

MODERATE DEVIATIONS OF MANY-SERVER QUEUES VIA IDEMPOTENT PROCESSES – CORRIGENDUM

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A correction to Section 4 in [1] is provided.

The expressions for $I_{x_0}^Q(q)$ in the statement of Theorem 5 and for $\bar{I}_{q_0, x_0}^Q(q)$ in [1] are incorrect as the solutions of the associated variational problems do not account for the constraints $\int_0^1 \dot{w}^0(x) dx = 0$ and $\int_0^1 \dot{k}(x, t) dx = 0$. The contents of Section 4 should be replaced with the following material.

4. Evaluating the deviation functions

This section is concerned with solving for $I_{x_0}^Q(q)$ and $\bar{I}_{q_0, x_0}^Q(q)$.

Theorem 5. Suppose that the c.d.f. F is an absolutely continuous function and $I_{x_0}^Q(q) < \infty$. Then q is absolutely continuous, $(\dot{q}(t) - \int_0^t \dot{q}(s) \mathbf{1}_q(s) > 0 F'(t-s) ds, t \in \mathbb{R}_+) \in \mathbb{L}_2(\mathbb{R}_+)$, the infimum in (2.5) is attained uniquely and

$$I_{x_0}^Q(q) = \frac{1}{2} \int_0^\infty \hat{p}(t) \left(\dot{q}(t) - \int_0^t \dot{q}(s) \mathbf{1}_q(s) > 0 F'(t-s) ds + (\beta - x_0^-) F'_0(t) \right) dt,$$

where $\hat{p}(t)$ represents the unique $\mathbb{L}_2(\mathbb{R}_+)$ solution $p(t)$ of the Fredholm equation of the second kind,

$$\begin{aligned} (\mu + \sigma^2)p(t) = & \dot{q}(t) - \int_0^t \dot{q}(s) \mathbf{1}_q(s) > 0 F'(t-s) ds + (\beta - x_0^-) F'_0(t) \\ & + F'_0(t) \int_0^\infty F'_0(s)p(s) ds + \sigma^2 \int_0^\infty F'(|s-t|)p(s) ds \\ & + (\mu - \sigma^2) \int_0^\infty \int_0^{s \wedge t} F'(s-\tilde{s})F'(t-\tilde{s}) d\tilde{s} p(s) ds, \end{aligned} \quad (4.1)$$

with $\dot{q}$, F'_0 and F' representing derivatives.

Proof. Using that

$$\int_{\mathbb{R}_+^2} \mathbf{1}_{x+s \leq t} \dot{k}(F(x), \mu s) dF(x) \mu ds = \int_0^t \int_0^{F(t-s)} \dot{k}(x, \mu s) dx \mu ds, \quad (4.2)$$

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(2.6) can be expressed in the form

$$q(t) = f(t) + \int_0^t q(t-s)^+ dF(s), \quad t \in \mathbb{R}_+,$$

with the functions $f(t)$ and $F(t)$ being absolutely continuous. The function $q(t)$ is absolutely continuous by Lemma 8. In addition, (4.2) implies that, almost everywhere,

$$\frac{d}{dt} \int_{\mathbb{R}_+^2} \mathbf{1}_x + s \leq t \dot{k}(F(x), \mu s) dF(x) \mu ds = \int_0^t \dot{k}(F(s), \mu(t-s)) F'(s) \mu ds. \quad (4.3)$$

The infimum in (2.5) is attained uniquely by coercitivity and strict convexity of the function being minimised, cf., Proposition II.1.2 in [5]. By differentiating equation (2.6), in light of equation (4.3), we find that, almost everywhere,

$$\begin{aligned} \dot{w}^0(F_0(t)) F'_0(t) + \sigma \dot{w}(t) - \int_0^t F'(t-s) \sigma \dot{w}(s) ds + \int_0^t \dot{k}(F(s), \mu(t-s)) F'(s) \mu ds \\ - \left(\dot{q}(t) - \int_0^t \dot{q}(s) \mathbf{1}_{q(s) > 0} F'(t-s) ds + (\beta - x_0^-) F'_0(t) \right) = 0. \end{aligned}$$

In addition, the requirements that $w^0(0) = w^0(1) = 0$ and $k(0, t) = k(1, t) = 0$ give rise to the constraints

$$\int_0^1 \dot{w}^0(x) dx = 0 \quad (4.4)$$

and

$$\int_0^1 \dot{k}(x, t) dx = 0. \quad (4.5)$$

Introduce the map

$$\begin{aligned} \Phi: (\dot{w}^0, \dot{w}, \dot{k}) \rightarrow \left(\dot{w}^0(F_0(t)) F'_0(t) + \sigma \dot{w}(t) - \int_0^t F'(t-s) \sigma \dot{w}(s) ds \right. \\ \left. + \int_0^t \dot{k}(F(s), \mu(t-s)) F'(s) \mu ds, t \in \mathbb{R}_+ \right). \end{aligned}$$

Since $F'_0(t)$ is bounded by (2.1), Φ maps $V = \mathbb{L}_2([0, 1]) \times \mathbb{L}_2(\mathbb{R}_+) \times \mathbb{L}_2([0, 1] \times \mathbb{R}_+)$ to $\mathbb{L}_2(\mathbb{R}_+)$. For instance, on using that $\int_0^\infty F'(s) ds = 1$,

$$\int_0^\infty \left(\int_0^t F'(t-s) \dot{w}(s) ds \right)^2 dt \leq \int_0^\infty \int_0^t F'(t-s) \dot{w}(s)^2 ds dt = \int_0^\infty \dot{w}(s)^2 ds < \infty$$

and

$$\begin{aligned} \int_0^\infty \left(\int_0^t \dot{k}(F(s), \mu(t-s)) F'(s) \mu ds \right)^2 dt &\leq \int_0^\infty \int_0^t \dot{k}(F(s), \mu(t-s))^2 F'(s) \mu^2 ds dt \\ &= \mu^2 \int_0^\infty \int_0^1 \dot{k}(x, t)^2 dx dt < \infty. \end{aligned}$$

By Lagrange multipliers (see, e.g., Proposition III.5.2 in [5]) with $Y = \mathbb{L}_2(\mathbb{R}_+)^2 \times \mathbb{R}$ and the set of componentwise nonnegative functions as the cone $\mathcal{C}$,

$$I_{x_0}^Q(q) = \sup_{(p, \tilde{p}, r) \in \mathbb{L}_2(\mathbb{R}_+)^2 \times \mathbb{R}} \inf_{\substack{(\dot{w}^0, \dot{w}, \dot{k}) \in \mathbb{L}_2([0, 1]) \\ \times \mathbb{L}_2(\mathbb{R}_+) \times \mathbb{L}_2([0, 1] \times \mathbb{R}_+)}} \left(\frac{1}{2} \int_0^1 \dot{w}^0(x)^2 dx + \frac{1}{2} \int_0^\infty \dot{w}(t)^2 dt \right. \\ \left. + \frac{1}{2} \int_0^\infty \int_0^1 \dot{k}(x, t)^2 dx dt + \int_0^\infty p(t) \left(\dot{q}(t) + F'(t)x_0^+ + (\beta - x_0^-)F'_0(t) \right. \right. \\ \left. \left. - \int_0^t \dot{q}(s) \mathbf{1}_{q(s) > 0} F'(t-s) ds - \dot{w}^0(F_0(t))F'_0(t) - \sigma \dot{w}(t) + \int_0^t F'(t-s) \sigma \dot{w}(s) ds \right. \right. \\ \left. \left. - \int_0^t \dot{k}(F(s), \mu(t-s))F'(s) \mu ds \right) dt + r \int_0^1 \dot{w}^0(x) dx + \int_0^\infty \tilde{p}(t) \int_0^1 \dot{k}(x, t) dx dt \right). \quad (4.6)$$

Minimising in (4.6) yields, with $(\dot{w}^0(t), \dot{w}(t), \dot{k}(x, t))$ being optimal,

$$\begin{aligned} \dot{w}^0(x) - p(F_0^{-1}(x)) + r &= 0, \\ \dot{w}(t) - \sigma p(t) + \sigma \int_0^\infty p(t+s)F'(s) ds &= 0, \\ \dot{k}(x, t) - p\left(\frac{t}{\mu} + F^{-1}(x)\right) + \tilde{p}(t) &= 0. \end{aligned}$$

For the latter, note that

$$\begin{aligned} \int_0^\infty p(t) \int_0^t \dot{k}(F(s), \mu(t-s))F'(s) \mu ds dt &= \int_0^\infty \int_s^\infty p(t) \dot{k}(F(s), \mu(t-s))F'(s) \mu dt ds \\ &= \int_0^\infty \int_0^\infty p(t+s) \dot{k}(F(s), \mu t)F'(s) \mu dt ds = \int_0^\infty \int_0^1 p\left(\frac{t}{\mu} + F^{-1}(x)\right) \dot{k}(x, t) dx dt. \end{aligned}$$

Hence,

$$\begin{aligned} I_{x_0}^Q(q) &= \sup_{(p, \tilde{p}, r) \in \mathbb{L}_2(\mathbb{R}_+)^2 \times \mathbb{R}} \left(\int_0^\infty p(t) \left(\dot{q}(t) - \int_0^t \dot{q}(s) \mathbf{1}_{q(s) > 0} F'(t-s) ds + (\beta - x_0^-)F'_0(t) \right) dt \right. \\ &\quad \left. - \frac{1}{2} \left(\int_0^1 (p(F_0^{-1}(x)) - r)^2 dx + \sigma^2 \int_0^\infty \left(p(t) - \int_0^\infty p(t+s)F'(s) ds \right)^2 dt \right. \right. \\ &\quad \left. \left. + \int_0^\infty \int_0^1 \left(p\left(\frac{t}{\mu} + F^{-1}(x)\right) - \tilde{p}(t) \right)^2 dx dt \right) \right). \end{aligned}$$

Given p , the optimal r is $\hat{r} = \int_0^1 p(F_0^{-1}(x)) dx$ and the optimal $\tilde{p}(t)$ is $\hat{\tilde{p}}(t) = \int_0^1 p(t/\mu + F^{-1}(x)) dx$. (As a by-product, $\hat{w}^0$ and $\hat{k}$ satisfy the constraints in (4.4) and (4.5).) Therefore,

$$\begin{aligned} I_{x_0}^Q(q) &= \sup_{p \in \mathbb{L}_2(\mathbb{R}_+)} \left(\int_0^\infty p(t) \left(\dot{q}(t) - \int_0^t \dot{q}(s) \mathbf{1}_{q(s) > 0} F'(t-s) ds + (\beta - x_0^-)F'_0(t) \right) dt \right. \\ &\quad \left. - \frac{1}{2} \left(\int_0^1 (p(F_0^{-1}(x)) - \int_0^1 p(F_0^{-1}(\tilde{x})) d\tilde{x})^2 dx + \sigma^2 \int_0^\infty \left(p(t) - \int_0^\infty p(t+s)F'(s) ds \right)^2 dt \right. \right. \\ &\quad \left. \left. + \int_0^\infty \int_0^1 \left(p\left(\frac{t}{\mu} + F^{-1}(x)\right) - \int_0^1 p\left(\frac{t}{\mu} + F^{-1}(\tilde{x})\right) d\tilde{x} \right)^2 dx dt \right) \right). \quad (4.7) \end{aligned}$$

As the function in the sup is strictly concave in p , a maximiser in (4.7) is specified uniquely; see, e.g., Proposition II.1.2 in [5]. The existence of the maximiser follows from Proposition III.5.2 in [5].

The function in the sup can be simplified if one notes that

$$\begin{aligned} \int_0^1 (p(F_0^{-1}(x)) - \int_0^1 p(F_0^{-1}(\tilde{x})) d\tilde{x})^2 dx &= \int_0^\infty p(s)^2 F'_0(s) ds - \left(\int_0^\infty p(s) F'_0(s) ds \right)^2, \\ \int_0^1 \left(p\left(\frac{t}{\mu} + F^{-1}(x)\right) - \int_0^1 p\left(\frac{t}{\mu} + F^{-1}(\tilde{x})\right) d\tilde{x} \right)^2 dx \\ &= \int_0^\infty p\left(\frac{t}{\mu} + s\right)^2 F'(s) ds - \left(\int_0^\infty p\left(\frac{t}{\mu} + s\right) F'(s) ds \right)^2, \end{aligned}$$

and that

$$\int_0^\infty p(s)^2 F'_0(s) ds + \mu \int_0^\infty \int_0^\infty p(t+s)^2 F'(s) ds dt = \mu \int_0^\infty p(s)^2 ds.$$

As a result,

$$\begin{aligned} I_{x_0}^Q(q) &= \sup_{p \in \mathbb{L}_2(\mathbb{R}_+)} \left(\int_0^\infty p(t) \left(\dot{q}(t) - \int_0^t \dot{q}(s) \mathbf{1}_{q(s) > 0} F'(t-s) ds + (\beta - x_0^-) F'_0(t) \right) dt \right. \\ &\quad - \frac{1}{2} \left(\mu \int_0^\infty p(s)^2 ds - \left(\int_0^\infty p(s) F'_0(s) ds \right)^2 + \sigma^2 \int_0^\infty \left(p(t) - \int_0^\infty p(t+s) F'(s) ds \right)^2 dt \right. \\ &\quad \left. \left. - \mu \int_0^\infty \left(\int_0^\infty p(t+s) F'(s) ds \right)^2 dt \right) \right). \quad (4.8) \end{aligned}$$

Varying p in (4.8) implies (4.1). As the maximiser in (4.8) is unique, so is an $\mathbb{L}_2(\mathbb{R}_+)$ solution of the Fredholm equation (4.1).

It is noteworthy that the integral operator on $\mathbb{L}_2(\mathbb{R}_+)$ that appears on the right-hand side of (4.1) is not generally either Hilbert–Schmidt or compact, so the existence and uniqueness of $\hat{p}(t)$ is not a direct consequence of the general theory. Solving Fredholm equations numerically, such as (4.1), is discussed at quite some length in the literature. For instance, the collocation method with a basis of ‘hat’ functions could be tried: for $i \in \mathbb{N}$ and $n \in \mathbb{N}$, let $t_i = i/n$ and $\ell_i(t) = (1 - |t - t_i|) \mathbf{1}_{t_{i-1} \leq t \leq t_i}$, with $t_0 = 0$. Then an approximate solution is

$$p_n(t) = \sum_{i=1}^{n^2} p_n(t_i) \ell_i(t)$$

where the $p_n(t_i)$, $i = 1, \dots, n^2$, satisfy the linear system

$$\begin{aligned} (\mu + \sigma^2) p_n(t_i) - \sigma^2 \sum_{j=1}^{n^2} p_n(t_j) \int_0^{t_i} \tilde{K}(t_i, s) \ell_j(s) ds \\ = \dot{q}(t_i) - \int_0^{t_i} \dot{q}(s) \mathbf{1}_{q(s) > 0} F'(t_i - s) ds + (\beta - x_0^-) F'_0(t_i), \end{aligned}$$

with $\tilde{K}$ denoting the kernel of the integral operator in (4.1). For more background, see [3] and references therein.

Evaluating $\bar{I}_{q_0, x_0}^Q$ is done similarly:

$$\begin{aligned} \bar{I}_{q_0, x_0}^Q(q) = & \sup_{(p, \tilde{p}, r) \in \mathbb{L}_2(\mathbb{R}_+)^2 \times \mathbb{R}} \inf_{\substack{(\dot{w}^0, \dot{w}, \dot{k}) \in \mathbb{L}_2([0, 1]) \\ \times \mathbb{L}_2(\mathbb{R}_+) \times \mathbb{L}_2([0, 1] \times \mathbb{R}_+)}} \left(\frac{1}{2} \int_0^1 \dot{w}^0(x)^2 dx + \frac{1}{2} \int_0^\infty \dot{w}(t)^2 dt \right. \\ & + \frac{1}{2} \int_0^\infty \int_0^1 \dot{k}(x, t)^2 dx dt + \int_0^\infty p(t) \left(\dot{q}(t) + x_0 F'_0(t) - \sqrt{q_0} \dot{w}^0(F_0(t)) F'_0(t) - \sigma \dot{w}(t) \right. \\ & \left. \left. + \int_0^t F'(t-s) \sigma \dot{w}(s) ds - \int_0^t \dot{k}(F(s), \lambda(t-s)) F'(s) \lambda ds \right) dt + r \int_0^1 \dot{w}^0(x) dx \right. \\ & \left. \left. + \int_0^\infty \tilde{p}(t) \int_0^1 \dot{k}(x, t) dx dt \right). \end{aligned}$$

Calculations as previously imply that

$$\begin{aligned} \bar{I}_{q_0, x_0}^Q(q) = & \sup_{p \in \mathbb{L}_2(\mathbb{R}_+)} \left(\int_0^\infty p(t) (\dot{q}(t) + x_0 F'_0(t)) dt - \frac{1}{2} \left(q_0 \int_0^\infty p(s)^2 F'_0(s) ds \right. \right. \\ & \left. \left. - q_0 \left(\int_0^\infty p(s) F'_0(s) ds \right)^2 + \lambda \int_0^\infty p(t+s)^2 F'(s) ds - \lambda \left(\int_0^\infty p(t+s) F'(s) ds \right)^2 \right. \right. \\ & \left. \left. + \sigma^2 \int_0^\infty \left(p(t) - \int_0^\infty p(t+s) F'(s) ds \right)^2 dt \right) \right) = \frac{1}{2} \int_0^\infty \bar{p}(t) (\dot{q}(t) + x_0 F'_0(t)) dt, \end{aligned}$$

with $\bar{p}(t)$ being the $\mathbb{L}_2(\mathbb{R}_+)$ solution $p(t)$ to the Fredholm equation of the second kind

$$\begin{aligned} (\lambda + \sigma^2) p(t) = & \dot{q}(t) + x_0 F'_0(t) + q_0 F'_0(t) \int_0^\infty F'_0(s) p(s) ds + \sigma^2 \int_0^\infty F'(|s-t|) p(s) ds \\ & + (\lambda - \sigma^2) \int_0^\infty \int_0^{s \wedge t} F'(s-\tilde{s}) F'(t-\tilde{s}) d\tilde{s} p(s) ds. \end{aligned}$$

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

References

- [1] PUHALSKII, A. (2024). Moderate deviations of many-server queues via idempotent processes. *Adv. Appl. Prob.* Published online 20 December 2024 DOI: <https://doi.org/10.1017/apr.2024.62>