Canad. Math. Bull. Vol. 18 (1), 1975

SOME ADMISSIBLE ESTIMATORS IN EXTREME VALUE DENSITIES

BY
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Let X be a random variable having the extreme value density of the form

$$f(x;\theta)=\begin{cases} q(\theta)r(x), & 0\le x\le \theta \\ 0\ , & \text{otherwise}\end{cases} \tag{1}$$

where r is assumed to be a positive Lebesgue measurable function of x and the function q is defined by

$$1/q(\theta)=\int_0^\theta r(x)\,dx<\infty$$

for all θ in $\Omega=(0,\infty)$. It is further assumed that $q(\theta)$ approaches zero as $\theta\to\infty$.

In this note we are concerned with estimating parametric functions $g(\theta)$ of the form $[1/q(\theta)]^\alpha$, α any real number. The loss function is assumed to be squared error and the estimators are assumed to be functions of a single observation X. The case of estimators based on a sample of size $n\geq 1$ is discussed in Remark 1.

In our search for a 'good' estimator for $g(\theta)=[1/q(\theta)]^\alpha$ we calculate $E[1/q(X)]^\alpha=\int_0^\theta[1/q(x)]^\alpha q(\theta)r(x)\,dx$. Since $r(x)=-q'(x)/q^2(X)$ almost everywhere we find that for every $\alpha>-1$, $E[1/q(X)]^\alpha$ exists and is given by $E[1/q(X)]^\alpha=(1/\alpha+1)[1/q(\theta)]^\alpha$. This leads us to consider the class $\Lambda_\alpha=\{\delta_K(X)=K[1/q(X)]^\alpha : K \text{ real}\}$ of estimators, which are constant multiples of $[1/q(X)]^\alpha$, for estimating the given parametric function $[1/q(\theta)]^\alpha$. Which of these estimators in Λ_α has the smallest risk uniformly for all θ in Ω? Since $E[1/q(X)]^l=[1/(l+1)]^2[1/q(\theta)]^l$ if $l>-1$ and $=\infty$ if $l\leq -1$, it follows easily that for any δ_K in Λ_α,

$$\begin{aligned} R(\delta_K,\theta) &= E[K(1/q(X))^\alpha-(1/q(\theta))^\alpha]^2 \\ &= \begin{cases} [1/q(\theta)]^{2\alpha}, & K=0, \quad \text{all } \alpha \\ \left(\dfrac{K^2}{2\alpha+1}-\dfrac{2K}{\alpha+1}+1\right)[1/q(\theta)]^{2\alpha}, & \alpha>-\tfrac{1}{2}, \quad \text{all } K \\ \infty, \quad \alpha\leq -\tfrac{1}{2}, & K\neq 0 \end{cases}\end{aligned} \tag{2}$$

where throughout this paper ∞ stands for $+\infty$. If $\alpha>-\frac{1}{2}$, the quadratic expression $[K^2/(2\alpha+1)]-[2K/(\alpha+1)]+1$ in K achieves its minimum at $K=(2\alpha+1)/(\alpha+1)$. It follows from this that for estimating $[1/q(\theta)]^\alpha$, $\alpha>-\frac{1}{2}$, the minimum risk estimator in Λ_α is $T_\alpha(X)=[(2\alpha+1)/(\alpha+1)][1/q(X)]^\alpha$ corresponding to $K=(2\alpha+1)/$

Received by the editors July 7, 1972 and, in revised form, October 15, 1973.

$(\alpha+1)$ with risk

(3) $$R(\delta_K, \theta) = [\alpha/(\alpha+1)]^2[1/q(\theta)]^{2\alpha}.$$

Is T_α an admissible estimator of $[1/q(\theta)]^\alpha$ for all α? We have the following

THEOREM 1. *Let the random variable X have density* (1) *and let the loss be quadratic. Then the estimator*

$$T_\alpha(X) = \frac{2\alpha+1}{\alpha+1}[1/q(X)]^\alpha$$

is admissible for estimating $[1/q(\ )]^\alpha$ *for every* $\alpha > -\frac{1}{2}$ *and is inadmissible for all* $\alpha \leq -\frac{1}{2}$.

Proof. Assume $\alpha > -\frac{1}{2}$. Let T be any estimator satisfying the inadmissibility inequality for T_α:

(4) $$E[T-(1/q(\theta))^\alpha]^2 \leq E[T_\alpha-(1/q(\theta))^\alpha]^2$$

Writing $m(\theta)$ for $E(T)$ and $m^*(\theta)$ for $E(T_\alpha)$ we have the following equivalent inequalities:

(5) $$E[T-m(\theta)]^2+[m(\theta)-(1/q(\theta))^\alpha]^2 \leq [\alpha/(\alpha+1)]^2[q(\theta)]^{-2\alpha},$$

and

(6) $$E(T-T_\alpha]^2+2E\{[T-T_\alpha][T_\alpha-(1/q(\theta))^\alpha]\} \leq 0.$$

Inequality (5) implies that

$$[m(\theta)-(1/q(\theta))^\alpha]^2 \leq [\alpha/(\alpha+1)]^2[1/q(\theta)]^{2\alpha}$$

from which we get the bounds for the function m as

$$\frac{2\alpha+1}{\alpha+1}\frac{1}{[q(\theta)]^\alpha} \leq m(\theta) \leq \frac{1}{\alpha+1}\frac{1}{[q(\theta)]^\alpha} \quad \text{if} \quad -\tfrac{1}{2} < \alpha < 0$$

(7) $$\frac{1}{\alpha+1}\frac{1}{[q(\theta)]^\alpha} \leq m(\theta) \leq \frac{2\alpha+1}{\alpha+1}\frac{1}{[q(\theta)]^\alpha} \quad \text{if} \quad \alpha \geq 0.$$

Since $1/q(\theta)$ tends to zero as θ tends to zero, it is clear from (7) that $m(\theta)/[q(\theta)]^{\delta-\alpha} \to 0$ for every $\delta > 0$. Now the hypothesis $\alpha > -\frac{1}{2}$ guarantees some $\delta > 0$ such that $\alpha = (\delta/2)-(\frac{1}{2})$ i.e., $2\alpha+1=\delta$, i.e., $\alpha+1=\delta-\alpha$. Thus it follows that

(8) $$m(\theta)/[q(\theta)]^{\alpha+1} \to 0 \quad \text{as} \quad \theta \to 0$$

The rest of the proof consists in showing that the only solution of the inadmissibility inequality (6) is $m=m^*$. For this it is enough to show that $m=m^*$ is the only solution to the inequality

(9) $$[m(\theta)-m^*(\theta)]^2+2E\{[T-T_\alpha][T_\alpha-(1/q(\theta))^\alpha]\} \leq 0$$

which is relaxation of (6) obtained after replacing its LHS by something smaller. But (9) still has T in it. To express it in terms of m we use the identity $m(\theta)=q(\theta)\int_0^\theta T(x)r(x)\,dx$ to provide us the relation

$$T(x) = \frac{m'(x)}{q(x)r(x)}+m(x) \tag{10}$$

Substituting this value of T in (9) and performing the expectation of the expression therein, we obtain the inequality

$$[m(\theta)-m^*(\theta)]^2+\frac{2\alpha}{\alpha+1}\frac{m(\theta)}{[q(\theta)]^\alpha}-\frac{2\alpha(2\alpha+1)}{\alpha+1}E\left\{\frac{m(X)}{[q(X)]^\alpha}\right\}\leq 0 \tag{11}$$

where in this derivation integration by parts and result (8) is used. This inequality still contains the integral $E\{m(X)/[q(X)]^\alpha\}$. If we write

$$u(\theta) = E\left\{\frac{m(X)}{[q(X)]^\alpha}\right\} = q(\theta)\int_0^\theta \frac{m(x)}{[q(x)]^\alpha}\,r(x)\,dx$$

we have

$$m(\theta) = u(\theta)q^\alpha(\theta)-\left[\frac{q^{\alpha+1}(\theta)}{q'(\theta)}\right]u'(\theta). \tag{12}$$

Introducing $u(\theta)$ in (11) we have the inequality

$$[m(\theta)-m^*(\theta)]^2-\left[\frac{4\alpha^2}{\alpha+1}\right]u(\theta)-\frac{2\alpha}{\alpha+1}\left[\frac{q(\theta)}{q'(\theta)}\right]u'(\theta)\leq 0 \tag{13}$$

wherein $m(\theta)$ is to be replaced by its value in terms of $u(\theta)$ from (12). It is now shown that $u^*(\theta)=[1/(1+\alpha)^2][1/q(\theta)]^{2\alpha}$, corresponding to $m=m^*$, is the unique solution of (13). For convenience we write

$$[1/q(\theta)]^{2\alpha}v(\theta) = u(\theta)-\frac{1}{(1+\alpha)^2}[1/q(\theta)]^{2\alpha}$$

in (13) which becomes

$$\left[(1+2\alpha)v(\theta)-\frac{q(\theta)}{q'(\theta)}\,v'(\theta)\right]^2-\frac{2\alpha}{\alpha+1}\frac{q(\theta)}{q'(\theta)}\,v'(\theta)\leq 0. \tag{14}$$

The proof now consists in showing that $v(\theta)\equiv 0$ is the only solution of (14). This is done by using typical Hodges-Lehmann argument as follows:

(a) $v'(\theta)\geq 0$ for $-\frac{1}{2}<\alpha<0$ and ≤ 0 for $\alpha>0$. If $v'(\theta)<0$, then, using the fact that $q'(\theta)<0$, we find that the expression $-[2\alpha/(\alpha+1)][q(\theta)/q'(\theta)]v'(\theta)$ is positive for $-\frac{1}{2}<\alpha<0$. But then inequality (14) is violated.

Hence the assertion for $-\frac{1}{2}<\alpha<0$ follows. The conclusion for $\alpha>0$ follows likewise.

(b) $v(\theta)$ is bounded. The inequality (7) for $-\frac{1}{2}<\alpha<0$ can be written as

$$\frac{2\alpha+1}{\alpha+1}\left[\frac{1}{q(x)}\right]^\alpha\leq m(x)\leq\frac{1}{\alpha+1}\left[\frac{1}{q(x)}\right]^\alpha$$

https://doi.org/10.4153/CMB-1975-019-0 Published online by Cambridge University Press

which after multiplying through by $[q(\theta)r(x)]/[q^{\alpha}(x)]$ and integrating from 0 to θ becomes

$$\frac{2\alpha+1}{\alpha+1}\,q(\theta)\int_0^{\theta}\frac{-q(x)}{[q(x)]^{2\alpha+2}}\,dx \le u(\theta) \le \frac{q(\theta)}{\alpha+1}\int_0^{\theta}\frac{-q(x)}{[q(x)]^{2\alpha+2}}\,dx$$

i.e.

$$\frac{1}{\alpha+1}\,[1/q(\theta)]^{2\alpha} \le u(\theta) \le \frac{1}{(\alpha+1)(2\alpha+1)}\,[1/q(\theta)]^{2\alpha}$$

Expressed in terms of $v(\theta)$, it becomes

$$\alpha[1+\alpha]^{-2} \le v(\theta) \le -\alpha[(1+2\alpha)(1+\alpha)]^{-2}$$

showing that $v(\theta)$ is bounded. The boundedness of $v(\theta)$ for $\alpha \ge 0$ follows likewise.

(c) $[q(\theta)/q'(\theta)]v'(\theta)$ is not bounded away from zero as $\theta\to 0$. For suppose there exists $\varepsilon>0$ and $\theta_0>0$ such that $[q(\theta)/q'(\theta)]v'(\theta)<-\varepsilon$ for $\theta<\theta_0$. That is, $-v'(x)<\varepsilon[q'(x)/q(x)]$ for all $x<\theta_0$. Integrating this from θ to θ_0 we get $v(\theta)-v(\theta_0)<\varepsilon \ln[q(\theta_0)/q(\theta)]$ which shows that $v(\theta)\to-\infty$ as $\theta\to 0$. This violates (b). Thus there exists a sequence $\theta_i\to 0$ along which

$$[q(\theta_i)/q'(\theta_i)]v'(\theta_i) \to 0.$$

Similarly we can show

(d) $[q(\theta)/q'(\theta)]v'(\theta)$ is not bounded away from zero as $\theta\to\infty$.

Now from (c) and (d) there are sequences $\theta_i\to 0$ and $\theta_i\to\infty$ along which $[q(\theta)/q'(\theta)]v'(\theta)\to 0$. From (14) it follows that $v(\theta)\to 0$ along these sequences. Hence from (a) it follows that $v(\theta)\equiv 0$. This completes the proof of admissibility of T_α for $\alpha>-\frac{1}{2}$. That T_α is inadmissible for $\alpha\le-\frac{1}{2}$ follows from the fact that its risk (as shown in (2)) is finite for each such α.

REMARKS 1. If $X_1,\ldots,X_n$ are independent random variables each having density (1) then the sufficient statistic $T=\max X_i$ has density given by

$$[q(\theta)]^n n\left[\int_0^t r(x)\,dx\right]^{n-1} r(t) \quad \text{for} \quad 0\le t\le\theta$$

which is a density of the form (1) with $q(\theta)$ replaced by $[q(\theta)]^n$ and $r(x)$ replaced by $n[\int_0^x f(v)\,dv]^{n-1}r(x)$. So from Theorem 1 we have the conclusion that

$$\frac{2\beta+1}{\beta+1}\left\{n\left[\int_0^x r(v)\,dv\right]^{n-1} r(X)\right\}^{-\beta}$$

is an admissible estimator of $[q(\theta)]^{-n\beta}$ if and only if $\beta>-\frac{1}{2}$. That is, writing α for $n\beta$, we conclude that

$$\frac{2\alpha+n}{\alpha+n}\left\{n\left[\int_0^x r(v)\,dv\right]^{n-1} r(x)\right\}^{-\alpha/n}$$

https://doi.org/10.4153/CMB-1975-019-0 Published online by Cambridge University Press

is an admissible estimator of $[q(\theta)]^{-\alpha}$ if and only if $\alpha > -n/2$. So for a given α we have admissibility for all sufficiently large sample sizes n.

2. Proof of Theorem 1 parallels the Blyth-Roberts [2] proof of the special case of the density (1) as

$$(15) \qquad f(x;\theta) = \begin{cases} n\theta^{-n}x^{n-1}, & 0 \le x \le \theta \\ 0 & \text{otherwise} \end{cases}$$

In [2], the parametric function of interest is $g(\theta)=\theta$. If $g(\theta)=\theta^s$ then according to Theorem 1 the estimator $(n+2s)/(n+s)X^s$ is admissible (with respect to quadratic loss) for estimating θ^s for every $s > -n/2$ and is inadmissible for $s \le -n/2$.

3. In [5] Karlin proved Theorem 1 (of this paper) for all $\alpha > 0$ (see his Theorem 2, p. 418). His proof makes use of the fact that $\alpha > 0$. Theorem 1 of the present paper settles the question of the admissibility of T_α for all values of α.

4. An attempt was made in [6] to extend Karlin's Theorem 2 to all values of α but this was successful only for some special extreme value densities such as (15). The approach there is the limiting Bayes method, used by Blyth [1] and Karlin [5].

5. The following theorem extends Theorem 3 of Karlin [5] to all other values of α.

THEOREM 2. *Let X have density*

$$(16) \qquad f(x;\theta) = \begin{cases} q(\theta)r(x), & x \ge \theta \\ 0, & \theta_0 < x < \theta, \end{cases}$$

where $q^{-1}(\theta)=\int_\theta^\infty r(x)\,dx$ and $q(\theta_0)=0$. Then (with quadratic loss) the estimator $T_\alpha=[(2\alpha+1)/(\alpha+1)][1/q(x)]^\alpha$ is admissible for estimating $[1/q(\theta)]^\alpha$ for all $\alpha > -\frac{1}{2}$ and inadmissible for all $\alpha \le -\frac{1}{2}$.

6. If the loss function is given by $L_0(\delta, g)=[(\delta-g)/g]^2$, the estimator T_α is minimax and admissible for estimating $[1/q(\theta)]^\alpha$ for all $\alpha > -\frac{1}{2}$.

7. The estimator $(\alpha+1)[1/q(X)]^\alpha$ is the uniformly minimum variance unbiased estimator of $[1/q(\theta)]^\alpha$ for all $\alpha > -\frac{1}{2}$. This estimator, however, is inadmissible for it is uniformly improved upon by the estimator T_α.

8. In addition to the example of the density (15), Theorems 1 and 2 have the following applications:

(i) *Pareto distribution.* Let X have density of the form

$$17) \qquad f(x;\theta) = \begin{cases} c\theta^c \dfrac{1}{x^{c+1}}, & x \ge \theta \\ 0 & \text{otherwise,} \end{cases}$$

where $c \ge 0$ is known and $g(\theta)=\theta^s$. If we take $r(x)=c/x^{c+1}$ then (17) is a special case of (16).

(ii) Let X have density

$$f(x;\theta)=\begin{cases} e^{-(x-\theta)}, & x\geq\theta \\ 0 & \text{otherwise,} \end{cases} \tag{18}$$

where $\theta\in(-\infty,\infty)$ and $g(\theta)=\theta^s$. If we set $r(x)=e^{-x}$ the (18) is a special case of (16).

Acknowledgement. The author is extremely grateful to the referee for his many valuable comments and suggestions. The proof of the assertion '$v(\theta)$ is bounded' (near the end of the proof of Theorem 1) is the one suggested by him. It is shorter and neater than the one given by the author.

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