ON EXCEPTIONAL VALUES OF A MEROMORPHIC FUNCTION

MAKOTO OHTSUKA

1. M. Brelot [1] has shown that if \( u(z) \) is subharmonic in an open set \( D \) in the \( z \)-plane with boundary \( C \) and is bounded from above in a neighborhood of a boundary point \( z_0 \), which is contained in a set \( E \subset C \) of inner harmonic measure zero with respect to \( D \), and such that \( z_0 \) is a regular point for Dirichlet problem in \( D \), then

\[
\lim_{z \to z_0} u(z) = \lim_{z' \to z_0} \left( \lim_{z \to z'} u(z) \right).
\]

Furthermore, it was shown that if \( f(z) \) is meromorphic in \( D \), then, for any \( z_0 \) of \( E \), which is in the closure of \( C - E \), whether a regular point or not, the same relation holds when \( u(z) \) is replaced by \( |f(z)| \) whenever the left side of (1) is finite. It is easy to see that this last relation is equivalent to the relation:

\[
\text{boundary of } S^{(D)}_{z_0} \subset S^{(C-E)}_{z_0},
\]

where the cluster set \( S^{(D)}_{z_0} \) is the set of values approached sequencewise by \( f(z) \) in any neighborhood of \( z_0 \) and the boundary cluster set \( S^{(C-E)}_{z_0} \) from \( C - E \) is the limit of the closure of \( \bigcup_{z' \in (C-E)_r} S^{(D)}_{z_0} \) as \( r \to 0 \), \( (C-E)_r \) being that part of \( C-E \) in \( |z - z_0| < r \).

Later M. Tsuji [5] showed that in the special case that \( D \) is a domain and \( E \) is a closed set of logarithmic capacity zero, the exceptional values in \( \Omega = S^{(D)}_{z_0} - S^{(C-E)}_{z_0} \), that is, the set of values in \( \Omega \) which \( f(z) \) does not assume in some neighborhood of \( z_0 \) form a set of inner logarithmic capacity zero.

2. In this note we shall prove that this is true in the general case.

**Theorem.** Let \( D \) be an open set in the \( z \)-plane, \( C \) its boundary, \( E \subset C \) a set of inner harmonic measure zero with respect to \( D \), \( z_0 \) a point of \( E \) in the closure of \( C - E \), and \( f(z) \) a meromorphic function in \( D \). Then every value of

Received May 30, 1955.

1) See [4], for instance.
$S_{x_0}^{(p)} - S_{x_0}^{(c-R)}$ is assumed by $f(z)$ in any neighborhood of $z_0$ except for a set of (outer) logarithmic capacity zero.

Proof. If there is a disc: $|z - z_0| < r$ such that the part $C_r$ of $C$ in this disc is contained in $E$, this part is of harmonic measure zero with respect to this disc minus $C_r$ and hence of logarithmic capacity zero. Hence in any neighborhood of $z_0$ in $D f(z)$ takes on every complex value except for a closed set of logarithmic capacity zero, or else $f(z)$ is continuous at $z_0$, by a theorem of Kametani [3]. Thus our theorem is true in this case.

Next we consider the case when $z_0$ is in the closure of $C - E$ and suppose that the exceptional values in $S_{x_0}^{(p)} - S_{x_0}^{(c-R)}$ form a set of positive inner logarithmic capacity. Then there exists a closed bounded set $F$ of positive logarithmic capacity lying in a component $\Omega_1$ of $S_{x_0}^{(p)} - S_{x_0}^{(c-R)}$ such that the values of $F$ are not assumed by $f(z)$ in $D_{r_0}: D \cap \{|z - z_0| < r_0\}$. Let $K$ be a compact set in $\Omega_1$ containing a closed subset $F_1 \subset F$ of positive logarithmic capacity in its interior and bounded by a smooth curve $\gamma$. If we take $r_1 < r_0$ sufficiently small, $K$ is disjoint from the closure of $\bigcup S_{x_0}^{(p)}$. Brelot's result (2) shows that $S_{x_0}^{(p)} \cap K = \phi$ or $S_{x_0}^{(p)} \supset K$ at any regular point $z' \in E_{r_1}: E \cap \{|z - z_0| < r_1\}$. However, the latter case cannot occur. For, if we apply Brelot's result (1) to the composed function in $D_{r_1}$ of $f(z)$ with the equilibrium potential of $F_1$, we get a contradiction. Therefore if we exclude all regular points from $E$ and denote the remaining set by $E_1$, that component of $S_{x_0}^{(p)} - S_{x_0}^{(c-R)}$ containing $K$ is equal to $\Omega_1$; that is, this set remains unchanged. Let us consider the inverse image in $D_{r_1}$ of the interior of $K$ and denote it by $D_0$. The boundary of $D_0$ consists of (i) part of $|z - z_0| = r_1$, (ii) certain arcs in $D_{r_1}$ on which $f(z) \in \gamma$ and (iii) a closed subset $E_0$ of $E_1$ of logarithmic capacity zero. If there is no connected component of $D_0$ containing $z_0$ on its boundary, then $z_0$ is a regular point with respect to $D_0$ and the reasoning used above is applied again. If there is a domain containing $z_0$ on its boundary, we can apply the result of Tsuji, stated at the beginning, to obtain a contradiction to the fact that $K$ contains a closed set of exceptional values of positive logarithmic capacity. Since the set $B_n$ of values in $S_{x_0}^{(p)} - S_{x_0}^{(c-R)}$ not taken by $f(z)$ in $D_{r_1/n}$ is a countable union of closed sets of logarithmic capacity zero, it is of (outer) logarithmic capacity zero. Hence the set of exceptional values which is equal to the union $\bigcup B_n$ is of (outer) logarithmic capacity zero. Thus our proof is completed.
Finally we remark that if we use the ramified topology in $D$ (see [2] for this) and define the vanishing of harmonic measure and the cluster sets with respect to this topology, then we can extend our result to this case.

BIBLIOGRAPHY


Mathematical Institute,
Nagoya University