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ON THE CONCEPT OF LENGTH IN THE SENSE OF LAUSCH–NÖBAUER AND ITS GENERALIZATIONS

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Abstract

In this paper the concept of length as defined for groups by Lausch–Nöbauer in their book *Algebra of Polynomials* (North Holland, Amsterdam, 1973) is generalized in several ways. It turns out that the main results of Lausch–Nöbauer concerning it remain valid for this generalization.

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The concept of length as defined below has been introduced by Scott (1969) in the case that the variety $\mathscr{V}$ considered is the variety of groups, the set X of indeterminates is a one-element set and the subgroup $\bar{A}$ of the group A equals A. It was generalized by Lausch–Nöbauer (1973) to the case in which X is an arbitrary finite set. This concept turned out to be very useful in the investigation of certain problems concerning direct products of algebras of polynomial functions over groups. The purpose of this paper is to generalize some results of Lausch–Nöbauer (1973) and to exhibit the categorical aspects of the theory with the heavy use of diagrams.

1. Let $\mathscr{V}$ be a variety of Ω-algebras in the sense of Cohn (1965) (for the other concepts of universal algebra used in the following see also Cohn (1965) and Lausch–Nöbauer (1973)), A a $\mathscr{V}$-algebra and X a set. A $\mathscr{V}$-algebra B will be called a $\mathscr{V}$-polynomial algebra in X over A and denoted by $A(X, \mathscr{V})$ if and only if B is a free $\mathscr{V}$-composition (often also called coproduct or free union) of A and the free $\mathscr{V}$-algebra $F(X, \mathscr{V})$ with $\mathscr{V}$-free generating set X, that is, if and only if there are homomorphisms $\varphi_1: A \to B$ and $\varphi_2: F(X, \mathscr{V}) \to B$ such that for any $\mathscr{V}$-algebra C and any homomorphisms $\psi_1: A \to C$, $\psi_2: F(X, \mathscr{V}) \to C$ there exists a unique homomorphism $\rho: B \to C$ with $\psi_i = \rho\varphi_i$, $i = 1, 2$. It is easy to see that φ_1 is injective and—except in the trivial case that $|A| = 1$ and no $\mathscr{V}$-algebra with more than one element contains a one-element subalgebra—the restriction $\varphi_2|X$ is injective, too. Therefore, in the following, we usually identify $\varphi_1 A$ with A and $\varphi_2 X$ with X. Furthermore, it is easy to see that $A \cup X$ is a generating set of $A(X, \mathscr{V})$. If $\eta: A \to C$ is a homomorphism from A to the $\mathscr{V}$-algebra C, then by the definition of a $\mathscr{V}$-polynomial algebra there exists a unique homomorphism $\eta(X)$: $A(X, \mathscr{V}) \to C(X, \mathscr{V})$ with $\eta(X)a = \eta a$ for all $a \in A$ and $\eta(X)x = x$ for all $x \in X$.

Furthermore, let $\bar{A}$ be a subalgebra of A and $\alpha\colon X \to A$ an arbitrary mapping, then there exists a unique homomorphism $\sigma_\alpha\colon \bar{A}(X, \mathscr{V}) \to A$ with $\sigma_\alpha a = a$ for all $a \in \bar{A}$ and $\sigma_\alpha x = \alpha x$ for all $x \in X$. Let $\tilde{\sigma}$ be the homomorphism from $\bar{A}(X, \mathscr{V})$ into the direct power A^{A^X} defined by $\tilde{\sigma} p = (\sigma_\alpha p)_{\alpha \in A^X}$ for $p \in \bar{A}(X, \mathscr{V})$. The image of $\tilde{\sigma}$ will be called the algebra of X-place polynomial functions over A with coefficients in $\bar{A}$ and denoted by $P_X(A, \bar{A})$. The homomorphism

$$\sigma\colon \bar{A}(X, \mathscr{V}) \to P_X(A, \bar{A})$$

which is obtained from $\tilde{\sigma}$ by restriction of the codomain is called the canonical homomorphism from $\bar{A}(X, \mathscr{V})$ onto $P_X(A, \bar{A})$ and denoted by $\sigma_X(A, \bar{A})$ if it is necessary to indicate the dependence of σ on X, A and $\bar{A}$. Clearly $P_X(A, \bar{A})$ is generated by $\sigma \bar{A} \cup \sigma X$, i.e. the set of the constant functions from A^X to A with values in $\bar{A}$ (in the following constant function with value a will also be denoted by a) and the projections π_x, $x \in X$, from A^X onto A. This concept of polynomial function algebras includes the polynomial function algebras in Lausch–Nöbauer (1973) and the polynomial algebras of Grätzer (1968) as special cases.

If $\eta\colon A \to C$ is a surjective homomorphism from A onto the $\mathscr{V}$-algebra C and $\bar{C} = \eta \bar{A}$, then it is easy to see (cf. Lausch–Nöbauer, 1973) that there exists a unique homomorphism $P_X(\eta)\colon P_X(A, \bar{A}) \to P_X(C, \bar{C})$ such that $P_X(\eta)\pi_x = \pi_x$ for all $x \in X$ (the projections of A^X and C^X are denoted by the same symbol) and $P_X(\eta)\, a = \eta a$ for all $a \in \bar{A}$, that is, that the diagram

$$(1) \qquad \begin{array}{ccc} \bar{A}(X, \mathscr{V}) & \xrightarrow[\bar{\eta}(X)]{} & \bar{C}(X, \mathscr{V}) \\ \downarrow{\scriptstyle \sigma_X(A, \bar{A})} & & \downarrow{\scriptstyle \sigma_X(C, \bar{C})} \\ P_X(A, \bar{A}) & \xrightarrow{P_X(\eta)} & P_X(C, \bar{C}) \end{array}$$

is commutative where $\bar{\eta}\colon \bar{A} \to \bar{C}$ is the homomorphism obtained from η by restriction.

Let $(A_i)_{i \in I}$ and $(\bar{A}_i)_{i \in I}$ be families of $\mathscr{V}$-algebras such that $\bar{A}_i$ is a subalgebra of A_i for all $i \in I$. Furthermore let $A = \prod_{i \in I} A_i$, $\bar{A} = \prod_{i \in I} \bar{A}_i$ with projections π_i. Clearly the mapping $\tau_1\colon \bar{A}(X, \mathscr{V}) \to \prod_{i \in I} \bar{A}_i(X, \mathscr{V})$ defined by $\tau_1 p = (\pi_i(X) p)_{i \in I}$ for $p \in \bar{A}(X, \mathscr{V})$ is a homomorphism, and similarly we can define a homomorphism $\tau_2\colon P_X(A, \bar{A}) \to \prod_{i \in I} P_X(A_i, \bar{A}_i)$ by $\tau_2 f = (P_X(\pi_i) f)_{i \in I}$ for $f \in P_X(A, \bar{A})$. By the definition of τ_1 and τ_2 and by (1) the diagram

$$(2) \qquad \begin{array}{ccc} \bar{A}(X, \mathscr{V}) & \xrightarrow{\tau_1} & \prod_{i \in I} \bar{A}_i(X, \mathscr{V}) \\ \downarrow{\scriptstyle \sigma_X(A, \bar{A})} & & \downarrow{\scriptstyle \prod_{i \in I} \sigma_X(A_i, \bar{A}_i)} \\ P_X(A, \bar{A}) & \xrightarrow{\tau_2} & \prod_{i \in I} P_X(A_i, \bar{A}_i) \end{array}$$

is commutative showing that τ_2 is surjective if τ_1 is.

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The following assertion is a generalization of Proposition 3.53 in Lausch–Nöbauer (1973):

PROPOSITION 1. *τ_2 is injective.*

PROOF. Let $\tilde{\tau}_2\colon \prod_{i\in I} P_X(A_i,\bar{A}_i)\to A^{A^X}$ be the mapping defined by

$$(\tilde{\tau}_2(f_i)_{i\in I})(a_x)_{x\in X} = (f_i(\pi_i a_x)_{x\in X})_{i\in I}$$

for $f_i\in P_X(A_i,\bar{A}_i)$, $a_x\in A$. One easily checks that $\tilde{\tau}_2\tau_2$ is the inclusion mapping of $P_X(A,\bar{A})$ into A^{A^X}. This clearly implies the injectivity of τ_2.

REMARK. The question whether τ_1 is injective or not cannot be answered in general. By the use of normal form systems it can be shown that τ_1 is injective for the variety of commutative rings with identity and for the variety of abelian groups, whereas for the variety of groups τ_1 is injective only in the trivial case that $|A|=1$.

2. From now on let $\mathscr{V}$ be a variety of groups with multiple operators such that the group identity 0 is the only nullary operation. If there is no danger of confusion, in the following we shall drop the fixed variety $\mathscr{V}$ from our notations.

It is easy to see that the $\mathscr{V}$-polynomial algebra $\{0\}(X)$ is the free $\mathscr{V}$-algebra $F(X)$ with $\mathscr{V}$-free generating set X. Let A be a $\mathscr{V}$-algebra and $\lambda_X(A)\colon A(X)\to F(X)$ the unique homomorphism such that $\lambda_X(A)a=0$ for all $a\in A$ and $\lambda_X(A)x=x$ for all $x\in X$. Clearly $\lambda_X(A)$ is a retraction. For a subalgebra $\bar{A}$ of A the ideal $\lambda_X(\bar{A})\ker\sigma_X(A,\bar{A})$ of $F(X)$ is called the length of A with respect to $\bar{A}$ and X and denoted by $L_X(A,\bar{A})$. It is easy to see that $L_X(A,\bar{A})$ is fully invariant.

The following two propositions are generalizations of results in Lausch–Nöbauer (1973).

PROPOSITION 2. $\bar{A}(X)/(\ker\sigma_X(A,\bar{A})+\ker\lambda_X(\bar{A}))\cong F(X)/L_X(A,\bar{A})$.

PROOF.

$$\begin{aligned} &F(X)/L_X(A,\bar{A})\\ &\quad=\lambda_X(\bar{A})\,\bar{A}(X)/\lambda_X(\bar{A})\ker\sigma_X(A,\bar{A})\cong\bar{A}(X)/(\ker\sigma_X(A,\bar{A})+\ker\lambda_X(\bar{A}))\end{aligned}$$

by the second isomorphism theorem.

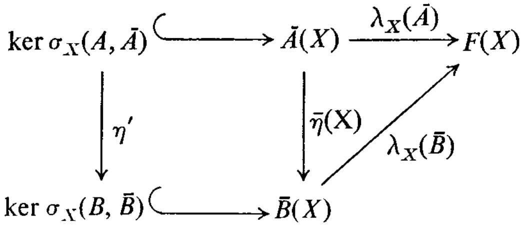
PROPOSITION 3. *If $\eta\colon A\to B$ is a surjective homomorphism, $\eta\bar{A}=\bar{B}$, then $L_X(A,\bar{A})\subseteq L_X(B,\bar{B})$, and equality holds if and only if*

$$\bar{\eta}(X)\ker\sigma_X(A,\bar{A})+\ker\sigma_X(B,\bar{B})\cap\ker\lambda_X(\bar{B})=\ker\sigma_X(B,\bar{B})$$

where $\bar{\eta}\colon\bar{A}\to\bar{B}$ is obtained from η by restriction.

PROOF. With the notations from above this is an easy consequence of the commutativity of the diagram

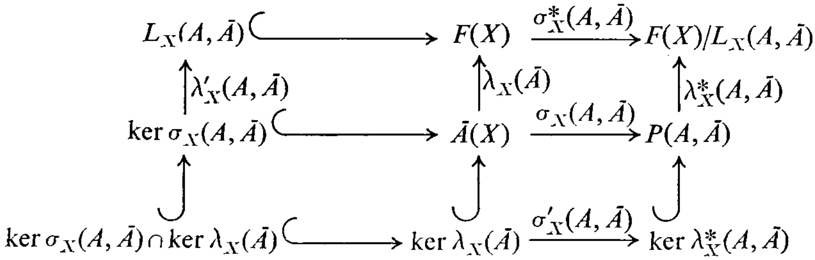
(3)



where η' is obtained from $\bar{\eta}(X)$ by restriction and arrows of the form $\hookrightarrow$ denote inclusion homomorphisms.

Let $\sigma^*_X(A,\bar{A})\colon F(X)\to F(X)/L_X(A,\bar{A})$ be the natural homomorphism and $\lambda^*_X(A,\bar{A})\colon P_X(A,\bar{A})\to F(X)/L_X(A,\bar{A})$ the mapping defined in the following way. If $f\in P_X(A,\bar{A})$ and $w(a_1,\dots,a_n,\pi_{x_1},\dots,\pi_{x_m})$ is a representation of f by a word, then $\lambda^*_X(A,\bar{A})f=w(0,\dots,0,x_1,\dots,x_m)+L_X(A,\bar{A})$. One easily checks that $\lambda^*_X(A,\bar{A})$ is well defined and a homomorphism. By these definitions, the following diagram

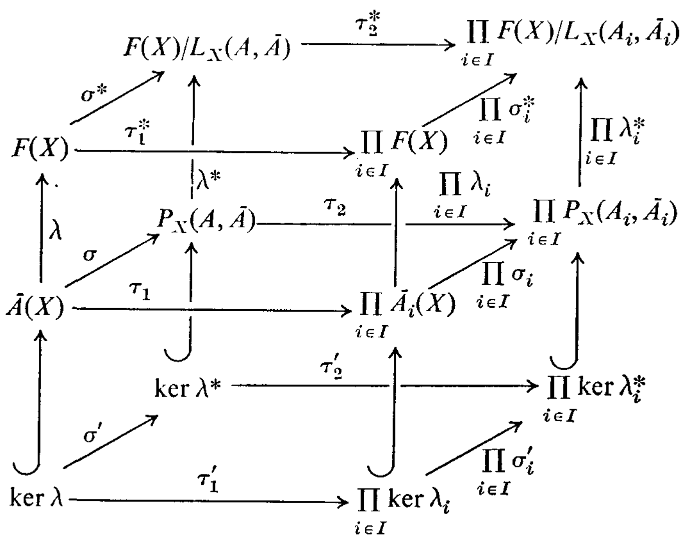
(4)



is commutative where $\lambda'_X(A,\bar{A})$ and $\sigma'_X(A,\bar{A})$ are obtained from $\lambda_X(\bar{A})$ and $\sigma_X(A,\bar{A})$, respectively, by restriction (and clearly are surjective).

3. One easily checks the commutativity of the diagram

(5)



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where τ_1, τ_2 have the same meaning as in Section 1, τ_1', τ_2' are obtained from τ_1, τ_2 respectively by restriction, τ_1^* is the diagonal mapping and τ_2^* is defined by $\tau_2^*(f+L_X(A,\bar{A})) = (f+L_X(A_i,\bar{A}_i))_{i\in I}$ for $f\in F(X)$ (in the diagram the notations of some mappings have been simplified in an unambiguous manner).

PROPOSITION 4. *If I is finite, then τ_1' is surjective.*

PROOF. Let $(p_i)_{i\in I}\in\prod_{i\in I}\ker\lambda_X(\bar{A}_i)$ be arbitrary. One easily checks that $\sum_{i\in I}\iota_i(X)p_i$ (ι_i is the canonical injection of $\bar{A}_i$ into $\bar{A}$) is an element of $\ker\lambda_X(\bar{A})$ and is mapped onto $(p_i)_{i\in I}$ by τ_1.

PROPOSITION 5. *If I is finite, then τ_2' is an isomorphism.*

PROOF. This is an immediate consequence of the injectivity of τ_2 and the surjectivity of τ_1' by the commutativity of diagram (5).

REMARK. Since $\ker\lambda_X(A,\bar{A})\cong(\ker\lambda_X(\bar{A})+\ker\sigma_X(A,\bar{A}))/\ker\sigma_X(A,\bar{A})$, Proposition 5 is a generalization of Proposition 1.21 in Lausch–Nöbauer (1973), chapter 5.

Another generalization of a result of Lausch–Nöbauer (1973) is the following.

THEOREM 6. *If I is finite, then τ_2^* is injective and as a consequence*

$$L_X(A,\bar{A}) = \bigcap_{i\in I} L_X(A_i,\bar{A}_i).$$

PROOF. This is again an easy consequence of the injectivity of τ_2 and the surjectivity of τ_2' by simple diagram chasing in the diagram (5) (actually, this is a very special case of the well-known 5-lemma).

The following theorem generalizes Corollary 1.22 in Lausch–Nöbauer (1973).

THEOREM 7. *If I is finite, say $I=\{1,\dots,n\}$, then the following conditions are equivalent:*

(1) *τ_2 is surjective.*

(2) *τ_2^* is surjective.*

(3) *The system of congruences $f\equiv f_i \bmod L_X(A_i,\bar{A}_i)$, $i\in I$, is solvable for any choice of $(f_i)_{i\in I}\in\prod_{i\in I}F(X)$.*

(4) $(\bigcap_{i\in I-\{j\}} L_X(A_i,\bar{A}_i))+L_X(A_j,\bar{A}_j) = F(X)$ for all $j\in I$.

(5) $(\bigcap_{i=1}^{j} L_X(A_i,\bar{A}_i))+L_X(A_{j+1},\bar{A}_{j+1}) = F(X)$ for $j=1,\dots,n-1$.

(6) $(\bigcap_{i=1}^{j} L_X(A_i,\bar{A}_i))+L_X(A_{j+1},\bar{A}_{j+1})\supseteq X$ for $j=1,\dots,n-1$.

Furthermore, if $X\neq\varnothing$ each of the conditions (1)–(6) *holds if and only if it holds for all one-element sets X.*

PROOF. (1)⇔(2): Trivially, (1) implies (2). The proof of the other direction is done by simple diagram chasing using the surjectivity of τ_2^* and τ_2' (which is again a special case of the 5-lemma).

(2)⇔(3): This is trivial by definition of τ_2^*.

(3)⇔(4): It is clear that (3) can be weakened to the condition that the system of congruences in (3) can be solved for all $(f_i)_{i\in I} \in \prod_{i\in I} F(X)$ with the property that all components are 0 with the possible exception of one, because these $(f_i)_{i\in I}$ form a generating set of $\prod_{i\in I} F(X)$. This is easily seen to be equivalent to condition (4).

(4)⇔(5): This is a well-known result of the theory of modular lattices (cf. Kuroš, 1963).

(5)⇔(6): Since X is a generating set of $F(X)$, the equivalence holds.

To prove the final assertion, we first observe that the unique extension $\bar{\alpha}: F(X) \to F(\bar{X})$ of a map $\alpha: X \to \bar{X}$ maps $L_X(A_i, \bar{A}_i)$ into $L_{\bar{X}}(A_i, \bar{A}_i)$, $i = 1, \ldots, n$. Suppose now that the condition (5) in Theorem 7 holds for an $X \neq \emptyset$. Applying $\bar{\alpha}: F(X) \to F(\bar{X})$ to (5) with $\bar{X}$ a one-element set yields the condition (5) for $\bar{X}$. Conversely, assume that (5) holds for all one-element sets X. For an arbitrary set $X \neq \emptyset$ one easily checks, using the same argument as above, that each $F(\bar{X})$, where $\bar{X}$ is a one-element subset of X, is contained in the left side of (5). Since $F(X)$ is generated by these $F(\bar{X})$, condition (5) holds for X, too. Since (1)–(6) are equivalent, this is all that is required for the proof.

In case that one (and therefore each) of the conditions (1)–(6) holds, we say that $A_1, \ldots, A_n$ are independent with respect to $\bar{A}_1, \ldots, \bar{A}_n$ and X (cf. Foster, 1955). If each two A_i, A_j, $i \neq j$, are independent with respect to $\bar{A}_i$, $\bar{A}_j$ and X, we speak of pairwise independence. Clearly, independence implies pairwise independence. We do not know whether the converse holds (for general varieties $\mathscr{V}$ a counter-example has been given by Froemke, 1971), but we shall prove

COROLLARY 1. *If $\mathscr{V}$ is one of the following varieties, then pairwise independence implies independence:*

(1) *the variety of groups* (cf. Froemke, 1971),

(2) *the variety of lattice ordered groups* (*regarded as algebras of type* (2, 1, 0, 2, 2)),

(3) *the variety of unitary left R-modules,*

(4) *the variety of rings.*

PROOF: If $X = \emptyset$ there is nothing to prove. Therefore, because of the final assertion of Theorem 7 we may assume that $X = \{x\}$. Then the statement in cases (1) and (2) is a simple consequence of the distributivity of the ideal lattice of $F(\{x\})$.

To prove (3), for notational convenience we set $L_i := L_{\{x\}}(A_i, \bar{A}_i)$, $i = 1, \ldots, n$. Since the L_i are fully invariant, they are ideals of $F(\{x\}) = {}_R R$ and pairwise independence means then that they are pairwise comaximal. Therefore we have

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$R = R^j = (L_1+L_{j+1})\dots(L_j+L_{j+1}) \subseteq L_1\dots L_j+L_{j+1} \subseteq L_1\cap\dots\cap L_j+L_{j+1} \subseteq R$ for all $j = 1,\dots,n-1$, which clearly implies independence by Theorem 7 (5).

For the proof of (4), we first observe that the elements of $F(\{x\})$ are all of the form $a_1x+a_2x^2+\dots+a_mx^m$, $a_i \in \mathbf{Z}$, $m \in \mathrm{N}$, where a_ix^i means the a_ith additive power of x^i. Let $j \in \{1,\dots,n-1\}$. Suppose we had already proved that $l(x)+l_{j+1}(x) = x$ for some $l(x) \in L_1\cap\dots\cap L_i$, $1 \leqslant i < j$, and $l_{j+1}(x) \in L_{j+1}$ (the case $i = 1$ is settled by the independence of A_1, A_{j+1} with respect to $\bar{A}_1, \bar{A}_{j+1}$ and X). Substituting $x = l_{i+1}(x)+l'_{j+1}(x)$ with $l_{i+1}(x) \in L_{i+1}$ and $l'_{j+1}(x) \in L_{j+1}$ (such a representation for x exists because of the assumed pairwise independence) and using the normal forms from above yields

$$l(l_{i+1}(x)+l'_{j+1}(x))+l_{j+1}(l_{i+1}(x)+l'_{j+1}(x)) = l(l_{i+1}(x))+l''_{j+1}(x)+l_{j+1}(l_{i+1}(x)+l'_{j+1}(x)) = x$$

for some $l''_{j+1}(x) \in L_{j+1}$. Since the L_k, $k = 1,\dots,n$, are fully invariant, we have $l(l_{i+1}(x)) \in L_1\cap\dots\cap L_{i+1}$ and $l''_{j+1}(x)+l_{j+1}(l_{i+1}(x)+l'_{j+1}(x)) \in L_{j+1}$, that is

$$x \in (L_1\cap\dots\cap L_{i+1})+L_{j+1}.$$

By induction it follows that condition (6) in Theorem 7 holds, which completes the proof.

Further examples for the application of Theorem 7 are given by the following three corollaries.

COROLLARY 2. Let $\mathscr{V}'$ be a variety of groups with multiple operators which arises from $\mathscr{V}$ by possibly adding non-nullary operations to the operation set Ω of $\mathscr{V}$ and (or) enlarging the set of laws of $\mathscr{V}$. Then the $\mathscr{V}'$-algebras $A_1,\dots,A_n$ are independent with respect to $\bar{A}_1,\dots,\bar{A}_n$ and X, if they are independent regarded as $\mathscr{V}$-algebras.

PROOF. Let $F(X,\mathscr{V})$ and $F(,\mathscr{V}')$ be the free algebras with free generating set X with respect to $\mathscr{V}$ and $\mathscr{V}'$, respectively. Regarding $F(X,\mathscr{V}')$ as a $\mathscr{V}$-algebra there exists a unique $\mathscr{V}$-homomorphism φ: $F(X,\mathscr{V}) \to F(X,\mathscr{V}')$ which fixes X elementwise. Clearly, φ maps the length $L_X(A_i,\bar{A}_i)$ with respect to $\mathscr{V}$ into the length $L_X(A_i,\bar{A}_i)$ with respect to $\mathscr{V}'$, $i = 1,\dots,n$. Applying φ to the inclusion in Theorem 7(6) yields the corresponding inclusion for $\mathscr{V}'$.

COROLLARY 3. Assume that among the operations of $\mathscr{V}$ there is a binary operation . satisfying the law $0.x = 0$. If A_i is a $\mathscr{V}$-algebra containing a left identity e_i with respect to . and $\bar{A}_i$ a subalgebra of A_i such that $e_i \in \bar{A}_i$, $i = 1,\dots,n$, then $A_1,\dots,A_n$ are independent with respect to $\bar{A}_1,\dots,\bar{A}_n$ and X.

PROOF. Let $x \in X$, then $x - e_i.x \in \ker\sigma_X(A_i,\bar{A}_i)$ and therefore $x \in L_X(A_i,\bar{A}_i)$ showing that $L_X(A_i,\bar{A}_i) = F(X)$ for $i = 1,\dots,n$.

Let the exponent $\exp A$ of a group A with multiple operators be defined to be the least common multiple of the group-orders of its elements if these orders are bounded, and 0 otherwise, then we obtain the following

COROLLARY 4. *If* $\exp A_i$, $i = 1, \ldots, n$, *are pairwise relatively prime, then* $A_1, \ldots, A_n$ *are independent with respect to any subalgebras* $\bar{A}_1, \ldots, \bar{A}_n$, *respectively, and* X. *Conversely,* $\exp A_i$, $i = 1, \ldots, n$, *are pairwise relatively prime, if either of the following hold*

(1) $\mathscr{V}$ *is the variety of groups and* $A_1, \ldots, A_n$ *are independent with respect to the subalgebras* $\bar{A}_i = \{0\}$, $i = 1, \ldots, n$, *and any set* $X \neq \varnothing$ (*cf. Froemke,* 1971).

(2) $\mathscr{V}$ *is the variety of abelian groups and* $A_1, \ldots, A_n$ *are independent with respect to any subalgebras* $\bar{A}_1, \ldots, \bar{A}_n$, *respectively, and any set* $X \neq \varnothing$.

PROOF. Suppose $e_i := \exp A_i$, $i = 1, \ldots, n$, are pairwise relatively prime and let $e^{(j)} = (\prod_{i=1}^{n} e_i)/e_j$, $j = 1, \ldots, n$. Then there are integers n_j, m_j such set

$$n_j e^{(j)} + m_j e^j = 1,$$

and therefore $x = (n_j e^{(j)})x + (m_j e_j)x$ for all $j = 1, \ldots, n$. Since

$$(n_j e^{(j)})x \in \bigcap_{i=1, i \neq j}^{n} L_X(A_i, \bar{A}_i)$$

and $(m_j e_j)x \in L_X(A_j, \bar{A}_j)$, we conclude from Theorem 7 (4) that τ_2 is an isomorphism.

The second part of the assertion is a simple consequence of Theorem 7 and the fact that in both cases mentioned in the corollary $L_{\{x\}}(A_i, \bar{A}_i) = \{(ke_i)x \mid k \in \mathbf{Z}\}$, $i = 1, \ldots, n$.

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