminus A_i) = 0$. Since the element x_i satisfies an integral equation over F, it follows that k_i does as well (as coefficients in the equation for k_i use functions with the appropriate element from F on A_i, and 0 elsewhere). But the sum of integral elements is again integrally dependent, and so the function which has arbitrary values of $\Omega(F)$ assigned to the elements of an arbitrary (finite) partition of X is integral over $C(X, F)$. The set of these functions is precisely $C(X, \Omega F)$.

$C(X, \Omega F)$ is Baer since it is essential over $C(X, F)$, a self-injective ring. One claims that every monic equation over $C(X, \Omega F)$ has a root in $C(X, \Omega F)$. Let

$$x^n + x^{n-1}g_{n-1} + \ldots + xg_1 + g_0 = 0, \qquad g_i \in C(X, \Omega F),$$

be such a monic equation for which one seeks a root. Each g_i is constant on the elements of a finite clopen partition of X. Let Π be the common refinement of all of these partitions. Π is clearly, itself, a finite clopen partition, say $X = \bigcup_j D_j$. Since each g_i is constant on each D_j, and since $\Omega(F)$ is an algebraically closed field, it follows that there is a root, say y_j in $\Omega(F)$, for the equation

$$\sum_{i=0}^{n} x^i g_i(D_j) = 0.$$

The function $y: X \to \Omega(F)$ which has value y_j on D_j is in $C(X, \Omega F)$, and it satisfies the equation in question. By 2.7, $C(X, \Omega F)$ is algebraically closed and the proof of the theorem is complete.

6.8. *Remark.* In view of Stone duality, any complete Boolean algebra can be represented as the ring of continuous functions from its spectrum to the two-element field. Thus 6.7 contains as a special case a representation for the algebraic closure of a complete Boolean ring. Since the embedding of any

Boolean ring into its complete ring of quotients is an algebraic embedding into a complete Boolean ring, it follows that this disposes of the algebraic closure of all Boolean rings.

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