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A NON-ABELIAN 2-GROUP WHOSE ENDOMORPHISMS GENERATE A RING, AND OTHER EXAMPLES OF E-GROUPS

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1. Introduction

Groups for which the distributively generated near-ring generated by the endomorphisms is in fact a ring are known as E-groups and are discussed in (3). R. Faudree in (1) has given the only published examples of non-abelian E-groups by presenting defining relations for a family of p-groups. However, as shown in (3), Faudree's group does not have the desired property when $p = 2$.

In this note, it is shown that most of the groups discussed by D. Jonah and M. Konvisser in (2) are actually E-groups. These groups, described below in Section 2 are proved by Jonah and Konvisser to be such that all their automorphisms are central. Here, it is shown that most of these groups are E-groups by proving that each strict endomorphism (i.e. an endomorphism that is not an automorphism) has its image in the centre of the group. Since one of the groups treated is a 2-group, this paper provides the only published example of a non-abelian 2-group which is an E-group.

E-groups are also discussed in (4) and (5). However, no examples are given in those papers.

2. The Groups of Jonah and Konvisser

The groups treated in (2) are described as follows. Let $\lambda = (\lambda_1, \lambda_2)$ be a vector with integer entries at least one of which is relatively prime to p. Then let $G_\lambda = \langle a_1, a_2, b_1, b_2\rangle$ be the p-group of class 2 with the additional relations:

$$a_1^p = [a_1, b_1],\ a_2^p = [a_1, b^\lambda],\ \text{where } b^\lambda = b_1^{\lambda_1} b_2^{\lambda_2},$$

$$b_1^p = [a_2, b_1 b_2],\ b_2^p = [a_2, b_2],\ \text{and } [a_1, a_2] = [b_1, b_2] = 1.$$

It is noted in (2) that $G = G_\lambda$ has order p^8 and exponent p^2, that $Z(G) = G' = \langle [a_1, b_1], [a_1, b_2], [a_2, b_1], [a_2, b_2]\rangle$ is elementary abelian of order p^4, and that G_μ and G_λ are isomorphic if and only if $\mu = k\lambda$ for some k relatively prime to p. Thus, for each prime p, there are $p+1$ non-isomorphic groups given as λ varies over the set $\{(0, 1), (1, 0), (1, 1), \ldots, (1, p-1)\}$. However, there is some difficulty with the defining relations when $\lambda = (1, 0)$ since $a_1^p = a_2^p$ so that G_λ does not have order p^8. Therefore, that case will not be considered in this paper.

If $p = 2$ and $\lambda = (1, 1)$, then application of the defining relations shows that $(a_1 a_2 b_2)^2 = 1$. Thus $G^2 \neq G'$. But Theorem 2 of (3) says that, in our context, $G^2 = G'$ is a necessary condition for G to be an E-group. Therefore, when $p = 2$, the case of $\lambda = (1, 1)$ will also not be considered in this paper.

In (2) it is noted that the normal subgroups $A = \langle a_1, a_2, Z(G)\rangle$ and $B = \langle b_1, b_2, Z(G)\rangle$ are the only abelian subgroups of order p^2 over the centre. Furthermore, $A^p \leq [x, G]$ for some x in A while there is no y in B such that $B^p \leq [y, G]$. For F a p-group as given in (1) by Faudree, Lemma 6 of (1) implies that the centralisers $C_F(a_2)$ and $C_F(a_3)$ are the only abelian subgroups of order p^2 over the centre. Thus, if F and G are to be isomorphic, the two centralisers in F must correspond, in some order, to A and B in G. From Lemma 6 we also have that any element of order p^2 in $C_F(a_2)$ has the form $a_2^r a_4^s$ and an element of order p^2 in $C_F(a_3)$ has the form $a_1^r a_3^s a_4^r$ with at least one of r and s relatively prime to p. But, the defining relations in F indicate there is no y in $C_F(a_2)$ such that $\langle a_2^p, a_4^p\rangle = (C_F(a_2))^p \leq [y, F]$ and no x in $C_F(a_3)$ such that $\langle a_3^p, a_1^p a_4^p\rangle = (C_F(a_3))^p \leq [x, F]$. Hence F and G are not isomorphic since neither centraliser of order p^6 in F can correspond to A in G.

3. The Strict Endomorphisms

The groups described in Section 2 for which p is odd and λ is in $\{(0, 1), (1, 1), (1, 2), \ldots, (1, p-1)\}$ or $p = 2$ and $\lambda = (0, 1)$ will be referred to as JK-groups.

Lemma 1. *For any JK-group, $Z(G) = G' = G^p = U_p(G)$, where $U_p(G)$ is the set of elements whose order divides p.*

Proof. From the defining relations it is immediate that $G' = \langle a_1^p, a_2^p, b_1^p, b_2^p\rangle = G^p$, $G' \leq Z(G)$, and $G' \leq U_p(G)$ with $|G| = p^8$ and $|G'| = p^4$. If $|Z(G)| \geq p^5$, then $|G/Z(G)| \leq p^3$. But $G/Z(G)$ is generated by $\{a_1Z(G), a_2Z(G), b_1Z(G), b_2Z(G)\}$. So two generators of G are congruent mod $Z(G)$ and $G/Z(G)$ is generated by at most three elements. Thus $|G'| \leq p^3$, a contradiction. Hence $Z(G) = G'$. Also, for x and y in G, $(xy)^p = x^p y^p [y, x]^{p(p-1)/2}$ by Proposition VI.1.k(4) of (6). Since, for odd p, G' has exponent p, it follows that $(xy)^p = x^p y^p$. Then $1 = (a_1^r a_2^s b_1^t b_2^u)^p = a_1^{rp} a_2^{sp} b_1^{tp} b_2^{up}$ implies p divides each of r, s, t, u so that $a_1^r a_2^s b_1^t b_2^u$ is in G^p. Hence $U_p(G) = G^p$ for odd p. For $p = 2$ we show directly from the generating relations that $G^2 = U_2(G)$. *Let* $g = a_1^{m_1} a_2^{m_2} b_1^{n_1} b_2^{n_2}$ be an element of order 2 in G. Then $e = g^2 = a_1^{2m_1(1+n_1)} a_2^{2(m_2+n_2m_1)} b_1^{2n_1(1-m_2)} b_2^{2(n_2+n_2m_2+n_1m_2)}$. Since g has order 2, each exponent in g^2 must be divisible by 4. If m_2 is odd, then the exponent of a_2 shows that n_2 and m_1 must also be odd. Then, from the exponent of a_1, n_1 must be odd. But now, the b_2 exponent is not divisible by 4. If m_2 is even, then m_1, n_1, and n_2 must also be even. Hence $U_2(G) \leq G^2$ and the Lemma is proved.

Lemma 2. *Let $m_1, m_2, n_1, n_2, \lambda_1, \lambda_2$ be integers modulo p and λ_2^{-1} be the inverse of λ_2 in the field of order p. Then the matrix A has rank 2 or greater over the field of order p if at least one of m_1, m_2, n_1, n_2 is not congruent to 0 modulo p.*

$$A = \begin{bmatrix} m_1 - \lambda_1\lambda_2^{-1}m_2 & \lambda_2^{-1}m_2 & 0 & 0 \\ 0 & 0 & m_1 & -m_1 + m_2 \\ -n_1 & 0 & -n_2 & n_2 \\ \lambda_1\lambda_2^{-1}n_1 & -\lambda_2^{-1}n_1 & 0 & -n_2 \end{bmatrix}$$

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Proof. If either m_1 or $m_2 \not\equiv 0(\text{mod. } p)$, the first two rows of A are linearly independent. If either n_1 or $n_2 \not\equiv 0(\text{mod. } p)$, the last two rows of A are linearly independent.

Theorem. *Any JK-group is an E-group.*

Proof. In (2) it is shown that all automorphisms of JK-groups are central. The theorem will be established if it is demonstrated that any strict endomorphism of one of these groups has its image in the centre of the group. This will be shown by arguing as was done in Lemma 5 of (1).

Let θ be a strict endomorphism of G. Then $(h)\theta \in G'$ for some $h \notin G'$ and $h = a_1^{n_1} a_2^{n_2} b_1^{m_1} b_2^{m_2}$ with at least one exponent $\not\equiv 0(\text{mod } p)$. Also, $([c, h])\theta = [(c)\theta, (h)\theta] = 1$ so that

$\langle [c, h] : c \in \{a_1, a_2, b_1, b_2\} \rangle \leqslant \text{Ker } \theta$. Note that

$$\begin{aligned}
[a_1, h] &= a_1^{pm_1 - p\lambda_1\lambda_2^{-1}m_2} a_2^{p\lambda_2^{-1}m_2}, \\
[a_2, h] &= b_1^{pm_1} b_2^{p(m_2 - m_1)}, \\
[b_1, h] &= a_1^{-pn_1} b_1^{-pn_2} b_2^{pn_2}, \\
[b_2, h] &= a_1^{p\lambda_1\lambda_2^{-1}n_1} a_2^{-p\lambda_2^{-1}n_1} b_2^{-pn_2}.
\end{aligned} \tag{$*$}$$

The matrix of the powers of a_1^p, a_2^p, b_1^p, b_2^p in $(*)$ is A. Hence $|\text{Ker } \theta \cap G'| \geqslant p^2$ and there exists $\{h_i : 1 \leqslant i \leqslant 4\}$ such that $G = \langle h_1, h_2, h_3, h_4 \rangle$ and $(h_1^p)\theta = (h_2^p)\theta = 1$. Thus for $i = 1$ or 2, $(h_i)\theta \in G'$ and $((G)\theta)' = \langle (h_3)\theta, (h_4)\theta \rangle'$. Hence $|(G')\theta| = |((G)\theta)'| \leqslant p$ and $|\text{Ker } \theta \cap G'| \geqslant p^3$. We can then additionally assume that $(h_3^p)\theta = 1$ and $(h_3)\theta \in G'$. But now, $((G)\theta)' = \langle 1 \rangle$. Hence $(G)\theta$ is abelian, $G' \leqslant \text{Ker } \theta$ and $(G)\theta \leqslant G' = Z(G)$.

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