ON EXCEPTIONAL VALUES OF A MEROMORPHIC FUNCTION

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1. M. Brelot [1] has shown that if \( u(z) \) is subharmonic in an open set \( D \) in the \( z \)-plane with boundary \( C \) and is bounded from above in a neighborhood of a boundary point \( z_0 \), which is contained in a set \( E \subset C \) of inner harmonic measure zero with respect to \( D \), and such that \( z_0 \) is a regular point for Dirichlet problem in \( D \), then

\[
\lim_{z \to z_0} \frac{u(z)}{z^2} = \lim_{z' \to z_0} \frac{u(z)}{z'^2}.
\]

Furthermore, it was shown that if \( f(z) \) is meromorphic in \( D \), then, for any \( z_0 \) of \( E \), which is in the closure of \( C - E \), whether a regular point or not, the same relation holds when \( u(z) \) is replaced by \( |f(z)| \) whenever the left side of (1) is finite. It is easy to see that this last relation is equivalent to the relation:

\[
\text{boundary of } S^{(D)}_{z_0} \subset S^{(C - E)}_{z_0},
\]

where the cluster set \( S^{(D)}_{z_0} \) is the set of values approached sequencewise by \( f(z) \) in any neighborhood of \( z_0 \) and the boundary cluster set \( S^{(C - E)}_{z_0} \) from \( C - E \) is the limit of the closure of \( \bigcup_{z' \in (C - E)_r} S^{(D)}_{z_0} \) as \( r \to 0 \), \( (C - E)_r \) being that part of \( C - E \) in \( |z - z_0| < r \).

Later M. Tsuji [5] showed that in the special case that \( D \) is a domain and \( E \) is a closed set of logarithmic capacity zero, the exceptional values in \( \Omega = S^{(D)}_{z_0} - S^{(C - E)}_{z_0} \), that is, the set of values in \( \Omega \) which \( f(z) \) does not assume in some neighborhood of \( z_0 \) form a set of inner logarithmic capacity zero.

2. In this note we shall prove that this is true in the general case.

**Theorem.** Let \( D \) be an open set in the \( z \)-plane, \( C \) its boundary, \( E \subset C \) a set of inner harmonic measure zero with respect to \( D \), \( z_0 \) a point of \( E \) in the closure of \( C - E \), and \( f(z) \) a meromorphic function in \( D \). Then every value of

Received May 30, 1955.

1) See [4], for instance.
$S_{2z}^{(p)} - S_{2z}^{(c-r)}$ is assumed by $f(z)$ in any neighborhood of $z_0$ except for a set of (outer) logarithmic capacity zero.

**Proof.** If there is a disc: $|z - z_0| < r$ such that the part $C_r$ of $C$ in this disc is contained in $E$, this part is of harmonic measure zero with respect to this disc minus $C_r$ and hence of logarithmic capacity zero. Hence in any neighborhood of $z_0$ in $D_f(z)$ takes on every complex value except for a closed set of logarithmic capacity zero, or else $f(z)$ is continuous at $z_0$, by a theorem of Kametani [3]. Thus our theorem is true in this case.

Next we consider the case when $z_0$ is in the closure of $C - E$ and suppose that the exceptional values in $S_{2z}^{(p)} - S_{2z}^{(c-r)}$ form a set of positive inner logarithmic capacity. Then there exists a closed bounded set $F$ of positive logarithmic capacity lying in a component $\Omega_1$ of $S_{2z}^{(p)} - S_{2z}^{(c-r)}$ such that the values of $F$ are not assumed by $f(z)$ in $D_{r_0}: D \cap \{|z - z_0| < r_0\}$. Let $K$ be a compact set in $\Omega_1$ containing a closed subset $F_1 \subset F$ of positive logarithmic capacity in its interior and bounded by a smooth curve $\gamma$. If we take $r_1 < r_0$ sufficiently small, $K$ is disjoint from the closure of $\bigcup \{ S_{2z}^{(p)} \}_{z' \in (c-r), \gamma}$. Brelot’s result (2) shows that $S_{2z}^{(p)} \cap K = \phi$ or $S_{2z}^{(p)} \cap \gamma$ at any regular point $z' \in E_{r_0}: E \cap \{|z - z_0| < r_1\}$. However, the latter case cannot occur. For, if we apply Brelot’s result (1) to the composed function in $D_{r_0}$ of $f(z)$ with the equilibrium potential of $F_1$, we get a contradiction. Therefore if we exclude all regular points from $E$ and denote the remaining set by $E_1$, that component of $S_{2z}^{(p)} - S_{2z}^{(c-r)}$ containing $K$ is equal to $\Omega_1$; that is, this set remains unchanged. Let us consider the inverse image in $D_{r_1}$ of the interior of $K$ and denote it by $D_0$. The boundary of $D_0$ consists of (i) part of $|z - z_0| = r_1$, (ii) certain arcs in $D_{r_1}$ on which $f(z) \in \gamma$ and (iii) a closed subset $E_0$ of $E_1$ of logarithmic capacity zero. If there is no connected component of $D_0$ containing $z_0$ on its boundary, then $z_0$ is a regular point with respect to $D_0$ and the reasoning used above is applied again. If there is a domain containing $z_0$ on its boundary, we can apply the result of Tsuji, stated at the beginning, to obtain a contradiction to the fact that $K$ contains a closed set of exceptional values of positive logarithmic capacity. Since the set $B_n$ of values in $S_{2z}^{(p)} - S_{2z}^{(c-r)}$ not taken by $f(z)$ in $D_{r_1/n}$ is a countable union of closed sets of logarithmic capacity zero, it is of (outer) logarithmic capacity zero. Hence the set of exceptional values which is equal to the union $\bigcup_n B_n$ is of (outer) logarithmic capacity zero. Thus our proof is completed.
Finally we remark that if we use the ramified topology in $D$ (see [2] for this) and define the vanishing of harmonic measure and the cluster sets with respect to this topology, then we can extend our result to this case.

BIBLIOGRAPHY


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