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COMPUTING L-FUNCTIONS AND SEMISTABLE REDUCTION OF SUPERELLIPTIC CURVES

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Abstract. We give an explicit description of the stable reduction of superelliptic curves of the form $y^n = f(x)$ at primes $\mathfrak{p}$ whose residue characteristic is prime to the exponent n. We then use this description to compute the local L-factor and the exponent of conductor at $\mathfrak{p}$ of the curve.

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1. Introduction

1.1. Let Y be a smooth projective curve of genus $g \geq 2$ over a number field K. The *L-function* of Y is defined as an Euler product

$$L(Y, s) := \prod_{\mathfrak{p}} L_{\mathfrak{p}}(Y, s),$$

where $\mathfrak{p}$ ranges over the prime ideals of K. The local L-factor $L_{\mathfrak{p}}(Y, s)$ is defined as follows. Choose a decomposition group $D_{\mathfrak{p}} \subset \mathrm{Gal}(K^{\mathrm{alg}}/K)$ of $\mathfrak{p}$. Let $I_{\mathfrak{p}} \subset D_{\mathfrak{p}}$ denote the inertia subgroup and let $\sigma_{\mathfrak{p}} \in D_{\mathfrak{p}}$ be an arithmetic Frobenius element (i.e. $\sigma_{\mathfrak{p}}(\alpha) \equiv \alpha^{\mathrm{N}\mathfrak{p}} \pmod{\mathfrak{p}}$). Then

$$L_{\mathfrak{p}}(Y, s) := \det\left(1 - (\mathrm{N}\mathfrak{p})^{-s}\sigma_{\mathfrak{p}}^{-1} | V^{I_{\mathfrak{p}}}\right)^{-1},$$

where

$$V := H^1_{\mathrm{et}}(Y \otimes_K K^{\mathrm{alg}}, \mathbb{Q}_\ell)$$

is the first étale cohomology group of Y (for some auxiliary prime ℓ distinct from the residue characteristic of $\mathfrak{p}$).

Another arithmetic invariant of Y closely related to $L(Y, s)$ is the *conductor of the L-function*. Similar to $L(Y, s)$, it is defined as a product over local factors (times a power of the discriminant δ_K of K):

$$N := \delta_K^{2g} \cdot \prod_{\mathfrak{p}} (\mathrm{N}\mathfrak{p})^{f_{\mathfrak{p}}},$$

where $f_{\mathfrak{p}}$ is a non-negative integer called the *exponent of conductor* at $\mathfrak{p}$. The integer $f_{\mathfrak{p}}$ measures the *ramification* of the Galois module V at the prime $\mathfrak{p}$. See Section 2.1 or [**24**], Section 2, for a precise definition.

Many spectacular conjectures and theorems concern these L-functions. For instance, it is conjectured that $L(Y, s)$ has a meromorphic continuation to the entire complex plane, and a functional equation of the form

$$\Lambda(Y, s) = \pm\Lambda(Y, 2 - s), \tag{1}$$

where

$$\Lambda(Y, s) := N^{s/2}(2\pi)^{-gs}\Gamma(s)^g L(Y, s).$$

This conjecture can be proved for certain special curves related to automorphic forms (like modular curves) and, as a consequence of the Taniyama–Shimura conjecture, for elliptic curves over $\mathbb{Q}$. Besides that, very little is known.

1.2. One motivation for this paper is the question how to compute the defining series for $L(Y, s)$ and the conductor N explicitly for a given curve Y. By definition, this is a local problem at each prime ideal $\mathfrak{p}$. So, we fix $\mathfrak{p}$ and aim at computing $L_{\mathfrak{p}}(Y, s)$ and $f_{\mathfrak{p}}$. Note that the residue field of $\mathfrak{p}$ is the finite field $\mathbb{F}_q$ with $q = \mathrm{N}(\mathfrak{p})$ elements. To study this problem, we construct suitable $\mathcal{O}_K$-*models* of Y. Recall that an $\mathcal{O}_K$-*model* of Y is a flat and proper $\mathcal{O}_K$-scheme $\mathcal{Y}$ with generic fibre Y.

Assume first that Y has *good reduction* at $\mathfrak{p}$. This means that there exists an $\mathcal{O}_K$-model $\mathcal{Y}$ whose special fibre $\bar{Y} = \bar{Y}_{\mathfrak{p}}$ at $\mathfrak{p}$ is a smooth $\mathbb{F}_q$-scheme. Standard theorems in étale cohomology show that the action of $\mathrm{Gal}(K^{\mathrm{alg}}/K)$ on $V = H^1_{\mathrm{et}}(Y_{K^{\mathrm{alg}}}, \mathbb{Q}_\ell)$ is unramified at $\mathfrak{p}$ (i.e. $I_{\mathfrak{p}}$ acts trivially) and therefore the exponent of conductor vanishes, $f_{\mathfrak{p}} = 0$. Furthermore, the local L-factor $L_{\mathfrak{p}}(Y, s)$ is equal to the inverse of the denominator of the zeta function of $\bar{Y}$, i.e.

$$Z(\bar{Y}, q^{-s}) = \frac{L_{\mathfrak{p}}(Y, s)^{-1}}{(1 - q^{-s})(1 - q^{1-s})},$$

where

$$Z(\bar{Y}, T) := \exp\left(\sum_{n\geq 1} |\bar{Y}(\mathbb{F}_{q^n})| \cdot \frac{T^n}{n}\right).$$

To compute $L_{\mathfrak{p}}(Y, s)$ for small prime ideals, we simply need to count the number of $\mathbb{F}_{q^n}$-rational points on $\bar{Y}$, for $n = 1, \ldots, g$.

If Y has bad reduction it is much harder to compute $L_{\mathfrak{p}}(Y, s)$ and $f_{\mathfrak{p}}$. To our knowledge, there are essentially three ways to proceed.

1. Compute a *regular model* of Y at $\mathfrak{p}$.
2. Compute the *semistable reduction* of Y at $\mathfrak{p}$.
3. Guess the local L-factors at *all* primes of bad reduction, and then verify this guess via the functional equation for $L(Y, s)$.

All three methods have certain advantages and drawbacks, and it is often a combination of them which works best. In this paper, we would like to advertise method (2), by

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demonstrating its simplicity and usefulness in a large class of examples (superelliptic curves).

1.3. Before we go into more details of methods (1) and (2), let us briefly describe method (3). Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the prime ideals of the number field K where Y has bad reduction. One can show the following.

- For $i = 1, \dots, r$ there are only finitely many possible choices for the local L-factor $L_{\mathfrak{p}_i}(Y, s)$ and the exponent $f_{\mathfrak{p}_i}$. In fact, the set of all choices depends only on the norm $q_i = \mathrm{N}\mathfrak{p}_i$ and the genus g.
- There is at most a unique choice for the conductor N and the local L-factors $L_{\mathfrak{p}_i}(Y, s)$ at the bad primes $\mathfrak{p}_i$ such that the L-function

$$L(Y, s) := \prod_{\mathfrak{p}} L_{\mathfrak{p}}(Y, s)$$

satisfies the functional equation (1).

This suggests the following strategy to determine $L(Y, s)$.

- Guess the exponents of conductor $f_{\mathfrak{p}_i}$ and the local L-factors $L_{\mathfrak{p}_i}(Y, s)$ at the bad primes $\mathfrak{p}_i$.
- Compute $L_{\mathfrak{p}}(Y, s)$ for all good primes $\mathfrak{p}$ with $\mathrm{N}\mathfrak{p} \leq C$ for some sufficiently large constant C. The constant C should be chosen large enough, so that knowing $L_{\mathfrak{p}}(Y, s)$ for all primes with $\mathrm{N}\mathfrak{p} \leq C$ yields a sufficiently good numerical approximation of the L-function. If C is not too large, computing $L_{\mathfrak{p}}(Y, s)$ for all such good primes can be done efficiently by simple point counting.
- Check numerically whether $L(Y, s) := \prod_{\mathfrak{p}} L_{\mathfrak{p}}(Y, s)$ satisfies the functional equation (1). By **[9]**, we need to choose $C \sim N^{1/2}$.

In practice, this can be done if $N \sim 10^{15}$. See e.g. **[10]**.

An obvious drawback of this method is that one can never prove that the guess one has made is correct.

1.4. Regular models We now describe the first method. Fix a prime ideal $\mathfrak{p}$ of K. Since the local L-factor $L_{\mathfrak{p}}(Y, s)$ and the exponent $f_{\mathfrak{p}}$ only depend on the base change of Y to the completion $\hat{K}_{\mathfrak{p}}$, we may and will from now on assume that K is a finite extension of $\mathbb{Q}_p$. We use the notation $L(Y/K, s)$ and $f_{Y/K}$ to denote the local L-factor and the exponent of conductor. We write $\mathbb{F}_K$ for the residue field of K, which is a finite field of characteristic p.

We may assume that Y has bad reduction. By resolution of singularities of two-dimensional schemes, there exists a *regular model* $\mathcal{Y}^{\mathrm{reg}}$, i.e. a flat and proper $\mathcal{O}_K$-model of Y which is regular. Since we assume $g \geq 2$ we may also assume that $\mathcal{Y}^{\mathrm{reg}}$ is the minimal regular model. Let $\bar{Y}^{\mathrm{reg}}$ denote the special fibre of $\mathcal{Y}^{\mathrm{reg}}$. Under an additional (relatively mild) assumption, it is still true that $L(Y/K, s)$ is the inverse of the denominator of the zeta function of the special fibre $\bar{Y}^{\mathrm{reg}}$ of $\mathcal{Y}^{\mathrm{reg}}$ as in the smooth case (see Proposition 2.8 below). Therefore, $L(Y/K, s)$ can be computed from $\bar{Y}^{\mathrm{reg}}$ by point counting.

By a result of Saito (**[21]**) it should also be possible to compute $f_{Y/K}$ from $\mathcal{Y}^{\mathrm{reg}}$. For curves of genus 2 this is achieved in **[16]**, and these methods probably extend to arbitrary hyperelliptic curves (see **[17]**). We are not aware of any attempt to explicitly compute $f_{Y/K}$ for non-hyperelliptic curves, using regular models.

Finding a regular model $\mathcal{Y}^{\rm reg}$ can be computationally challenging. The computer algebra system MAGMA has a built-in function to compute regular models of curves of genus $g \geq 2$, but it seems that there are still many restrictions on the types of curves for which it works. A similar function which should overcome these limitations is being prepared in SINGULAR.

1.5. Semistable reduction. We now describe the second method. For precise definitions and more details, we refer to Section 2.3. Since we assume that $g \geq 2$, the curve $Y_L := Y \otimes_K L$ admits a stable model $\mathcal{Y}^{\rm stab}$ over a finite extension L/K. The stable model $\mathcal{Y}^{\rm stab}$ is minimal with the property that its special fibre $\bar{Y}^{\rm stab}$ has at most ordinary double points as singularities. However, $\mathcal{Y}^{\rm stab}$ need not be regular.

We may assume that L/K is Galois. The Galois group $\Gamma := \mathrm{Gal}(L/K)$ has a natural semilinear action on $\mathcal{Y}^{\rm stab}$. Restricting this action to the special fibre, we obtain a natural, semilinear action of Γ on the special fibre $\bar{Y}^{\rm stab}$ of $\mathcal{Y}^{\rm stab}$. The quotient scheme $\bar{Z}^{\rm inert} := \bar{Y}^{\rm stab}/\Gamma$ is a semistable curve over the residue field $\mathbb{F}_K$ of K. We call it the *inertial reduction* of Y. The following result is certainly known to experts, but not so easy to find in the literature.

THEOREM 1.1. *The stable reduction $\bar{Y}^{\rm stab}$, together with its natural Γ-action, determines the local L-factor $L(Y/K, s)$ and the exponent of conductor $f_{Y/K}$. In particular:*

1. *The local L-factor $L(Y/K, s)$ is the inverse of the denominator of the zeta function of $\bar{Z}^{\rm inert}$ (which may be computed by point counting).*
2. *If, moreover, Y has semistable reduction over a tamely ramified extension of K then*

$$f_{Y/K} = 2g(Y) - \dim H^1_{\rm et}(\bar{Z}^{\rm inert}_k, \mathbb{Q}_\ell).$$

Here, k is the algebraic closure of $\mathbb{F}_K$.

The first statement of Theorem 1.1.(1) follows from Corollary 2.5. That corollary shows that one may use somewhat more general models of Y. The computational aspects are discussed in Section 2.4. Theorem 1.1.(2) is Corollary 2.6. An analogous statement in the wild case can be found in Section 2.6.

1.6. Let us compare the two methods discussed in Sections 1.4 and 1.5. If the curve Y already has semistable reduction, the minimal regular model of Y is also semistable. In this case, there is no essential difference between the two methods. In general, however, the two methods are quite different in nature.

From the theoretical point of view one may consider the method of stable reduction as “better” because it gives more information. For instance, unlike the regular model, the stable model is invariant under base change of the curve Y to any finite extension K'/K. Therefore, once the stable reduction of Y has been computed, we can directly compute $L(Y'/K', s)$ and $f_{Y'/K'}$, where $Y' := Y \otimes_K K'$.

From a computational point of view it may seem to be a lot easier to find a regular model. After all, to compute a semistable model is essentially equivalent to computing a regular model over a larger field L *and* to find the correct extension L/K in the first place. However, one goal of the present paper is to show that, at least for special classes of curves, it is actually rather easy to determine the stable reduction, even though the reduction behaviour can be arbitrarily complicated.

1.7. Superelliptic curves. We consider *superelliptic curves*, i.e. curves Y given by an equation of the form

$$y^n = f(x),$$

where n is a positive integer and $f(x)$ is a rational function over a p-adic number field K. The additional and crucial condition we impose is that the exponent n must be prime to the residue characteristic p of K.

Let L_0/K be the splitting field of $f(x)$, i.e. the smallest extension of K over which all poles and zeros of $f(x)$ become rational. Our main result in Section 4 says that Y has semistable reduction over an explicit and at most tamely ramified extension L/L_0. Moreover, the stable reduction $\bar{Y}^{\rm stab}$, together with the natural action of $\Gamma = {\rm Gal}(L/K)$, can be described easily and in a purely combinatorial manner. The only part which may be computationally difficult is the analysis of the extension L_0/K. Indeed, by choosing $f(x)$ appropriately we can make this extension as large and as complicated as we want. However, it is possible to construct examples where the computation of the stable reduction is still rather easy, but the standard algorithms for computing a regular model fail. We refer to **[4]** for a concrete case.

Starting from the description of the stable reduction, we give an explicit procedure to determine an equation for the inertial reduction $\bar{Z}^{\rm inert} = \bar{Y}^{\rm stab}/\Gamma$ in Section 5. This equation can then be used to compute the local L-factor of Y and the exponent of conductor $f_{Y/K}$, via Theorem 1.1.

We remark that our description of the stable reduction of superelliptic curves is based on a very special case of more general results on *admissible reduction* for covers of curves. These results are well known to experts. One of the goals of the present paper is to make these results more widely known and to demonstrate their usefulness for explicit computations. In a subsequent paper, we will present a software implementation of our results.

2. Stable and inertial reduction.

In this section, we prove Theorem 1.1.

2.1. Let p be a prime number and K a finite extension of $\mathbb{Q}_p$. The residue field of K is a finite field, which we denote by $\mathbb{F}_K$. The residue field of a finite extension L/K is denoted by $\mathbb{F}_L$.

We choose an algebraic closure $K^{\rm alg}$ of K and write $\Gamma_K = {\rm Gal}(K^{\rm alg}/K)$ for the absolute Galois group of K. The residue field of $K^{\rm alg}$ is denoted by k; it is the algebraic closure of $\mathbb{F}_K$.

Let $K^{\rm ur} \subset K^{\rm alg}$ be the maximal unramified extension of K and $I_K := {\rm Gal}(K^{\rm alg}/K^{\rm ur})$ the inertia group of K. We have a short exact sequence

$$1 \to I_K \to \Gamma_K \to \Gamma_{\mathbb{F}_K} \to 1,$$

where $\Gamma_{\mathbb{F}_K} = {\rm Gal}(k/\mathbb{F}_K)$ is the absolute Galois group of $\mathbb{F}_K$. This is the free profinite group of rank one generated by the *Frobenius element* σ_q, defined by $\sigma_q(\alpha) := \alpha^q$, where $q = |\mathbb{F}_K|$.

2.2. Let Y/K be a smooth projective and absolutely irreducible curve over K. We assume that the genus g of Y satisfies $g \geq 2$. We fix an auxiliary prime $\ell \neq p$. As explained in the introduction, we are interested in computing certain invariants of the natural action of Γ_K on the étale cohomology group

$$V = H^1_{\text{et}}(Y_{K^{\text{alg}}}, \mathbb{Q}_\ell) := \big(\varprojlim_n H^1_{\text{et}}(Y_{K^{\text{alg}}}, \mathbb{Z}/\ell^n)\big) \otimes \mathbb{Q}_\ell.$$

The *local L-factor* is the function $L(Y/K, s) := P_1(Y/K, q^{-s})^{-1}$, where

$$P_1(Y/K, T) := \det(1 - \sigma_q^{-1} \cdot T \mid V^{I_K}).$$

The *exponent of conductor* is defined as the integer

$$f = f_{Y/K} = \epsilon + \delta, \tag{2}$$

where

$$\epsilon := \dim V - \dim V^{I_K} \tag{3}$$

is the codimension of the I_K-invariant subspace and δ is the *Swan conductor* of V (see [**24**] Section 2, or [**27**], Section 3.1).

The invariant $f_{Y/K}$ depends only on the I_K-action on V, and vanishes if the I_K-action is trivial (i.e. if V is *unramified*). In general it gives a measure of "how bad" the ramification of V is.

2.3. A theorem of Deligne and Mumford ([**8**]) states the existence of a finite extension L/K such that the curve $Y_L = Y \otimes_K L$ has *semistable reduction*. This means that there exists a flat and proper $\mathcal{O}_L$-model $\mathcal{Y}$ of Y_L whose special fibre $\bar{Y}$ is reduced and has at most ordinary double points as singularities. The model $\mathcal{Y}$ is not unique, but the assumption $g \geq 2$ implies that there is a minimal semistable model $\mathcal{Y}^{\text{stab}}$, called the *stable model* of Y_L. The special fibre $\bar{Y}^{\text{stab}}$ of $\mathcal{Y}^{\text{stab}}$ is called the *stable reduction* of Y_L. It is a stable curve over the residue field $\mathbb{F}_L$, uniquely determined by the K-curve Y and the extension L/K. The dependence on L is very mild: if L'/L is a further finite extension then the stable reduction of Y corresponding to the extension L'/K is just the base change of $\bar{Y}^{\text{stab}}$ to the residue field of L'.

If $\mathcal{Y}$ is an arbitrary semistable model of Y_L, there exists a unique $\mathcal{O}_L$-morphism $c : \mathcal{Y} \to \mathcal{Y}^{\text{stab}}$ which is the identity on the generic fibre. The morphism c contracts the instable components of the special fibre of $\mathcal{Y}$ and is an isomorphism everywhere else. Here an irreducible component C of the special fibre of $\mathcal{Y}$ is called *instable* if C is smooth of genus zero and intersects the rest of the special fibre in at most two points.

After replacing L by a suitable finite extension, we may and will henceforth assume that L/K is a Galois extension. We also choose an embedding $L \subset K^{\text{alg}}$. Then the absolute Galois group Γ_K acts naturally on Y_L via its finite quotient $\Gamma := \mathrm{Gal}(L/K)$. Let $I \lhd \Gamma$ denote the inertia subgroup, i.e. the image of I_K in Γ. Note that the action of Γ on Y_L is only L/K-semilinear, but its restriction to I is L-linear.

DEFINITION 2.1. A semistable $\mathcal{O}_L$-model $\mathcal{Y}$ of Y_L is called *quasi-stable* if the tautological action of Γ on Y_L extends to an action on $\mathcal{Y}$.

The uniqueness of the stable model shows that it is quasi-stable. For our purposes it is more convenient to work with an arbitrary quasi-stable model $\mathcal{Y}$. Let $\bar{Y}$ denote the special fibre of $\mathcal{Y}$. Restricting the canonical Γ-action on $\mathcal{Y}$ to $\bar{Y}$ yields a canonical action of Γ on $\bar{Y}$. This action is again semilinear, meaning that the structure map $\bar{Y} \to \mathrm{Spec}\mathbb{F}_L$ is Γ-equivariant. However, the action of the inertia group I on $\bar{Y}$ is $\mathbb{F}_L$-linear.

We let $\bar{Z} := \bar{Y}/\Gamma$ denote the quotient scheme. It has a natural structure of an $\mathbb{F}_K$-scheme, and as such we have $\bar{Z}_{\mathbb{F}_L} := \bar{Z} \otimes_{\mathbb{F}_K} \mathbb{F}_L = \bar{Y}/I$. Since the quotient of a semistable curve by a finite group of geometric automorphisms is semistable, it follows that $\bar{Z} \otimes_{\mathbb{F}_K} \mathbb{F}_L$ is a semistable curve over $\mathbb{F}_L$. We conclude that $\bar{Z}$ is a semistable curve over $\mathbb{F}_K$. We denote by $\bar{Z}_k := \bar{Z} \otimes_{\mathbb{F}_K} k$ the base change of $\bar{Z}$ to the algebraic closure k of $\mathbb{F}_K$.

DEFINITION 2.2. The $\mathbb{F}_K$-curve $\bar{Z} = \bar{Y}/\Gamma$ is called the *inertial reduction* of Y, corresponding to the quasi-stable model $\mathcal{Y}$.

REMARK 2.3. In Section 1.5, we considered the inertial reduction $\bar{Z}^{\mathrm{inert}}$ corresponding to the stable model $\mathcal{Y}^{\mathrm{stab}}$. It is canonically associated with the K-curve Y and does not depend on the choice of the Galois extension L/K.

An arbitrary quasi-stable model $\mathcal{Y}$ admits a contraction map $c : \mathcal{Y} \to \mathcal{Y}^{\mathrm{stab}}$, which is Γ-equivariant. The inertial reduction $\bar{Z}$ corresponding to $\mathcal{Y}$ admits therefore a map $\bar{Z} \to \bar{Z}^{\mathrm{inert}}$ contracting the components of $\bar{Z}$ which are the image of the instable components of $\bar{Y}$. The image of a stable component of $\bar{Y}$ may be an instable component of $\bar{Z}$. So in general, $\bar{Z}$ is not a stable curve.

The following theorem is the main result of this section.

THEOREM 2.4. *Let $\bar{Z}$ be the inertial reduction of Y corresponding to some quasi-stable model $\mathcal{Y}$. We have a natural, Γ_K-equivariant isomorphism*

$$H^1_{\mathrm{et}}(Y_{K^{\mathrm{alg}}}, \mathbb{Q}_\ell)^{I_K} \cong H^1_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell).$$

COROLLARY 2.5. *In the situation of Theorem 2.4, the local L-factor $L(Y/K, s)$ is equal to the numerator of the local zeta function of $\bar{Z}$, i.e.*

$$L(Y/K, s) = P_1(\bar{Z}, q^{-s})^{-1},$$

where

$$P_1(\bar{Z}, T) := \det\left(1 - \mathrm{Frob}_q \cdot T | H^1_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell)\right)$$

and $\mathrm{Frob}_q : \bar{Z} \to \bar{Z}$ *is the relative q-Frobenius endomorphism and $q = |\mathbb{F}_K|$.*

Proof. The action of Γ_K on $H^1_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell)$ factors through the quotient $\Gamma_K \to \Gamma_{\mathbb{F}_K}$. The resulting $\Gamma_{\mathbb{F}_K}$-action is the same as the action induced by the identification $\bar{Z}_k = \bar{Z} \otimes k$. It follows that the action of an arithmetic Frobenius element $\sigma_q \in \Gamma_K$ on $H^1_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell)$ is induced by the map $\mathrm{Id}_{\bar{Z}} \otimes \sigma_q$. But the composition $(\mathrm{Id}_{\bar{Z}} \otimes \sigma_q) \circ (\mathrm{Frob}_q \otimes \mathrm{Id}_k)$ is equal to the *absolute q-Frobenius* of $\bar{Z}_k$. Since the absolute Frobenius induces the identity on étale cohomology, it follows that $\mathrm{Frob}_q = \sigma_q^{-1}$ on $H^1_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell)$. (This is a standard argument, see e.g. [**7**], Proposition 4.8 (ii) or [**6**].) The claim is now a consequence of Theorem 2.4 and the definition of $L(Y/K, s)$. □

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Corollary 2.5 implies that we can compute the local L-factor $L(Y/K, s)$ from the explicit knowledge of the inertial reduction $\bar{Z}$. In a special case, this is also enough to determine the exponent of conductor $f_{Y/K}$. The computation of $f_{L/K}$ without the tameness assumption is described in Section 2.6.

COROLLARY 2.6. *In the situation of Theorem 2.4, we additionally assume that L/K is at most tamely ramified. Then*

$$f_{Y/K} = 2g(Y) - \dim H^1_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell).$$

Proof. If the extension L/K is at most tamely ramified, the action of Γ_K on $H^1_{\mathrm{et}}(Y_{K^{\mathrm{alg}}}, \mathbb{Q}_\ell)$ is tame. The definition of the Swan conductor implies that $\delta = 0$ in (2). The claim is now a direct consequence of Theorem 2.4 and the definition of the conductor $f_{Y/K}$ in (2). □

2.4. Corollary 2.5 reduces the calculation of the local L-factor to the calculation of the relative Frobenius endomorphism on the étale cohomology of the semistable curve $\bar{Z}$. The following well-known lemma describes this action.

In this subsection, we let $\bar{Z}/\mathbb{F}_K$ be an arbitrary semistable curve defined over the finite field $\mathbb{F}_K$. Let k be the algebraic closure of $\mathbb{F}_K$ and $\bar{Z}_k$ the base change to k. Denote by $\pi : \bar{Z}_k^{(0)} \to \bar{Z}_k$ the normalization. Then $\bar{Z}_k^{(0)}$ is the disjoint union of its irreducible components, which we denote by $(\bar{Z}_j)_{j\in J}$. These correspond to the irreducible components of $\bar{Z}_k$. The components $\bar{Z}_j$ are smooth projective curves. The absolute Galois group $\Gamma_{\mathbb{F}_K}$ of $\mathbb{F}_K$ naturally acts on the set of irreducible components. We denote the permutation character of this action by χ_{comp}.

Let $\xi \in \bar{Z}_k$ be a singular point. Then $\pi^{-1}(\xi) \subset \bar{Z}_k^{(0)}$ consists of two points. We define a one-dimensional character ε_ξ on the stabilizer $\Gamma_{\mathbb{F}_K(\xi)} \subset \Gamma_{\mathbb{F}_K}$ of ξ as follows. If the two points in $\pi^{-1}(\xi)$ are permuted by $\Gamma_{\mathbb{F}_K(\xi)}$, then ε_ξ is the unique character of order two. Otherwise, $\varepsilon_\xi = \mathbf{1}$ is the trivial character. Denote by χ_ξ the character of the induced representation

$$\mathrm{Ind}_{\Gamma_{\mathbb{F}_K(\xi)}}^{\Gamma_{\mathbb{F}_K}} \varepsilon_\xi.$$

In the case that $\varepsilon_\xi = 1$ this is just the character of the permutation representation of the orbit of ξ. Define

$$\chi_{\mathrm{sing}} = \sum_\xi \chi_\xi.$$

Here the sum runs over a system of representatives of the orbits of $\Gamma_{\mathbb{F}_K}$ acting on the set of the singularities of $\bar{Z}_k$ (these correspond exactly to the singularities of $\bar{Z}$).

We denote by $\Delta_{\bar{Z}_k}$ the graph of components of $\bar{Z}_k$.

LEMMA 2.7. *Let $\bar{Z}/\mathbb{F}_K$ be a semistable curve and ℓ a prime with $\ell \nmid q$.*

1. *We have a decomposition*

$$H^1_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell) = \oplus_{j\in J} H^1_{\mathrm{et}}(\bar{Z}_j, \mathbb{Q}_\ell) \oplus H^1(\Delta_{\bar{Z}_k})$$

   as $\Gamma_{\mathbb{F}_K}$-representation.

2. *The character of $H^1(\Delta_{\bar{Z}_k})$ as $\Gamma_{\mathbb{F}_K}$-representation is $1 + \chi_{\mathrm{sing}} - \chi_{\mathrm{comp}}$.*

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Proof. As before, we let $\pi : \bar{Z}_k^{(0)} \to \bar{Z}_k$ be the normalization. We have a short exact sequence

$$0 \to \mathbb{Q}_\ell \to \pi_*(\mathbb{Q}_\ell) \to Q \to 0$$

of sheaves on $\bar{Z}_k$, where $Q := \pi_*(\mathbb{Q}_\ell)/\mathbb{Q}_\ell$ is a skyscraper sheaf with support in the singular points. This induces

$$0 \to H^0_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell) \to H^0_{\mathrm{et}}(\bar{Z}_k, \pi_*(\mathbb{Q}_\ell)) \to H^0_{\mathrm{et}}(\bar{Z}_k, Q) \to \\ H^1_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell) \to H^1_{\mathrm{et}}(\bar{Z}_k, \pi_*(\mathbb{Q}_\ell)) \to 0.$$

Identifying $H^0_{\mathrm{et}}(\bar{Z}_k, \pi_*(\mathbb{Q}_\ell))$ with $\mathbb{Q}_\ell^J$, we find that the kernel of the map $H^1_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell) \to H^1_{\mathrm{et}}(\bar{Z}_k, \pi_*(\mathbb{Q}_\ell))$ equals $H^0_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell) \oplus H^0_{\mathrm{et}}(\bar{Z}_k, Q)/\mathbb{Q}_\ell^J$ as $\Gamma_{\mathbb{F}_K}$-representation. It is easy to see that the character of $H^0_{\mathrm{et}}(\bar{Z}_k, Q)$ is equal to χ_{sing}. This proves (2). Since $H^1_{\mathrm{et}}(\bar{Z}_k, \pi_*(\mathbb{Q}_\ell)) = \oplus_{j\in J} H^1_{\mathrm{et}}(\bar{Z}_j, \mathbb{Q}_\ell)$, part (1) follows as well. □

The irreducible components of $\bar{Z}$ are in general not absolutely irreducible. An irreducible component $\bar{Z}_{[j]}$ of $\bar{Z}$ decomposes in $\bar{Z}_k$ as a finite disjoint union of absolutely irreducible curves, which form an orbit under $\Gamma_{\mathbb{F}_K}$. Let $\bar{Z}_j$ be a representative of the orbit. Let $\Gamma_j \subset \Gamma_{\mathbb{F}_K}$ be the stabilizer of $\bar{Z}_j$ and $\mathbb{F}_{q_j} = k^{\Gamma_j}$. We may identify $\bar{Z}_{[j]}$ and $\bar{Z}_j/\Gamma_j$ as absolute schemes. The natural $\mathbb{F}_K$-structure of $\bar{Z}_{[j]}$ (which is missing from $\bar{Z}_j/\Gamma_j$) is given by

$$\bar{Z}_j/\Gamma_j \to \mathrm{Spec}(\mathbb{F}_{q_j}) \to \mathrm{Spec}(\mathbb{F}_K).$$

With this interpretation, the contribution of $\bar{Z}_{[j]}$ to the local zeta function in Corollary 2.5 can be computed explicitly using point counting. We refer to Section 7.2 for an example where $\mathbb{F}_{q_j} \neq \mathbb{F}_K$.

Summarizing, we see that to compute the local L-factor it suffices to describe the irreducible components of the normalization $\bar{Z}^{(0)}$ of $\bar{Z}$ using equations over $\mathbb{F}_K$, together with the inverse image $\bar{Z}^{(1)} \subset \bar{Z}^{(0)}$ of the singular locus of $\bar{Z}$. When L/K is at most tamely ramified the same information also yields the exponent of conductor. In the general case, we need somewhat more information (Theorem 2.9 below), which may be calculated in an equally explicit way. For superelliptic curves this will be done in Section 5.

2.5. The proof of Theorem 2.4 relies on the following (well known) proposition.

PROPOSITION 2.8. *Let K be a henselian local field. Let k denote the algebraic closure of the residue field of K. Let Y be a smooth projective curve over K and $\mathcal{Y}$ be an $\mathcal{O}_K$-model of Y which is semistable or regular. If $\mathcal{Y}$ is regular, we assume moreover that the gcd of the multiplicities of the components of the special fibre $\bar{Y}$ of $\mathcal{Y}$ is one. Then the cospecialization map induces an isomorphism*

$$H^1_{\mathrm{et}}(Y_{K^{\mathrm{alg}}}, \mathbb{Q}_\ell)^{I_K} \cong H^1_{\mathrm{et}}(\bar{Y}_k, \mathbb{Q}_\ell).$$

Proof. By **[19]**, Corollary 4.18, we have isomorphisms

$$H^1_{\mathrm{et}}(Y_{K^{\mathrm{alg}}}, \mathbb{Q}_\ell(1)) \cong V_\ell(\mathrm{Pic}^0(Y)), \quad H^1_{\mathrm{et}}(\bar{Y}_k, \mathbb{Q}_\ell(1)) \cong V_\ell(\mathrm{Pic}^0(\bar{Y})), \tag{4}$$

where $V_\ell(\,\cdot\,)$ denotes the rational ℓ-adic Tate module.

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Let $\mathcal{J}$ denote the Néron model of the Jacobian of Y and $\bar{\mathcal{J}}^0$ the connected component of its special fibre. Then by [**12**], 6.4 (see also [**25**], Lemma 2) we have

$$V_\ell(\mathrm{Pic}^0(Y))^{I_K} \cong V_\ell(\bar{\mathcal{J}}^0). \tag{5}$$

Under the conditions imposed on $\mathcal{Y}$ we have an isomorphism

$$\bar{\mathcal{J}}^0 \cong \mathrm{Pic}^0(\bar{Y}) \tag{6}$$

by [**5**], Theorem 9.5.4 and Corollary 9.7.2. The proposition follows by combining (4), (5) and (6). □

Proof. We prove Theorem 2.4. Let L/K be a finite Galois extension over which Y has semistable reduction. Let $\mathcal{Y}$ be a quasi-stable model of Y_L and $\bar{Y}$ its special fibre. Proposition 2.8 yields an isomorphism

$$H^1_{\mathrm{et}}(Y_{K^{\mathrm{alg}}}, \mathbb{Q}_\ell)^{I_L} \cong H^1_{\mathrm{et}}(\bar{Y}_k, \mathbb{Q}_\ell)$$

which is canonical, and therefore Γ_K-invariant. Taking I_K-invariants and using the Hochschild–Serre spectral sequence ([**19**], III.2.20), we conclude that

$$H^1_{\mathrm{et}}(Y_{K^{\mathrm{alg}}}, \mathbb{Q}_\ell)^{I_K} \cong H^1_{\mathrm{et}}(\bar{Y}_k, \mathbb{Q}_\ell)^{I_K} \cong H^1_{\mathrm{et}}(\bar{Y}_k/I_K, \mathbb{Q}_\ell).$$

Since $\bar{Y}_k/I_K = \bar{Z}_k$, Theorem 2.4 follows. □

2.6. We give a formula for the exponent of conductor $f_{Y/K}$ in terms of the stable reduction $\bar{Y}$ that works in general, i.e. without the tameness assumption of Corollary 2.6.

The exponent of conductor is defined in (2) as $f_{Y/K} = \epsilon + \delta$. Theorem 2.4 and (3) imply that

$$\epsilon = 2g_Y - \dim H^1_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell). \tag{7}$$

Therefore ϵ may be computed from the inertial reduction $\bar{Z}$.

The following result expresses the Swan conductor δ in terms of the special fibre $\bar{Y}$ of a quasi-stable model $\mathcal{Y}$. Let $(\Gamma_i)_{i\geq 0}$ be the filtration of $\Gamma = \mathrm{Gal}(L/K)$ by higher ramification groups. Then $\Gamma_0 = I$ is the inertia group and $\Gamma_1 = P$ its Sylow p-subgroup ([**23**], Chapter 4). Moreover, $\Gamma_i = 1$ for $i \gg 0$. Let $\bar{Y}_i := \bar{Y}/\Gamma_i$ be the quotient curve. Then $\bar{Y}_0 = \bar{Y}/I = \bar{Z}_{\mathbb{F}_L}$ and $\bar{Y}_i = \bar{Y}$ for $i \gg 0$.

THEOREM 2.9. *The Swan conductor is*

$$\delta = \sum_{i=1}^{\infty} \frac{|\Gamma_i|}{|\Gamma_0|} \cdot (2g_Y - 2g_{\bar{Y}_i}).$$

Here $g_{\bar{Y}_i}$ denotes the arithmetic genus of $\bar{Y}_i$.

Proof. Let $I_K^w \subset \Gamma_K$ denote the wild inertia subgroup. The image of I_K^w in the finite quotient $\Gamma = \mathrm{Gal}(L/K)$ is equal to Γ_1. It follows from [**1**], Theorem 1.5, that the action of I_K^w on $V = H^1_{\mathrm{et}}(Y_{K^{\mathrm{alg}}}, \mathbb{Q}_\ell)$ factors over the Γ_1-action. (Note that this is not true for the action of the full inertia group I_K.) To compute δ, we may therefore use the *Hilbert*

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formula of [**20**], page 3, which says that

$$\delta = \sum_{i=1}^{\infty} \frac{|\Gamma_i|}{|\Gamma_0|} \cdot \dim_{\mathbb{Q}_\ell} V/V^{\Gamma_i}. \tag{8}$$

Although *loc.cit* is an expression for the Swan conductor of the mod-ℓ-representation $\bar{V} = H^1_{\mathrm{et}}(Y_{K^{\mathrm{alg}}}, \mathbb{F}_\ell)$, we can use the same formula for V as well. This follows from [**27**], Proposition 3.1.42. To finish the proof it remains to show that

$$\dim_{\mathbb{Q}_\ell} V^{\Gamma_i} = 2g_{\bar{Y}_i} \tag{9}$$

for $i \geq 1$. Note again that (9) does not hold for $i = 0$: by Theorem 2.4 we have $V^{\Gamma_0} = H^1_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell)$, and the dimension of this space is equal to $2g_{\bar{Z}}$ only if the graph of components of $\bar{Z}$ is a tree.

The results of [**1**], Section 3, imply that V decomposes, as a Γ_1-module, into the direct sum

$$V = H^1_{\mathrm{et}}(\bar{Y}^{(0)}) \oplus H_1(\Delta_{\bar{Y}}) \oplus H^1(\Delta_{\bar{Y}}), \tag{10}$$

where $\bar{Y}^{(0)}$ is the normalization of $\bar{Y}$, $\Delta_{\bar{Y}}$ is the graph of components of $\bar{Y}$ and $H_1(\Delta_{\bar{Y}})$ (resp. $H^1(\Delta_{\bar{Y}})$) denotes the (co)homology of $\Delta_{\bar{Y}}$ with $\mathbb{Q}_\ell$-coefficients. Using the Hochschild–Serre spectral sequence, it follows from (10) that

$$V^{\Gamma_i} = H^1_{\mathrm{et}}(\bar{Y}_i^{(0)}) \oplus H_1(\Delta_{\bar{Y}_i}) \oplus H^1(\Delta_{\bar{Y}_i}), \tag{11}$$

for $i \geq 1$. The dimension of the right-hand side of (11) is equal to $2g_{\bar{Y}_i}$, proving (9). The theorem follows. □

Remark 2.10. The results of this section yield the following "trivial" upper bound for the exponent of conductor, which is easily computed in the case that the ramification of the extension L/K is known.

If L/K is at most tamely ramified we have already seen that $\delta = 0$, hence we have that $f_{Y/K} = \epsilon \leq 2g(Y)$.

Suppose that L/K is wildly ramified. Let h be the last jump in the filtration of higher ramification groups, i.e. $h = i$ is maximal with $\Gamma_i \neq \{0\}$. Then Theorem 2.9 implies that $\delta \leq 2g(Y)h|P|/|\Gamma_0|$. It follows that

$$f_{Y/K} = \epsilon + \delta \leq 2g(Y)(1 + h|P|/|\Gamma_0|).$$

3. Admissible covers

3.1. Let $K/\mathbb{Q}_p$ be a p-adic number field as before and $\phi : Y \to X = \mathbb{P}^1_K$ a finite cover over K. We assume that Y is smooth, absolutely irreducible and of genus $g \geq 2$.

Let L/K be a finite extension over which Y has semistable reduction. There exists a unique semistable model $\mathcal{X}$ of X_L such that ϕ extends to a finite $\mathcal{O}_L$-morphism $\mathcal{Y}^{\mathrm{stab}} \to \mathcal{X}$ ([**18**]). Moreover, the stable model $\mathcal{Y}^{\mathrm{stab}}$ is the normalization of $\mathcal{X}$ inside the function field of Y_L. If ϕ is a Galois cover with Galois group G, then the G-action on Y_L extends to $\mathcal{Y}^{\mathrm{stab}}$ and the quotient scheme $\mathcal{X} := \mathcal{Y}^{\mathrm{stab}}/G$ has the desired property.

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Our strategy for computing the stable reduction of Y is to try to reverse the process described above: we try to find a semistable model $\mathcal{X}$ of X whose normalization $\mathcal{Y}$ with respect to Y is again semistable. In [**3**], a general method for finding such semistable model $\mathcal{X}$ is developed. This approach has been made algorithmic in [**2**] for cyclic covers $\phi : Y \to \mathbb{P}^1_K$ of degree p, were p is the residue characteristic.

The case that ϕ is a Galois cover where p does not divide the order of the Galois group G is much easier than the "wild" case. In this case it is well known how to compute the stable reduction of Y. The main insight goes back to Harris–Mumford ([**14**]) and is based on the notion of *admissible covers*. We describe the result in Section 3.3.

3.2. We first need a generalization of the notion of a (semi)stable model.

DEFINITION 3.1. Let S be a scheme, $\mathcal{X} \to S$ a semistable curve over S and $s_1, \ldots, s_r : S \to \mathcal{X}^{\mathrm{sm}}$ disjoint sections supported in the smooth locus of $\mathcal{X} \to S$. Then $(\mathcal{X}/S, s_1, \ldots, s_r)$ is called a *pointed semistable curve* over S (cf. [**15**]). Since we are usually not interested in ordering the sections s_i, we write $\mathcal{D} \subset \mathcal{X}$ for the relative divisor composed of the images of the s_i and call $(\mathcal{X}, \mathcal{D})$ a *marked semistable curve*. The divisor $\mathcal{D} \subset \mathcal{X}$ is called a *marking* of X/S.

Let K be a local field as before and X/K a smooth projective curve. Let $D \subset X$ be a smooth relative divisor of degree d over Spec$K$. We say that D *splits* over K if D consist of d distinct K-rational points. We say that the marked curve (X, D) has *semistable reduction* if D splits and the pair (X, D) extends to a marked semistable curve $(\mathcal{X}, \mathcal{D})$ over $\mathcal{O}_K$. If this is the case, $(\mathcal{X}, \mathcal{D})$ is called a *semistable model* of (X, D).

The semistable reduction theorem extends to the marked case, as follows.

PROPOSITION 3.2. *Let (X, D) be as above.*

1. *There exists a finite extension L/K such that (X_L, D_L) has semistable reduction.*
2. *Assume, moreover, that $2g(X) - 2 + d > 0$. Then there exists a unique minimal semistable model $(\mathcal{X}, \mathcal{D})$ (which we call the* stable model *of (X, D)).*
3. *If $g = 0$ and D splits then (X, D) has semistable reduction.*
4. *Assume that $g = 0$, $d \geq 3$ and that D splits. Let $(\bar{X}, \bar{D})$ be the special fibre of the stable model $(\mathcal{X}, \mathcal{D})$ of (X, D). Then $\bar{X}$ is a tree of projective lines. Every irreducible component $\bar{X}_v$ of $\bar{X}$ has at least three points which are either singular points of $\bar{X}$ or belong to the support of the divisor $\bar{D}$.*

Proof. Statements (1) and (2) follow from the Semistable Reduction Theorem (Section 2.3) combined with the main result of [**15**].

Statements (3) and (4) are proved in [**11**]. In that paper one also finds a much more direct proof for (1) and (2) in the case that $g = 0$. □

3.3. We return to the situation from the beginning of this section. Let $\phi : Y \to X = \mathbb{P}^1_K$ be a finite cover of the projective line, where Y is smooth and absolutely irreducible over K.

Let $D \subset X$ be the *branch locus* of ϕ, i.e. the reduced closed subscheme exactly supporting the branch points of ϕ. Then $D \to \mathrm{Spec}K$ is a finite flat morphism. Since the characteristic of K is zero and D is reduced by definition, $D \to \mathrm{Spec}K$ is actually étale. The geometric points of D are exactly the branch points of $\phi_{K^{\mathrm{alg}}}$. Let d denote the degree of D, i.e. the number of branch points of $\phi_{K^{\rm alg}}$. We make the following additional assumptions on ϕ.

ASSUMPTION 3.3.

(a) The cover ϕ is *potentially Galois*, i.e. the base change $\phi_{K^{\rm alg}} : Y_{K^{\rm alg}} \to X_{K^{\rm alg}}$ is a Galois cover.

(b) The characteristic p of the residue field of K does not divide the order of the Galois group G of $\phi_{K^{\rm alg}}$.

(c) We have $g(Y) \geq 2$.

Assumption 3.3.(c) implies that $d \geq 3$.

Let L/K be a finite extension which splits D. Then (X, D) has semistable reduction over L (Proposition 3.2.(3)). Let $(\mathcal{X}, \mathcal{D})$ denote the stable model of (X_L, D_L) and $\mathcal{Y}$ the normalization of $\mathcal{X}$ in the function field of Y. Then $\mathcal{Y}$ is a normal integral model of Y over $\mathcal{O}_L$. Let $\bar{Y} := \mathcal{Y} \otimes \mathbb{F}_L$ be the special fibre and $\bar{\phi} : \bar{Y} \to \bar{X}$ the induced map.

An irreducible component W of $\bar{Y}$ corresponds to a discrete valuation η_W of the function field of Y_L (since W is a prime divisor on $\mathcal{Y}$). Let m_W denote the ramification index of η_W in the extension of function fields induced by ϕ. The integer m_W is called the *multiplicity* of the component W. (Alternatively, one can define m_W as the length of $\mathcal{O}_{\mathcal{Y},W}/(\pi)$, where $\mathcal{O}_{\mathcal{Y},W}$ is the local ring at the generic point of W and π is a prime element of $\mathcal{O}_L$.)

THEOREM 3.4. *Let L/K and $\mathcal{Y}$ be as above. Assume that $\phi_L : Y_L \to X_L$ is a Galois cover and that $m_W = 1$ for every irreducible component W of $\bar{Y}$. Then $\mathcal{Y}$ is a quasi-stable model of Y_L. In particular, $\mathcal{Y}$ is semistable.*

Proof. The proof is a straightforward adaptation of the proof of [**18**], Theorem 2.3 to our situation. □

REMARK 3.5.

1. The quasi-stable model $\mathcal{Y}$ from Theorem 3.4 is in general not the stable model of Y. Furthermore, the extension L/K is in general not the minimal extension over which Y has semistable reduction.
2. A key step in the proof of Theorem 3.4 is showing that $\mathcal{Y} \to \mathcal{X}$ is an admissible cover (see [**14**] or [**26**]). For the purpose of the present paper, it suffices to know that this implies that smooth (resp. singular) points of $\bar{Y}$ map to smooth (resp. singular) points of $\bar{X}$. Since the irreducible components of $\bar{X}$ are smooth (see Section 4.2 below), it follows that the same holds for the irreducible components of $\bar{Y}$.

COROLLARY 3.6. *Let $\phi : Y \to X = \mathbb{P}^1_K$ be a cover satisfying Assumption 3.3, with branch locus $D \subset X$. Let L_0/K be a finite extension which splits D and such that ϕ_{L_0} is Galois. There exists a tamely ramified extension L/L_0 over which Y has semistable reduction.*

Proof. Let $(\mathcal{X}_0, \mathcal{D}_0)$ be the stable model of the marked curve (X_{L_0}, D_{L_0}) and $\mathcal{Y}_0$ the normalization of $\mathcal{X}_0$ in Y_{L_0}. Let e be the lcm of all multiplicities m_W, where W runs over the irreducible components of the special fibre of $\mathcal{Y}_0$. It is clear that e divides the order of the Galois group of ϕ_{L_0} and is therefore prime to p.

Let L/L_0 be a tamely ramified extension with ramification index divisible by e. Let $(\mathcal{X}, \mathcal{D})$ be the base change of $(\mathcal{X}_0, \mathcal{D}_0)$ to $\mathcal{O}_L$; this is the stable model of the marked

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curve (X_L, D_L). Let $\mathcal{Y}$ be the normalization of $\mathcal{X}$ in Y_L. It follows from Abhyankar's lemma ([**13**], Exposé X, Lemma 3.6, p. 297) that the multiplicities of the irreducible components of $\bar{Y}$ are one. Theorem 3.4 implies that $\mathcal{Y}$ is semistable. This proves the corollary. □

4. Superelliptic curves

4.1. As before, $K/\mathbb{Q}_p$ is a finite extension. Let $\phi : Y \to X := \mathbb{P}^1_K$ be the cover of curves which is birationally determined by an equation of the form

$$y^n = f(x),$$

where $f \in K[x]$ is a non-constant polynomial in the natural parameter x of the projective line $X = \mathbb{P}^1_K$ and $\phi(x, y) = x$. In other words, Y is the smooth projective curve with function field $F(Y) := K(x, y \mid y^n = f(x))$. We assume that f has no non-trivial factor which is an nth power in $K[x]$. This implies that every zero of f corresponds to a branch point of ϕ.

Let L_0/K be the splitting field of f and $S \subset L_0$ the set of roots of f. Then we can write

$$f = c \prod_{\alpha \in S} (x - \alpha)^{a_\alpha},$$

with $c \in K^\times$ and $a_\alpha \in \mathbb{N}$. We impose the following conditions on f and n.

ASSUMPTION 4.1.
(a) We have $\gcd(n, a_\alpha \mid \alpha \in S) = 1$.
(b) The exponent n is ≥ 2 and prime to p.
(c) We have $g(Y) \geq 2$.

We note that Assumption 4.1 implies Assumption 3.3. In fact, the base change of ϕ to $K^{\rm ur}$ is a Galois cover with cyclic Galois group of order n, branched over the roots of f and possibly also over ∞. The ramification index of the points of $\phi^{-1}(\infty)$ is $n/\gcd(n, \sum_{\alpha \in S} a_\alpha)$.

Our goal is to compute the stable reduction of Y in terms of the data f and n, following the procedure suggested by Theorem 3.4 and Corollary 3.6.

4.2. Let $D \subset X$ be the branch divisor of ϕ. Let L/L_0 be a finite extension. Then D splits over L, and $D_L \subset \mathbb{P}^1(L)$ satisfies

$$D_L = \begin{cases} S & \text{if } \sum_{\alpha \in S} a_\alpha \equiv 0 \pmod{n}, \\ S \cup \{\infty\} & \text{otherwise.} \end{cases}$$

Assumption 4.1.(c) implies that $|D_L| \geq 3$. Therefore the marked curve (X_L, D_L) has a stable model $(\mathcal{X}, \mathcal{D})$ (Proposition 3.2). In the rest of this section, we describe the special fibre $(\bar{X}, \bar{D})$ of $(\mathcal{X}, \mathcal{D})$ explicitly, in terms of the divisor $D_L \subset X_L$.

We first introduce some notation. Let $\Delta = (V(\Delta), E(\Delta))$ denote the graph of components of $\bar{X}$. This is a finite, undirected tree whose vertices $v \in V(\Delta)$ correspond the irreducible components $\bar{X}_v \subset \bar{X}$. Two vertices v_1, v_2 are adjacent if and only if

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the components $\bar{X}_{v_1}$ and $\bar{X}_{v_2}$ meet in a (necessarily unique) singular point of $\bar{X}$. For an element $\alpha \in D_L$, we denote by $\bar{\alpha} \in \bar{D} \subset \bar{X}$ its specialization. We obtain a map $\psi : D_L \to V(\Delta)$ defined by $\bar{\alpha} \in \bar{X}_{\psi(\alpha)}$. Proposition 3.2.(4) states that (Δ, ψ) is a *stably marked tree* ([**11**], Definition 1.2). By this we mean that Δ is an undirected tree and for each vertex $v \in V(\Delta)$ we have

$$\mathrm{val}(v) := |\psi^{-1}(v)| + |\{v' \in V(\Delta) \mid \{v, v'\} \in E(\Delta)\}| \geq 3.$$

Let us call an L-linear isomorphism $\lambda : X_L \xrightarrow{\sim} \mathbb{P}^1_L$ a *chart*. Since $X_L = \mathbb{P}^1_L$ by definition, a chart may be represented by an element in $\mathrm{PGL}_2(L)$. We call two charts λ_1, λ_2 *equivalent* if the automorphism $\lambda_2 \circ \lambda_1^{-1} : \mathbb{P}^1_L \xrightarrow{\sim} \mathbb{P}^1_L$ extends to an automorphism of $\mathbb{P}^1_{\mathcal{O}_L}$, i.e. corresponds to an element of $\mathrm{PGL}_2(\mathcal{O}_L)$. In other words, an equivalence class of charts corresponds to a right coset in $\mathrm{PGL}_2(\mathcal{O}_L)\backslash\mathrm{PGL}_2(L)$.

Let T denote the set of triples $t = (\alpha, \beta, \gamma)$ of pairwise distinct elements of D_L. For $t = (\alpha, \beta, \gamma)$, we let λ_t denote the unique chart such that

$$\lambda_t(\alpha) = 0, \quad \lambda_t(\beta) = 1, \quad \lambda_t(\gamma) = \infty.$$

Explicitly, we have

$$\lambda_t(x) = \frac{\beta - \gamma}{\beta - \alpha} \cdot \frac{x - \alpha}{x - \gamma}, \tag{12}$$

where we interpret this formula in the obvious way if $\infty \in \{\alpha, \beta, \gamma\}$. The equivalence relation $\sim$ on charts defined above induces an equivalence relation on T, which we denote by $\sim$ as well.

PROPOSITION 4.2. *Let $(\mathcal{X}, \mathcal{D})$ be the stable model of (X_L, D_L).*

1. *For all $t \in T$ the chart λ_t extends to a proper $\mathcal{O}_L$-morphism $\lambda_t : \mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_L}$. Its reduction to the special fibre is a contraction morphism*

$$\bar{\lambda}_t : \bar{X} \to \mathbb{P}^1_{\mathbb{F}_L}$$

which contracts all but one component of $\bar{X}$ to a closed point.
2. *For every component $\bar{X}_v$ there exists $t \in T$ such that $\bar{\lambda}_t$ does not contract $\bar{X}_v$ (and hence induces an isomorphism $\bar{X}_v \xrightarrow{\sim} \mathbb{P}^1_{\mathbb{F}_L}$).*
3. *The equivalence class of the chart λ_t in (2) is uniquely determined by the component $\bar{X}_v$. We therefore obtain a bijection $V(\Delta) \cong T/_\sim$.*

Proof. By combining Lemma 5 with the corollary to Lemma 4 of [**11**], we see that for every $t = (\alpha, \beta, \gamma)$ there exists a unique proper $\mathcal{O}_L$-morphism $\lambda : \mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_L}$ such that $\lambda(\alpha) = 0$, $\lambda(\beta) = 1$, $\lambda(\gamma) = \infty$. Clearly, the restriction of λ to the generic fibre is equal to the chart λ_t. From now on, we write $\lambda = \lambda_t$.

The restriction of λ_t to the special fibre is a proper $\mathbb{F}_L$-morphism $\bar{\lambda}_t : \bar{X} \to \mathbb{P}^1_{\mathbb{F}_L}$. Since $(\bar{X}, \bar{D})$ is stably marked, the morphism $\bar{\lambda}_t$ is uniquely determined by its restriction to $\bar{D}$. For $\delta \in D_L$, we have $\bar{\lambda}_t(\bar{\delta}) = \overline{\lambda_t(\delta)}$ by construction. Therefore, $\bar{\lambda}_t$ is equal to the *generalized cross-ratio map* defined in [**11**], Section 1. Statements (1)–(3) follow immediately from the properties of this map proved in *loc.cit.* □

REMARK 4.3. For $t = (\alpha, \beta, \gamma) \in T$, consider the map

$$\phi_t : D_L \to \mathbb{P}^1_{\mathbb{F}_L}, \quad \delta \mapsto \bar{\lambda}_t(\bar{\delta}),$$

where $\bar{\lambda}_t : \bar{X} \to \mathbb{P}^1_{\mathbb{F}_L}$ is the map defined by Proposition 4.2.(1). By the proof of the proposition, we have

$$\phi_t(\delta) = \overline{\lambda_t(\delta)},$$

where $\overline{\cdot}$ stands for the reduction map $\mathbb{P}^1_L \to \mathbb{P}^1_{\mathbb{F}_L}$. Together with formula (12), this shows that the collection of maps (ϕ_t) (which constitute a finite amount of data) can be computed explicitly. By **[11]**, Proposition 1, the stably marked curve $(\bar{X}, \bar{D})$ can be reconstructed effectively from the data $(\phi_t)_{t\in T}$. In particular, the following facts are shown in *loc.cit.*

1. We have $t \sim t'$ if and only if $\phi_t = \phi_{t'}$. The maps ϕ_t determine the set $V(\Delta)$, via the bijection of Proposition 4.2.(3).
2. For every $\delta \in D_L$ there exists a $t \in T$, unique up to $\sim$, such that $|\phi_t^{-1}(\phi_t(\delta))| = 1$. Moreover, $\bar{\delta} \in \bar{X}_v$, where $v \in V(\Delta)$ corresponds to t via the correspondence in (1). It follows that we can recover the map $\psi : D_L \to V(\Delta)$ from the maps ϕ_t.
3. Fix $t \in T$ and let $v \in V(\Delta)$ correspond to t via (1). Then the isomorphism $\bar{X}_v \xrightarrow{\sim} \mathbb{P}^1_{\mathbb{F}_L}$ induced by $\bar{\lambda}_t$ sends $\bar{\delta}$ to $\phi_t(\delta)$, for all $\delta \in D_L$. In this way, we can recover the divisor $\bar{D} \subset \bar{X}$.

NOTATION 4.4. For every vertex $v \in V(\Delta)$, we choose $t \in T$ corresponding to v via the bijection of Proposition 4.2.(3). Let $x_v := \lambda_t^*(x) \in L(x)$ be the pullback of the standard coordinate x of $X_L = \mathbb{P}^1_L$ via the chart λ_t. Equation (12) expresses x_v in terms of the original coordinate x and the triple $t = (\alpha, \beta, \gamma)$.

Since $\mathcal{X}$ is an integral, normal scheme and $\bar{X}_v \subset \mathcal{X}$ is an irreducible closed subset of codimension one, the local ring of $\mathcal{X}$ at the generic point of $\bar{X}_v$ is a discrete valuation ring. We denote the corresponding discrete valuation on $L(x)$ by η_v, where we normalize η_v such that $\eta_v|_L$ is the standard valuation on L. Then η_v is simply the Gauss valuation of $L(x_v)$ with respect to the parameter x_v. The residue field of η_v is naturally identified with the function field of $\bar{X}_v$. We have that

$$F(\bar{X}_v) = \mathbb{F}_L(\bar{x}_v),$$

where $\bar{x}_v$ denotes the image of x_v in the function field $F(\bar{X}_v)$. In fact, $\bar{x}_v$ is the pullback of the standard parameter of $\mathbb{P}^1_{\mathbb{F}_L}$ via the isomorphism $\bar{X}_v \xrightarrow{\sim} \mathbb{P}^1_{\mathbb{F}_L}$ induced by $\bar{\lambda}_t$.

4.3. As in Section 4.2, we denote by $(\mathcal{X}, \mathcal{D})$ the stable model of the marked curve (X_L, D_L), where L is a finite extension of the splitting field L_0 of f. Let $\mathcal{Y}$ be the normalization of $\mathcal{X}$ in the function field of Y_L. Corollary 3.6 states that $\mathcal{Y}$ is a semistable model of Y if L is a sufficiently large tame extension of L_0. The following proposition quantifies the degree of L/L_0 and describes the special fibre $\bar{Y}$ of $\mathcal{Y}$.

Choose a prime element π of $\mathcal{O}_L$. Consider $v \in V(\Delta)$ and let x_v be the corresponding coordinate as in Notation 4.4. Define

$$N_v := \eta_v(f)/\eta_v(\pi), \qquad f_v := \pi^{-N_v} f.$$

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Then $\eta_v(f_v) = 0$ and we may consider the image $\bar{f}_v$ of f_v in the residue field $\mathbb{F}_L(\bar{x}_v)$ of the valuation η_v. Let n_v denote the order of the image of $\bar{f}_v$ in the group $\mathbb{F}_L(\bar{x}_v)^\times/(\mathbb{F}_L(\bar{x}_v)^\times)^n$.

PROPOSITION 4.5.

1. *Assume that the field L contains all nth roots of unity. Then the model $\mathcal{Y}$ of Y_L is semistable if and only if $n \mid N_v$ for all $v \in V(\Delta)$.*
2. *Assume that the condition in (1) holds, and fix $v \in V(\Delta)$. Then there is a bijection between the set of irreducible components of $\bar{Y}_v := \bar{Y}|_{\bar{X}_v}$ and the set of elements $\bar{g} \in \mathbb{F}_L(\bar{x}_v)^\times$ satisfying*

$$\bar{g}^{n/n_v} = \bar{f}_v.$$

The restriction of $\bar{\phi}$ to the irreducible component corresponding to $\bar{g}$ is the Kummer cover with equation

$$\bar{y}_v^{n_v} = \bar{g},$$

where $y_v = \pi^{-N_v/n} y$.

Proof. By Theorem 3.4 and the proof of Corollary 3.6, $\mathcal{Y}$ is semistable if and only if the valuation η_v is unramified in the extension of function fields $F(Y_L)/F(X_L)$ for all $v \in V(\Delta)$. If this is the case, the irreducible components of $\bar{Y}_v$ are in bijection with the discrete valuations on $F(Y_L)$ extending η_v. The irreducible component corresponding to an extension ξ_v of η_v to $F(Y_L)$ is the smooth projective curve whose function field is the residue field of ξ_v. This reduces the proof of the proposition to standard facts on the behaviour of valuations in Kummer extensions. For convenience, we give the main argument.

Assume that $n \mid N_v$ for some v. Then the element $y_v := \pi^{-N_v/n} y \in F(Y_L)$ generates the extension $F(Y_L)/F(X_L)$ and is a root of the irreducible polynomial $F_v := T^n - f_v \in L(x_v)[T]$. The polynomial F_v is integral with respect to η_v. Its reduction is separable and is the product of n/n_v irreducible factors of degree n_v, as follows:

$$\bar{F}_v = \prod_{\bar{g}^{n/n_v} = \bar{f}_v} (T^{n_v} - \bar{g}).$$

(Here the hypothesis $\zeta_n \in L$ is used.) It follows that η_v is unramified in the extension $F(Y_L)/F(X_L)$. Furthermore, the extensions of η_v are in bijection with the irreducible factors of $\bar{F}_v$. For each extension the residue field extension is generated by the image of y_v, which is a root of the corresponding irreducible factor of $\bar{F}_v$. This proves (2) and the backward implication in (1). The forward implication in (1) is left to the reader. □

COROLLARY 4.6. *Assume that L contains the nth roots of unity and that the ramification index of L/L_0 is divisible by n. Then Y_L has semistable reduction. The irreducible components of the reduction $\bar{Y}$ are absolutely irreducible.*

5. Computing the inertial reduction

We continue using the notation of the previous section. In particular, $\phi : Y \to X = \mathbb{P}^1_K$ is a Kummer cover given by the equation

$$y^n = f(x)$$

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satisfying Assumption 4.1, L_0/K is the splitting field of f and L/L_0 is a sufficiently large finite extension. The precise meaning of “sufficiently large” is given by the condition of Proposition 4.5.(1). In this section, we assume that the possibly stronger condition from Corollary 4.6 holds.

Let $(\mathcal{X}, \mathcal{D})$ be the stable model of the marked curve (X_L, D_L) and $\mathcal{Y}$ the normalization of $\mathcal{X}$ in the function field of Y_L. By Proposition 4.5 and Remark 3.5.(1), $\mathcal{Y}$ is a quasi-stable model of Y_L. After enlarging L we may also assume that L/K is a Galois extension. The following assumption summarizes the requirements on L.

Assumption 5.1. We consider a finite extension L/K satisfying
- L contains the splitting field L_0 of f over K,
- L contains a primitive nth root of 1 and an nth root of a uniformizing element of L_0,
- the extension L/K is Galois.

As before we let $\Gamma = \mathrm{Gal}(L/K)$ denote the Galois group of L/K and $I \lhd \Gamma$ the inertia subgroup. The group Γ has a natural semilinear action on the special fibre $\bar{Y} := \mathcal{Y} \otimes_{\mathcal{O}_L} \mathbb{F}_L$. Recall that the *inertial reduction* of Y (with respect to the quasi-stable model $\mathcal{Y}$) is defined as the quotient $\bar{Z} := \bar{Y}/\Gamma$. In this section, we give a concrete recipe how to compute $\bar{Z}$. Our assumption is that the extension L/K together with the Galois group $\Gamma = \mathrm{Gal}(L/K)$ and its action on a chosen prime element π of L are known explicitly.

Our strategy to compute $\bar{Z}$ may be summarized as follows. It is clear that the cover $\phi : Y \to X$ extends to a finite Γ-equivariant morphism $\mathcal{Y} \to \mathcal{X}$. Its restriction to the special fibre is a finite Γ-equivariant map $\bar{\phi} : \bar{Y} \to \bar{X}$ between semistable curves over $\mathbb{F}_L$. It induces a finite map $\bar{Z} \to \bar{W} := \bar{X}/\Gamma$ between semistable curves over $\mathbb{F}_K$. We also write $\bar{Z}_{\mathbb{F}_L} := \bar{Y}/I$ and $\bar{W}_{\mathbb{F}_L} := \bar{X}/I$ for the quotients by the action of the inertia group. Diagram (13) shows the relevant maps. Our strategy is to first describe the

$$\begin{tikzcd}
\bar{Y} \arrow[dr] \arrow[ddd] & & \\
& \bar{Z}_{\mathbb{F}_L} = \bar{Y}/I \arrow[dr] \arrow[ddd] & \\
& & \bar{Z} = \bar{Y}/\Gamma \arrow[ddd] \\
\bar{X} \arrow[dr] \arrow[ddd] & & \\
& \bar{W}_{\mathbb{F}_L} = \bar{X}/I \arrow[dr] \arrow[ddd] & \\
& & \bar{W} = \bar{X}/\Gamma \arrow[ddd] \\
\mathrm{Spec}\mathbb{F}_L \arrow[dr] & & \\
& \mathrm{Spec}\mathbb{F}_L \arrow[dr] & \\
& & \mathrm{Spec}\mathbb{F}_K
\end{tikzcd} \tag{13}$$

action of Γ on $\bar{X}$ (Sections 5.1 and 5.2), and then the maps $\bar{Z}_{\mathbb{F}_L} \to \bar{W}_{\mathbb{F}_L}$ (Section 5.3) and $\bar{Z} \to \bar{W}$ (Section 5.4).

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5.1. Recall that $(\bar{X}, \bar{D})$ is the special fibre of the stable model $(\mathcal{X}, \mathcal{D})$ of the marked curve (X_L, D_L). In particular, $\bar{X}$ is a semistable curve of genus zero. Let Δ denote the tree of components associated with $\bar{X}$. In Section 4.2, we gave a description of the tree Δ in terms of the divisor $D_L \subset X_L$. It is clear from this description that the action of Γ on $V(\Delta)$ is determined, in an explicit way, by the action of Γ on D_L. (We refer to Section 6.4 for an explicit example.) We may therefore consider the action of Γ on Δ as known.

For a vertex $v \in V(\Delta)$, we let $\Gamma_v \subset \Gamma$ be the stabilizer of the component $\bar{X}_v$ of $\bar{X}$ corresponding to v. The subgroup Γ_v consists exactly of those elements of Γ leaving invariant the set $\psi^{-1}(v)$ consisting of the branch points $\alpha \in D_L$ specializing to $\bar{X}$.

The curve $\bar{W} = \bar{X}/\Gamma$ is a semistable curve over $\mathbb{F}_K$ with component graph Δ/Γ. Then $\bar{W}_v := \bar{X}_v/\Gamma_v$ is the irreducible component of $\bar{W}$ corresponding to the Γ-orbit of v. In order to compute $\bar{W} = \bar{X}/\Gamma$, it therefore suffices to compute $\bar{W}_v = \bar{X}_v/\Gamma_v$, for each v.

5.2. Let us fix a vertex $v \in V(\Delta)$. The goal of Lemma 5.2 below is to describe the action of Γ_v on the curve $\bar{X}_v$. We retain Notation 4.4 and write

$$x_v = A(x) = \frac{ax+b}{cx+d}, \qquad \text{with } A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(L). \tag{14}$$

LEMMA 5.2. *For $\sigma \in \Gamma_v$, the matrix $B_\sigma := \sigma(A)A^{-1}$ lies in $\mathrm{PGL}_2(\mathcal{O}_L)$. Furthermore, if $\psi_\sigma \in \mathrm{Aut}(\mathbb{F}_L(\bar{x}_v))$ denotes the automorphism induced by the action of σ on $\bar{X}_v$, then*

$$\psi_\sigma(\bar{x}_v) = \bar{B}_\sigma(\bar{x}_v).$$

Here $\bar{B}_\sigma \in \mathrm{PGL}_2(\mathbb{F}_L)$ denotes the reduction of B_σ.

Proof. An element $\sigma \in \Gamma = \mathrm{Gal}(L/K)$ acts canonically on $L(x)$, the function field of $X_L = \mathbb{P}^1_L$, by fixing the generator x. Therefore,

$$\sigma(x_v) = \sigma(A(x)) = \sigma(A)(x) = \sigma(A)(A^{-1}(x_v)) = B_\sigma(x_v).$$

If $\sigma \in \Gamma_v$ then σ fixes the Gauss valuation corresponding to x_v and hence $B_\sigma \in \mathrm{PGL}_2(\mathcal{O}_L)$. The equality $\psi_\sigma(\bar{x}_v) = \bar{B}_\sigma(\bar{x}_v)$ is a direct consequence. □

REMARK 5.3. Clearly, the map $\Gamma_v \to \mathrm{Aut}(\mathbb{F}_L(\bar{x}_v))$, $\sigma \mapsto \psi_\sigma$, is a group homomorphism. However, the map $\Gamma_v \to \mathrm{PGL}_2(\mathbb{F}_L)$, $\sigma \mapsto \bar{B}_\sigma$, is *not* a group homomorphism. A straightforward computation shows that it obeys the rule

$$\bar{B}_{\sigma\tau} = \sigma(\bar{B}_\tau) \cdot \bar{B}_\sigma.$$

The reason is that the restriction of ψ_σ to $\mathbb{F}_L$ need not be trivial if $\sigma \notin I_v$. It follows that the map $\sigma \mapsto \bar{B}_\sigma$ defines an element of the set of non-abelian cocycles

$$Z^1(\Gamma, \mathrm{PGL}_2(\mathbb{F}_L)^{\mathrm{opp}}),$$

as defined in **[22]**, I, Section 5.1. Of course, the restriction of this cocycle to the inertia group $I \subset \Gamma$ is a group homomorphism.

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LEMMA 5.4. *For a suitable choice of the chart λ_v we have*

$$\psi_\sigma(\bar{x}_v) = a_\sigma \bar{x}_v + b_\sigma,$$

with $a_\sigma, b_\sigma \in \mathbb{F}_L$, for all $\sigma \in \Gamma_v$. In other words, ψ_σ is an affine linear transformation for all $\sigma \in \Gamma_v$.

Proof. To prove the lemma, we need to show the existence of an $\mathbb{F}_L$-rational point $p_1 \in \bar{X}_v$ which is fixed by all $\sigma \in \Gamma_v$. Let $p_0 := \bar{\infty} \in \bar{X}$ denote the specialization of the point $\infty \in X_L = \mathbb{P}^1_L$. It is clear that p_0 is an $\mathbb{F}_L$-rational point fixed by Γ. If $p_0 \in \bar{X}_v$ then $p_1 := p_0$ satisfies the requirements.

Otherwise, we let $p_1 \in \bar{X}_v$ be the unique singular point of $\bar{X}$ such that p_0 is contained in the connected component of $\bar{X} - \{p_1\}$ not containing $\bar{X}_v - \{p_1\}$. In other words, p_1 is the unique singular point of $\bar{X}$ contained in $\bar{X}_v$ which is "nearest" to p_0. Since $\psi_\sigma \in \mathrm{Aut}(\mathbb{F}_L(\bar{x}_v))$, it follows that $p_1 \in \bar{X}_v$ is an $\mathbb{F}_L$-rational point which is fixed by the action of Γ_v. We now choose the chart λ_v such that p_1 is the point $\bar{x}_v = \infty$. This shows the statement of the lemma. □

5.3. We now describe how to compute the quotient $\bar{Z}_{\mathbb{F}_L} = \bar{Y}/I$ of $\bar{Y}$ by the action of the inertia group, together with the map $\bar{Z}_{\mathbb{F}_L} \to \bar{W}_{\mathbb{F}_L} = \bar{X}/I$. By what was explained in Section 5.1, it suffices to consider the subcurve $\bar{Y}_v := \bar{Y}|_{\bar{X}_v}$.

We choose a chart for $\bar{X}_v$ as in Lemma 5.4. Recall that this means that $\sigma \in I_v$ acts on the coordinate $\bar{x}_v$ as $\psi_\sigma(\bar{x}_v) = a_\sigma \bar{x}_v + b_\sigma$ with $a_\sigma, b_\sigma \in \mathbb{F}_L$. Abusing notation, we also write $\psi_\sigma(\bar{x}_v, \bar{y}_v)$ for the automorphism on $\bar{Y}_v$ induced by σ.

Recall that $\bar{Y}_v$ is given by the Kummer equation

$$\bar{y}_v^n = \bar{f}_v(\bar{x}_v), \tag{15}$$

where $y_v = \pi^{-N_v/n} y$ (Proposition 4.5.(2)). The curve $\bar{Y}_v$ is in general reducible. We prefer to work with the reducible equation (15) rather than the equation for the irreducible components. This means that we work with the function algebra $\mathbb{F}_L(\bar{x}_v)[\bar{y}_v]/(\bar{y}_v^n - \bar{f}_v)$ instead of with the function field of one of the irreducible components.

We have assumed that L contains a primitive nth root of unity (Assumption 5.1). It follows that the groups G and I_v commute inside $\mathrm{Aut}_{\mathbb{F}_L}(\bar{Y}_v)$. The quotient cover

$$\bar{Z}_{v,\mathbb{F}_L} = \bar{Y}_v/I_v \to \bar{W}_{v,\mathbb{F}_L} = \bar{X}_v/I_v$$

is therefore still Galois with Galois group $G/(I_v \cap G)$. In Propositions 5.5 and 5.6 below we compute a Kummer equation for this cover.

Our next goal is to compute an equation for the quotient curve of $\bar{Y}_v$ by the finite group I_v explicitly. Being an inertia group, $I_v = P_v \rtimes C_v$ is an extension of a cyclic group C_v of order prime to p by its Sylow p-subgroup P_v. The following proposition describes the action of P_v on $\bar{Y}_v$.

PROPOSITION 5.5. *Write* $\bar{P}_v = \{\psi_\sigma \mid \sigma \in P_v\}$ *for the image of* P_v *in* $\mathrm{Aut}_{\mathbb{F}_L}(\bar{Y}_v)$.

1. *For every* $\sigma \in P_v$ *we have that*

$$\psi_\sigma(\bar{x}_v, \bar{y}_v) = (\bar{x}_v + b_\sigma, \bar{y}_v)$$

for some $b_\sigma \in \mathbb{F}_L$.

2. *The group* $\bar{P}_v$ *is an elementary abelian* p*-group.*

Proof. Let $\sigma \in P_v$. The definition $y_v = \pi^{-N_v/n} y$ implies that $\psi_\sigma(\bar{y}_v) = \gamma_\sigma \bar{y}_v$. Since σ has p-power order, it follows that γ_σ is trivial. We have chosen the chart λ_v such that ψ_σ acts on $\bar{X}_v$ as affine linear transformation (Lemma 5.4). Statement (1) follows. Moreover, we may identify $\bar{P}_v$ with a subgroup of $\mathbb{F}_L$. This implies (2). □

Proposition 5.5 allows us to compute the quotient cover $\bar{Y}_v/P_v \to \bar{X}_v/P_v$. The coordinates $\bar{y}_v$ and

$$\bar{u}_v := \prod_{\sigma \in \bar{P}_v} \psi_\sigma(\bar{x}_v) = \prod_{\sigma \in \bar{P}_v} (\bar{x}_v + b_\sigma)$$

are $\bar{P}_v$-invariant and generate the function algebra of $\bar{Y}_v/P_v$. The rational function $\bar{f}_v(\bar{x}_v)$ is an element of $\mathbb{F}_L(\bar{u}_v)$, hence we may write $\bar{f}_v(\bar{x}_v) = \bar{g}_v(\bar{u}_v)$. The function $\bar{g}_v$ is easily determined explicitly by comparison of coefficients. We conclude that the curve $\bar{Y}_v/P_v$ is given by the Kummer equation

$$\bar{y}_v^n = \bar{g}_v(\bar{u}_v).$$

The Kummer cover $\bar{Y}_v/P_v \to \bar{X}_v/P_v$ is given by $(\bar{u}_v, \bar{y}_v) \mapsto \bar{u}_v$. Note that the degree of this cover is still n, since the intersection $G \cap \bar{P}_v \subset \mathrm{Aut}_{\mathbb{F}_L}(\bar{Y}_v)$ is trivial.

It remains to consider the quotient of $\bar{Y}_v/P_v$ by $I_v/P_v = C_v$, which is cyclic of order prime to p. We choose an element $\sigma \in I_v$ whose image generates C_v, this defines a section $C_v \to I_v$. Define μ as the order of ψ_σ considered as automorphism of $\bar{Y}_v$ and m as the order of $\psi_\sigma \in \mathrm{Aut}(\bar{X}_v)$. Then $m \mid \mu$. Moreover, $(\mu/m) \mid n$ since $\psi_\sigma^m \in \mathrm{Aut}(\bar{Y}_v)$ is an element of G, which is cyclic of order n. In particular, we have that

$$|G \cap \langle \psi_\sigma \rangle| = \frac{\mu}{m}.$$

The cover $\bar{Y}_v/I_v \to \bar{X}_v/I_v$ is a Galois cover with Galois group $G/(G \cap I_v)$, which is cyclic of order $\bar{n} := n/(\mu/m) = nm/\mu$.

If $m = 1$, we have that $\psi_\sigma \in G$ and the cover $\bar{Z}_{v,\mathbb{F}_L} \to \bar{W}_{v,\mathbb{F}_L} = \bar{X}_v/I_v$ is given by

$$\bar{z}_v^{n/\mu} = \bar{g}_v(\bar{u}_v), \qquad \text{where } \bar{z}_v = \bar{y}_v^\mu.$$

We consider the case $m \neq 1$. Recall from Lemma 5.4 that $\psi_\sigma \in \mathrm{Aut}_{\mathbb{F}_L}(\bar{X}_v)$ is an affine linear transformation of order m with at least one $\mathbb{F}_L$-rational fixed point (which we assumed to be $\bar{x}_v = \infty$). It follows that the second fixed point is also $\mathbb{F}_L$-rational. After a further normalization of the chart, we may assume that it is $\bar{x}_v = 0$. With this choice of chart we have that

$$\psi_\sigma(\bar{x}_v, \bar{y}_v) = (c\bar{x}_v, \gamma \bar{y}_v)$$

for some $c, \gamma \in \mathbb{F}_L$. The definitions of μ and m imply that $m = \mathrm{ord}(c)$ and $\mu = \mathrm{lcm}(m, \mathrm{ord}(\gamma))$. It follows that $\gamma^{\mu/m} = c^s \in \mathbb{F}_L$ for some integer s.

Since P_v is a normal subgroup of I_v, the automorphism ψ_σ descends to an automorphism of $\bar{X}_v/P_v$, which we still denote by ψ_σ. The definition of the coordinate $\bar{u}_v$ of $\bar{X}_v/P_v$ implies that the fixed points $\bar{x}_v = 0, \infty$ map to $\bar{u}_v = 0, \infty$, respectively. It follows that $\psi_\sigma(\bar{u}_v) = \tilde{c}\bar{u}_v$. Since the order of σ is prime to p and hence prime to $|P_v|$, we have that $\mathrm{ord}(\tilde{c}) = \mathrm{ord}(c) = m$. We conclude that the functions

$$\bar{z}_v := \bar{y}_v^{\mu/m}\bar{u}_v^{-s}, \qquad \bar{w}_v := \bar{u}_v^m$$

are invariant under I_v. We find the following Kummer equation:

$$\bar{Z}_{v,\mathbb{F}_L}: \qquad \bar{z}^{\bar{n}} = \frac{\bar{y}_v^n}{\bar{u}_v^{s\bar{n}}} = \frac{\bar{f}_v(\bar{x}_v)}{\bar{x}_v^{s\bar{n}}}. \tag{16}$$

Since the function algebra of the quotient curve $\bar{Z}_{v,\mathbb{F}_L}$ is generated by $\bar{z}_v$ and $\bar{w}_v$, it follows that the right-hand side of (16) is a rational function $\bar{h}_v(\bar{w}_v) \in \mathbb{F}_L(\bar{w}_v)$. As in the previous step, it is easy to calculate $\bar{h}_v$.

The following proposition summarizes the above discussion.

PROPOSITION 5.6.

1. *We may choose the chart λ_v such that*

$$\psi_\sigma(\bar{x}_v, \bar{y}_v) = (c\bar{x}_v, \gamma\bar{y}_v),$$

for suitable constants $c, \gamma \in \mathbb{F}_L^\times$.

2. *The cover $\bar{Z}_{v,\mathbb{F}_L} \to \bar{W}_{v,\mathbb{F}_L}$ is given by a Kummer equation*

$$\bar{z}_v^{\bar{n}} = \bar{h}_v(\bar{w}_v),$$

where

$$\bar{u}_v := \prod_{\sigma\in\bar{P}_v} \psi_\sigma(\bar{x}_v), \quad \bar{w}_v := \bar{u}_v^m, \quad \bar{z}_v := \bar{y}_v^{\mu/m}\bar{u}_v^{-s}.$$

Moreover, we have $m = \mathrm{ord}(c)$, $\mu = \mathrm{lcm}(m, \mathrm{ord}(\gamma))$, $c^s = \gamma^{\mu/m}$ and $\bar{n} = n/(\mu/m)$.

In Section 6, we give an example where the degree $\bar{n}$ of the quotient Kummer cover is strictly smaller than n (Remark 6.1).

REMARK 5.7. In the case that $\mu/m = n$, the Galois group G of the cover $\bar{Y}_v \to \bar{X}_v$ is contained in $\langle\psi_\sigma\rangle \subset I_v$. In this case the quotient curve $\bar{Z}_{v,\mathbb{F}_L} = \bar{Y}_v/I_v$ is a disjoint union of curves of genus 0, since each component is isomorphic to a quotient of $\bar{X}_v$. It follows that v does not contribute to the L-function, and we may disregard v in the rest of the calculation. An example can be found in Section 6.4.

5.4. In this section, we describe how to compute the quotient curve $\bar{Z} = \bar{Y}/\Gamma = \bar{Z}_{\mathbb{F}_L}/(\Gamma/I)$, together with the map $\bar{Z} \to \bar{W} = \bar{X}/\Gamma$. We write $\bar{\Gamma} := \Gamma/I \simeq \mathrm{Gal}(\mathbb{F}_L/\mathbb{F}_K)$.

In Section 5.2, we have already described the action of Γ on $\bar{X}$, and therefore on the set of irreducible components. This action is induced by the action of Γ on the roots of the polynomial f, which is assumed to be known. As a result, the action of $\bar{\Gamma} = \Gamma/I$ on the irreducible components of $\bar{W}_{\mathbb{F}_L} = \bar{X}/I$ may therefore be considered as known.

Let us choose $v \in V(\Delta)$. As before, we denote by $\bar{W}_{v,\mathbb{F}_L}$ (resp. $\bar{W}_v$) the irreducible component of $\bar{W}_{\mathbb{F}_L}$ (resp. of $\bar{W}$) corresponding to the I-orbit (resp. to the Γ-orbit) of v. Similarly, we write $\bar{Z}_{v,\mathbb{F}_L} = \bar{Z}_{\mathbb{F}_L}|_{\bar{W}_{v,\mathbb{F}_L}}$ and $\bar{Z}_v := \bar{Z}|_{\bar{W}_v}$.

Recall from Proposition 5.6 that the cover $\bar{Z}_{v,\mathbb{F}_L} \to \bar{W}_{v,\mathbb{F}_L}$ is given birationally by a Kummer equation

$$\bar{z}_v^{\bar{n}} = \bar{h}_v,$$

where $\bar{h}_v \in \mathbb{F}_L(\bar{w}_v)$ is a rational function in the coordinate $\bar{w}_v$ for the projective line $\bar{W}_{v,\mathbb{F}_L}$.

PROPOSITION 5.8. *Let $\bar{\Gamma}_v \subset \bar{\Gamma}$ be the stabilizer of $\bar{W}_v$ and put $\mathbb{F}_v = \mathbb{F}_L^{\bar{\Gamma}_v}$.*

1. *The curve $\bar{W}_v$ is isomorphic to the projective line over $\mathbb{F}_v$, and a coordinate $\bar{w}_v'$ corresponding to such an isomorphism can be explicitly computed.*
2. *The cover $\bar{Z}_v \to \bar{W}_v$ is birationally given by a Kummer equation*

$$(\bar{z}_v')^{\bar{n}} = \bar{h}_v',$$

where $\bar{h}_v'$ is a polynomial in $\bar{w}_v'$ with $\mathbb{F}_v$-coefficients. The polynomial $\bar{h}_v'$ can be explicitly computed.

Proof. Since $\bar{W}_v$ is a curve of genus zero over $\mathbb{F}_v$, the first part of (1) follows from the fact that the Brauer group of the finite field $\mathbb{F}_v$ is trivial. However, in order to justify the second claim in (1) it is better to give a more direct proof which does not use the Brauer group (and therefore does not depend on $\mathbb{F}_v$ being finite).

By Proposition 5.6, the function field of $\bar{W}_{v,\mathbb{F}_L}$ is $\mathbb{F}_L(\bar{w}_v)$, where $\bar{w}_v$ is an explicit polynomial in the chosen coordinate $\bar{x}_v$ on $\bar{X}_v$. The semilinear action of $\bar{\Gamma}_v$ is therefore given by a cocycle

$$(A_\tau)_\tau \in Z^1(\bar{\Gamma}_v, \mathrm{PGL}_2(\mathbb{F}_L)^{\mathrm{opp}}),$$

which can be explicitly computed from the knowledge of the cocycle from Remark 5.3. Moreover, since $\bar{w}_v$ is a polynomial in $\bar{x}_v$, Lemma 5.4 shows that A_τ corresponds to an affine linear transformation, i.e.

$$A_\tau = \begin{pmatrix} \bar{a}_\tau & \bar{b}_\tau \\ 0 & 1 \end{pmatrix},$$

with $\bar{a}_\tau, \bar{b}_\tau \in \mathbb{F}_L$. To prove (1), we need to find a coordinate $\bar{w}_v'$ which is $\bar{\Gamma}_v$-invariant. In other words, we need to find a matrix

$$B = \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F}_L)$$

such that $A_\tau = \tau(B)B^{-1}$ for all $\tau \in \bar{\Gamma}_v$. This translates to

$$\frac{\alpha}{\tau(\alpha)} = \bar{a}_\tau, \qquad \beta - \tau(\beta) = \bar{b}_\tau \tau(\alpha).$$

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In fact, it suffices to solve this equation for a generator τ of $\bar{\Gamma}_v$. Clearly, solutions $\alpha, \beta \in \mathbb{F}_L$ may be found explicitly as in the proof of the additive and multiplicative versions of Hilbert's Theorem 90. This completes the proof of (1).

It remains to prove (2). By (1) we can write $\bar{h}_v$ as a rational function in $\bar{w}'_v$ with coefficients in $\mathbb{F}_L$. There exists a rational function $\bar{h}''_v \in \mathbb{F}_L(\bar{w}'_v)$ such that

$$\bar{h}'_v = \bar{h}_v \cdot (\bar{h}''_v)^{\bar{n}},$$

is a polynomial in $\mathbb{F}_L[\bar{w}'_v]$ which does not have any non-trivial factors which are $\bar{n}$th powers. We set $\bar{z}'_v := (\bar{h}''_v)\bar{z}_v$. The cover $\bar{Z}_{v,\mathbb{F}_L} \to \bar{W}_{v,\mathbb{F}_L}$ is now given by the Kummer equation

$$(\bar{z}'_v)^{\bar{n}} = \bar{h}'_v. \tag{17}$$

For $\tau \in \bar{\Gamma}_v$, we write ψ_τ for the (semilinear) automorphism of $\bar{Z}_{v,\mathbb{F}_L}$ induced by τ. We claim that for any element $\tau \in \bar{\Gamma}_v$ we have

$$\psi_\tau(\bar{z}'_v) = \bar{q}_\tau \cdot \bar{z}'_v, \qquad \text{with } \bar{q}_\tau \in \mathbb{F}_L[\bar{w}'_v]. \tag{18}$$

To see this, note that the extension

$$\mathbb{F}_L(\bar{Z}_{v,\mathbb{F}_L}) \supset \mathbb{F}_v(\bar{W}_v) \simeq \mathbb{F}_v(\bar{w}'_v) \tag{19}$$

of functions rings is a Galois extension. Recall that the Galois group $\bar{G} := \mathrm{Gal}(\bar{Z}_{v,\mathbb{F}_L}/\bar{W}_{v,\mathbb{F}_L})$ is cyclic of order $\bar{n}$. Since $\mathbb{F}_L$ contains the $\bar{n}$th roots of unity, $\bar{G}$ is a normal subgroup of the Galois group of the extension (19), which is a quotient of Γ_v. It follows that $\psi_\tau(\bar{z}'_v)$ is a Kummer generator of $\bar{Z}_{v,\mathbb{F}_L}/\bar{W}_{v,\mathbb{F}_L}$. Kummer theory implies that

$$\psi_\tau(\bar{z}'_v) = \bar{q}_\tau \cdot (\bar{z}'_v)^{m_\tau}, \tag{20}$$

where $m_\tau \in \{1, \ldots, \bar{n}-1\}$ represents the character $\chi : \bar{\Gamma}_v \to (\mathbb{Z}/\bar{n}\mathbb{Z})^\times$ which determines the action of $\bar{\Gamma}_v$ on $\bar{G}$ by conjugation. The claim (18) states that the character χ is trivial.

To prove that χ is trivial, we consider the action of ψ_τ on the polynomial $\bar{h}'_v$. Recall that $\bar{h}'_v$ is a polynomial which does not have any non-trivial factors that are $\bar{n}$th powers. It follows that the roots of $\bar{h}'_v$ are branched in the Kummer cover $\bar{Z}_{v,\mathbb{F}_L} \to \bar{W}_{v,\mathbb{F}_L}$. More precisely, the roots of $\bar{h}'_v$ are the images of the branch points of the cover $Y \to X$ that specialize to $\bar{X}_v$. In particular, it follows that Γ_v acts on the set of roots of $\bar{h}'_v$.

It also follows that the order of vanishing of a zero of $\bar{h}'_v$ is equivalent $\pmod{\bar{n}}$ to the order of vanishing of the corresponding zero of the polynomial f describing the Kummer cover $Y \to X$. Since $Y \to X$ is defined over K it follows that any two roots of $\bar{h}'_v$ which are conjugate under the action of Γ_v have the same order of vanishing in $\bar{h}'_v$. The coordinate $\bar{w}'_v$ is already invariant under τ. We conclude that

$$\psi_\tau(\bar{h}'_v) = \bar{q}_\tau \cdot \bar{h}'_v \qquad \text{with } \bar{q}_\tau \in \mathbb{F}_L^\times,$$

for all $\tau \in \bar{\Gamma}_v$. With (17) it follows that m_τ in (20) is trivial for all $\tau \in \bar{\Gamma}_v$, and hence that the character χ is trivial. This proves the claim (18).

Replacing $\bar{z}'_v$ with $\gamma\bar{z}'_v$, for some $\gamma \in \mathbb{F}_L^\times$, has the effect of replacing $\bar{q}_\tau$ with $\bar{q}_\tau\tau(\gamma)\gamma^{-1}$. Using again Hilbert's Theorem 90, we may assume that $\bar{q}_\tau = 1$, i.e. that $\bar{z}'_v$ is invariant under the action of $\bar{\Gamma}_v$.

The extension of function algebras $F(\bar{Z}_v)/F(\bar{W}_v) = \mathbb{F}_v(\bar{w}'_v)$ has degree $\bar{n}$, which is the same as the degree of the Kummer equation for $\bar{z}'_v$. We conclude that $\bar{z}'_v$ is a generator of the extension of function algebras $F(\bar{Z}_v)/F(\bar{W}_v) = \mathbb{F}_v(\bar{w}'_v)$. The proof of the proposition is now complete. $\square$

Proposition 5.8 gives an explicit description of the (possibly reducible) curves $\bar{Z}_v = \bar{Z}|_{\bar{W}_v}$. Remark 3.5.(2) implies that $\bar{Z}_v$ is smooth. It follows that the normalization $\pi : \bar{Z}^{(0)} \to \bar{Z}$ is the disjoint union of the curves $\bar{Z}_v$, where v runs over a subset of $V(\Delta)$ representing the Γ-orbits. We therefore have an explicit description of the normalization $\bar{Z}^{(0)}$ as well.

As explained in Section 2.4, it remains to describe the singular locus $\bar{Z}^{(1)} := \pi^{-1}(\bar{Z}^{\rm sing}) \subset \bar{Z}^{(0)}$. Remark 3.5 implies that $\bar{Z}^{(1)}$ is the inverse image of $\bar{W}^{(1)} \subset \bar{W}$ under the map $\bar{Z} \to \bar{W}$, where $\bar{W}^{(i)}$ is defined analogously to $\bar{Z}^{(i)}$ for $i = 0, 1$. Since the map $\bar{Z}^{(0)} \to \bar{W}^{(0)}$ has an explicit description as a disjoint union of Kummer covers, it suffices to describe the closed subset $W^{(1)} \subset \bar{W}^{(0)}$. Since $\bar{W} = \bar{X}/\Gamma$, an explicit description of $\bar{W}^{(1)} \subset \bar{W}^{(0)}$ can immediately be derived from the inclusion $\bar{X}^{(1)} \subset \bar{X}^{(0)}$. This is easy using the description of $\bar{X}$ as a tree of projective lines in Section 4.2.

6. Example I In this section and the next, we compute the local L-factor and the exponent of conductor of two superelliptic curves.

6.1. We consider the Kummer cover $\phi : Y \to X = \mathbb{P}^1_K$ over $K := \mathbb{Q}_3$ given by the equation

$$y^4 = f(x) = (x^2 - 3)(x^2 + 3)(x^2 - 6x - 3).$$

The branch points of ϕ are the six roots of f (with ramification index 4) and the point at ∞ (with ramification index 2). The Riemann–Hurwitz formula shows that the genus of Y is 7.

The splitting field of f over K is the biquadratic extension $L_0 := K(i, 3^{1/2})$, where i is a fourth root of unity and $3^{1/2}$ is a square root of three. In fact, the roots of f are

$$\pm 3^{1/2}, \pm i3^{1/2}, \alpha, \alpha',$$

where $\alpha = 3 - 2 \cdot 3^{1/2}, \alpha' = 3 + 2 \cdot 3^{1/2} \in L_0$ are the two roots of $x^2 - 6x - 3$. Note that $K(i)/K$ is the maximal unramified subextension and that the residue field of $K(i)$ (and of L_0) is the field $\mathbb{F}_9$ with nine elements.

Let $L := L_0(3^{1/4})$ be the extension obtained by adjoining a square root $3^{1/4}$ of $3^{1/2}$. Since $K(i)$ already contains all fourth roots of unity, we see that L/K is a Galois extension whose Galois group Γ is the dihedral group of order 8. The inertia subgroup $I \lhd \Gamma$ is the unique cyclic subgroup of order 4. Moreover, L satisfies Assumption 5.1 and Y_L has semistable reduction over L.

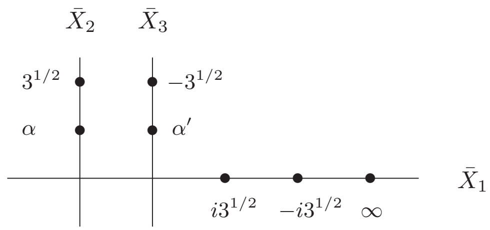
6.2. Let $(\mathcal{X}, \mathcal{D})$ denote the stably marked model of (X_L, D_L) and $(\bar{X}, \bar{D})$ the special fibre of $(\mathcal{X}, \mathcal{D})$, see Section 3.2. We note that

$$\frac{\alpha - 3^{1/2}}{3^{1/2}} \equiv 0 \pmod{3^{1/4}}, \qquad \frac{\alpha' - (-3^{1/2})}{3^{1/2}} \equiv 0 \pmod{3^{1/4}},$$

and that there are no further congruences between the elements of D_L. Following Remark 4.3 one easily sees that $\mathcal{X}$ is given by the three charts $\lambda_i : X_L \to \mathbb{P}^1_L$, $i = 1, 2, 3$ corresponding to the parameters

$$x_1 := 3^{-1/2}x, \quad x_2 := \frac{x - 3^{1/2}}{3}, \quad x_3 := \frac{x + 3^{1/2}}{3}.$$

Let $\bar{X}_i \subset \bar{X}$ be the irreducible component corresponding to λ_i. Then $\bar{X}$ looks as follows:



In this picture the dots indicate the position of the points $\bar{\alpha}_i \in \bar{D} \subset \bar{X}$. Next to the dots one finds the value of the corresponding point $\alpha_i \in D_L \subset X_L = \mathbb{P}^1_L$.

6.3. Let $\mathcal{Y}$ denote the normalization of $\mathcal{X}$ in the function field of Y_L. We use Proposition 4.5 to show that $\mathcal{Y}$ is a semistable model of Y_L and to describe its special fibre $\bar{Y}$.

Let η_i denote the discrete valuation corresponding to the component $\bar{X}_i$ on the function field $F(X_L) = L(x)$, where we normalize η_i by $\eta_i(3) = 1$. Set $N_i := \eta_i(f)$. For $i = 1$, we write

$$f(x) = f(3^{1/2}x_1) = 3^3(x_1^2 - 1)(x_1^2 + 1)(x_1^2 - 2 \cdot 3^{1/2}x_1 - 1),$$

from which we conclude that

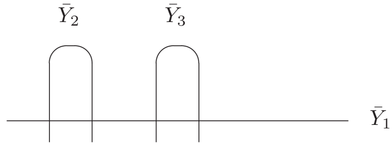
$$\eta_1(f) = 3, \quad \bar{f}_1 = (\bar{x}_1^2 - 1)^2(\bar{x}_1^2 + 1).$$

Similarly, we check that for $i = 2, 3$ we have

$$\eta_i(f) = 4, \quad \bar{f}_i = 2\bar{x}_i(\bar{x}_i - 1).$$

By the first part of Proposition 4.5 it follows that $\mathcal{Y}$ is semistable. The second part of the proposition implies that there is a unique irreducible component $\bar{Y}_i$ of $\bar{Y}$ lying above $\bar{X}_i$. The restriction $\bar{Y}_i \to \bar{X}_i$ is the Kummer cover with equation $\bar{y}_i^4 = \bar{f}_i$, for $i = 1, 2, 3$. Note that the genus of $\bar{Y}_1$ is equal to 3, whereas $\bar{Y}_2$ and $\bar{Y}_3$ have genus 1.

To describe $\bar{Y}$ it remains to describe the singular locus of $\bar{Y}$. By Remark 3.5.(2), the singular locus of $\bar{Y}$ is precisely the inverse image of the singular locus of $\bar{X}$. The latter is contained in the component $\bar{X}_1$, and consists of the two points with $\bar{x}_1 = \pm 1$. Note that the points above $\bar{x}_1 = \pm 1$ have ramification index 2 in the cover $\bar{Y}_1 \to \bar{X}_1$. Hence $\bar{Y}$ contains $2 \cdot (4/2) = 4$ singular points: two intersection points of $\bar{Y}_2$ with $\bar{Y}_1$ and two intersection points of $\bar{Y}_3$ with $\bar{Y}_1$. The curve $\bar{Y}$ therefore looks as follows.



Note that the arithmetic genus of $\bar{Y}$ equals $3+1+1+2=7$, which is equal to the genus of Y, as it should be.

6.4. We now look at the action of $\Gamma = \mathrm{Gal}(L/K)$ on $\bar{Y}$. Let $\sigma, \tau \in \Gamma$ be the two generators given by

$$\begin{aligned} \sigma(3^{1/4}) &= i \cdot 3^{1/4}, \quad & \sigma(i) &= i, \\ \tau(3^{1/4}) &= 3^{1/4}, \quad & \tau(i) &= -i. \end{aligned}$$

Recall that the inertia subgroup group $I \subset \Gamma$ is cyclic of order 4, hence I is generated by σ.

Following the strategy of Section 5, we first study the action of $I = \langle\sigma\rangle$ on $(\bar{X}, \bar{D})$, which is determined by its action on the set D_L.

The element $\sigma \in I$ acts as an involution on D_L, as follows:

$$3^{1/2} \leftrightarrow -3^{1/2}, \quad i3^{1/2} \leftrightarrow -i3^{1/2}, \quad \alpha \leftrightarrow \alpha'.$$

It follows that the automorphism ψ_σ of $\bar{X}$ maps the component $\bar{X}_1$ of $\bar{X}$ to itself and interchanges the two components $\bar{X}_2, \bar{X}_3$. We conclude that ψ_σ also fixes the component $\bar{Y}_1$ of $\bar{Y}$ and interchanges $\bar{Y}_2$ with $\bar{Y}_3$.

As a second step, we determine the quotients $\bar{Z}_{\mathbb{F}_L} = \bar{Y}/I \to \bar{W}_{\mathbb{F}_L} = \bar{X}/I$. The definition of x_1 as $x_1 = x/3^{1/2}$ implies that the restriction of ψ_σ to $\bar{X}_1$ is given by $\psi_\sigma(\bar{x}_1) = -\bar{x}_1$. The coordinate $\bar{y}_1$ is the image in $\mathbb{F}_L(\bar{Y}_1)$ of $y_1 := \pi^{-N_1/n} y = 3^{-3/4} y$ (Proposition 4.5.(2)). It follows that

$$\psi_\sigma(\bar{x}_1, \bar{y}_1) = (-\bar{x}_1, i\bar{y}_1).$$

Therefore the Kummer equation for $\bar{Y}_1/I_1 \to \bar{X}_1$ from Proposition 5.6.(2) is given by

$$\bar{z}_1^2 = \bar{w}_1(\bar{w}_1 + 1), \quad \bar{w}_1 = \bar{x}_1^2,\ \bar{z}_1 = \bar{y}_1^2 \bar{x}_1/(\bar{x}_1^2 - 1). \tag{21}$$

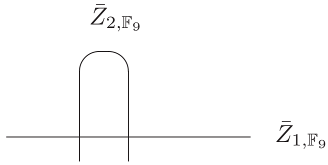
This implies that $\bar{Z}_{1,\mathbb{F}_9} \cong \mathbb{P}^1_{\mathbb{F}_9}$ has genus zero.

REMARK 6.1. Note that ψ_σ considered as automorphism of $\bar{Y}_1$ has order 4, which is strictly larger than the order of the corresponding automorphism of $\bar{X}_1$. This is the reason why the quotient Kummer cover $\bar{Z}_{1,\mathbb{F}_L} \to \bar{W}_{1,\mathbb{F}_L}$ has degree $\bar{n} = 4/2 = 2$.

A similar analysis shows that $\psi_\sigma(\bar{x}_2) = \bar{x}_3$ and $\psi_{\sigma^2}(\bar{x}_2) = \bar{x}_2$. The restriction of ψ_{σ^2} to $\bar{Y}_2 \cup \bar{Y}_3$ is the identity since $y_2 = y/3$. We have already seen that ψ_σ interchanges $\bar{Y}_2$ and $\bar{Y}_3$. We conclude that $\bar{Z}_{2,\mathbb{F}_9} := (\bar{Y}_2 \cup \bar{Y}_3)/I$ is an isomorphic copy of $\bar{Y}_2$ (or $\bar{Y}_3$). The quotient cover $\bar{Z}_{2,\mathbb{F}_L} \to \bar{W}_{2,\mathbb{F}_L}$ is the same as the original cover $\bar{Y}_2 \to \bar{X}_2$, i.e.

$$\bar{z}_2^4 = 2\bar{w}_2(\bar{w}_2 - 1), \qquad \bar{z}_2 := \bar{y}_2 = \bar{y}_3,\ \bar{w}_2 = \bar{x}_2 + \bar{x}_3. \tag{22}$$

It follows that the quotient curve $\bar{Z}_{\mathbb{F}_9} := \bar{Y}/I$ is a semistable curve over $\mathbb{F}_9$ consisting of two irreducible components $\bar{Z}_{1,\mathbb{F}_9}$ and $\bar{Z}_{2,\mathbb{F}_9}$ intersecting each other in two points, as follows.



The arithmetic genus of $\bar{Z}_{\mathbb{F}_9}$ is equal to $g(\bar{Z}_{\mathbb{F}_9}) = g(\bar{Z}_{1,\mathbb{F}_9}) + g(\bar{Z}_{2,\mathbb{F}_9}) + 1 = 0 + 1 + 1 = 2$.

6.5. It remains to determine the semilinear action of $\bar{\Gamma} = \Gamma/I = \langle\bar{\tau}\rangle$ on $\bar{Z}_{\mathbb{F}_L}$ and the quotient $\bar{Z} := \bar{Z}_{\mathbb{F}_L}/\bar{\Gamma} = \bar{Y}/\Gamma$. By considering the action of τ on the branch points of ϕ as in Section 6.4, we see that $\psi_{\bar{\tau}}$ acts trivially on the graph Δ of components of $\bar{X}$. Since there is a unique irreducible component of $\bar{Y}$ above $\bar{X}$, $\psi_{\bar{\tau}}$ also acts trivially on the graph of components of $\bar{Y}$.

From the proof of Proposition 5.8 it follows that $\bar{\tau}$ leaves the coordinates $\bar{z}_i$, $\bar{w}_i$ defined in (21) and (22) invariant. We conclude that $\bar{Z}_{\mathbb{F}_L}$ is already the correct model over $\mathbb{F}_3$. Note that the $\langle\bar{\tau}\rangle \simeq \mathrm{Gal}(\mathbb{F}_9/\mathbb{F}_3)$ acts semilinearly on $\bar{Z} = \bar{Z}_{\mathbb{F}_L}$. For example, the singular locus of $\bar{Z}$ consists of two geometric points which are conjugate over the quadratic extension $\mathbb{F}_9/\mathbb{F}_3$. This completes our description of $\bar{Z}$.

6.6. We can now write down the local L-factor of the curve $Y/\mathbb{Q}_3$. By Corollary 2.5, the local factor is

$$L_3(Y, s) = P_1(\bar{Z}, 3^{-s}),$$

where

$$P_1(\bar{Z}, T) := \det\left(1 - \mathrm{Frob}_3 \cdot T | H^1(\bar{Z}, \mathbb{Q}_\ell)\right)$$

and where $\mathrm{Frob}_3 : \bar{Z}_{\mathbb{F}_3} \to \bar{Z}_{\mathbb{F}_3}$ is the $\mathbb{F}_3$-Frobenius endomorphism.

The normalization of $\bar{Z}$ is equal to the disjoint union of $\bar{Z}_1 \cong \mathbb{P}^1_k$ and $\bar{Z}_2$. Lemma 2.7.(1) implies that

$$H^1_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell) = H^1(\Delta_{\bar{Z}_k}) \oplus H^1_{\mathrm{et}}(\bar{Z}_{2,k}, \mathbb{Q}_\ell).$$

In Section 6.5, we have seen that Frob_3 fixes the two irreducible components $\bar{Z}_1$ and $\bar{Z}_2$ of $\bar{Z}$ and interchanges the two singular points. Lemma 2.7.(2) therefore implies that the corresponding factor of $P_1(\bar{Z}, T)$ is equal to

$$1 + T.$$

The second factor is the numerator of the zeta function of the genus-one curve $\bar{Z}_2$ given by (22). Since the number of $\mathbb{F}_3$-rational points is

$$|\bar{Z}_2(\mathbb{F}_3)| = 4 = 1 + 3,$$

it follows that

$$P_1(\bar{Z}, T) = (1+T)(1+3T^2).$$

6.7. We use our description of the stable reduction of Y to compute the exponent of the conductor of the Γ_K-representation $H^1(Y_{\bar{K}}, \mathbb{Q}_\ell)$. Since Y achieves semistable reduction over a tame extension of $K = \mathbb{Q}_3$ it follows from Corollary 2.6 and the above calculations that

$$f_{Y/K} = 2g(Y) - \dim H^1_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell) = 14 - 3 = 11.$$

7. Example II As a second example, we consider the curve Y over $K = \mathbb{Q}_2$ given by

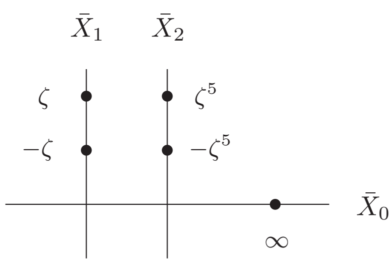
$$y^3 = f(x) := x^4 - x^2 + 1. \tag{23}$$

We will see that in this case any extension $L/\mathbb{Q}_2$ over which Y acquires stable reduction is wildly ramified.

We curve over $\mathbb{Q}$ given by (23) has good reduction outside $p = 2, 3$. The reduction at three together with the local L-faction and the exponent of conductor are computed in Section 4.3 of the follow-up paper [**4**].

7.1. The ramification divisor $D \subset X := \mathbb{P}^1_K$ has degree 5 and consists of the zero set of f together with ∞, hence $g(Y) = 3$. As f is the 12th cyclotomic polynomial, its zero set is $\{\pm\zeta, \pm\zeta^5\}$, where ζ is a chosen primitive 12th root of unity. The splitting field of f is $L_0 := \mathbb{Q}_2(\zeta)$. We set $L := L_0(2^{1/3})$, where $2^{1/3}$ is a third root of 2. Since L_0 contains the third root of unity $\zeta_3 := \zeta^4$, the extension L/K is Galois and its Galois group $\Gamma := \mathrm{Gal}(L/K)$ is the dihedral group of order 12. Its inertia subgroup is $I := \mathrm{Gal}(L/K(\zeta_3))$, which is the cyclic subgroup of Γ of order 6. In particular, L/K is wildly ramified. The residue field $\mathbb{F}_L$ of L is $\mathbb{F}_4$, and is generated over $\mathbb{F}_2$ by the image $\bar{\zeta}$ of ζ_3. Assumption 5.1 is satisfied, therefore the curve Y_L has semistable reduction over L.

As in Section 6.2 , we find that the special fibre $\bar{X}$ of the stable model $(\mathcal{X}, \mathcal{D})$ of (X_L, D_L) looks as follows:



We may choose the parameters x_i for the components $\bar{X}_i$ as follows

$$x_0 := x, \qquad x_1 := \frac{x-\zeta}{2}, \qquad x_2 := \frac{x-\zeta^5}{2}. \tag{24}$$

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The choice of x_1 differs from the convention in Notation 4.4 by a unit. This leads to slightly easier formulas afterwards.

Proposition 4.5 yields as Kummer equation for $\bar{Y}_i := \bar{Y}|_{\bar{X}_i}$:

$$y_0 := y, \qquad \bar{y}_0^3 = \bar{f}_0(\bar{x}_0) := (\bar{x}_0^2 + \bar{x}_0 + 1)^2, \tag{25}$$
$$y_1 := 2^{2/3}y, \qquad \bar{y}_1^3 = \bar{f}_1(\bar{x}_1) := \bar{x}_1(\bar{x}_1 + \bar{\zeta}), \tag{26}$$
$$y_2 := 2^{2/3}y, \qquad \bar{y}_2^3 = \bar{f}_2(\bar{x}_2) := \bar{x}_2(\bar{x}_2 + \bar{\zeta}^2). \tag{27}$$

Note that $\bar{Y}_i$ is irreducible and has genus 1 for $i = 1, 2, 3$.

7.2. We now describe the action of $\Gamma = \mathrm{Gal}(L/K)$ on $\bar{X}$ and $\bar{Y}$ and determine the quotient curve $\bar{Z} = \bar{Y}/\Gamma$. For convenience, we choose generators σ, τ of Γ as follows

$$\sigma(i) = -i, \qquad \sigma(2^{1/3}) = \zeta_3 2^{1/3}, \qquad \sigma(\zeta_3) = \zeta_3, \tag{28}$$
$$\tau(i) = i, \qquad \tau(2^{1/3}) = 2^{1/3}, \qquad \tau(\zeta_3) = \zeta_3^2. \tag{29}$$

Note that σ generates I and the image of τ generates $\bar{\Gamma} := \Gamma/I$.

Since $x_0 = x$ and $y_0 = y$ it follows that Γ leaves these coordinates invariant. We conclude that $\bar{W}_0 := \bar{X}_0/\Gamma$ is isomorphic to the projective line over $\mathbb{F}_2$ with parameter $\bar{x}_0$. Similarly, $\bar{Z}_0 := \bar{Y}_0/\Gamma$ is simply the $\mathbb{F}_2$-model of $\bar{Y}_0$ given by the equation (25).

We describe the action of Γ on the graph Δ of irreducible components of $\bar{X}$. Since Γ permutes the primitive 12th roots of unity, the components $\bar{X}_1$ and $\bar{X}_2$ are interchanged. The choice of coordinates in (24) implies that $\psi_\tau(\bar{x}_1) = \bar{x}_2$, and conversely. Since $\zeta^5 = \zeta_3 \cdot \zeta$, the stabilizer Γ_i of $\bar{X}_i$ is the inertia group I for $i = 1, 2$.

Obviously, Γ permutes the components $\bar{Y}_1$ and $\bar{Y}_2$ as well. We are reduced to computing the quotient $\bar{Z}_1 := \bar{Y}_1/I$. The definition of the coordinates in (24) and (26) implies that

$$\psi_\sigma(\bar{x}_1, \bar{y}_1) = (\bar{x}_1, \bar{\zeta}\bar{y}_1),$$

since $(\zeta - \sigma(\zeta))/2 = (\zeta - \zeta^7)/2 = \zeta \equiv \zeta_3 \pmod 2$. Therefore, ψ_{σ^2} generates the Galois group of $\bar{Y}_1 \to \bar{X}_1$ and $\bar{W}_1 = \bar{Y}_1/I$ is a projective line over $\mathbb{F}_4$ with coordinate $\bar{w}_1 := \bar{x}_1(\bar{x}_1 + \bar{\zeta})$.

The corresponding component of $\bar{Z} = \bar{Y}/\Gamma$ is $\bar{Z}_3 := (\bar{Z}_1 \coprod \bar{Z}_2)/\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. The curve $\bar{Z}_3$ is isomorphic to $\mathbb{P}^1_{\mathbb{F}_4}$ considered as a curve over $\mathbb{F}_2$ and is not absolutely irreducible. Since $\bar{Z}_1$ has genus 0, the curve $\bar{Z}_3$ does not contribute to the étale cohomology of $\bar{Z}$. Since there are no loops, the contraction map $\bar{Z} \to \bar{Z}_0$ induces an isomorphism on H^i_{et}.

The curve $\bar{Z}_0 = \bar{Y}_0/\Gamma$ is the smooth curve of genus one over $\mathbb{F}_2$ given by (25) with $|\bar{Z}_0(\mathbb{F}_2)| = 3$. We conclude that the zeta function of $\bar{Z}$ is

$$Z(\bar{Z}, T) = \frac{1 + 2T^2}{(1-T)(1-2T)}.$$

7.3. It remains to compute the exponent of conductor $f_{Y/K}$. Since the extension L/K is wildly ramified, Corollary 2.6 does not apply and we have to use the formula

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of Theorem 2.9. Recall that

$$f_{Y/K} = \epsilon + \delta,$$

where $\epsilon = 2g_Y - \dim H^1_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell)$ and δ is the Swan conductor. The results from Section 7.2 show that $\dim H^1_{\mathrm{et}}(\bar{Z}_k, \mathbb{Q}_\ell) = 2$ and therefore that $\epsilon = 4$.

Let $(\Gamma_i)_{i\geq 0}$ be the filtration of Γ by higher ramification groups. Then $\Gamma_0 = I$ is the inertia group and $\Gamma_1 = P$ is the Sylow p-subgroup of I. In our case, $I = \langle\sigma\rangle$ is cyclic of order 6 and $P \subset I$ is generated by the element σ^3. A simple computation using (28) shows that

$$\Gamma_1 = \Gamma_2 = \Gamma_3 = P, \quad \Gamma_4 = \{1\}.$$

Theorem 2.9 implies that

$$\delta = 2(g_{\bar{Y}} - g_{\bar{Z}^w}), \tag{30}$$

where $g_{\bar{Y}}$ (resp. $g_{\bar{Z}^w}$) is the arithmetic genus of $\bar{Y}$ (resp. of the quotient curve $\bar{Z}^w := \bar{Y}/P$).

The curve $\bar{Y}$ has genus 3. The computations of Section 7.2 show that the curve $\bar{Z}^w$ is a semistable curve over $\mathbb{F}_4$ with three smooth irreducible components $\bar{Z}^w_0, \bar{Z}^w_1, \bar{Z}^w_2$, where $\bar{Z}^w_1$ and $\bar{Z}^w_2$ each intersect $\bar{Z}^w_0$ in a unique point. The curve $\bar{Z}^w_0$ is canonically isomorphic to the genus-one curve $\bar{Y}_0$ (since I acts trivially on $\bar{Y}_0$), while $\bar{Z}^w_1$ and $\bar{Z}^w_2$ are curves of genus zero. We conclude that $g(\bar{Z}^w) = 1$, and hence $\delta = 4$ by (30). All in all, we obtain

$$f_{Y/K} = \epsilon + \delta = 4 + 4 = 8.$$

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