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A KOROVKIN TYPE THEOREM FOR WEIGHTED SPACES OF CONTINUOUS FUNCTIONS

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We prove a Korovkin type approximation theorem for positive linear operators on weighted spaces of continuous real-valued functions on a compact Hausdorff space X. These spaces comprise a variety of subspaces of $C(X)$ with suitable locally convex topologies and were introduced by Nachbin 1967 and Prolla 1977. Some early Korovkin type results on the weighted approximation of real-valued functions in one and several variables with a single weight function are due to Gadzhiev 1976 and 1980.

1. WEIGHTED SPACES OF FUNCTIONS

Throughout this paper, let X be a locally compact Hausdorff space. A real-valued function f on X is said to *vanish at infinity* if for every $\varepsilon > 0$ the set $\{x \in X \mid |f(x)| \geqslant \varepsilon\}$ is relatively compact. As usual, we denote by

$C(X)$ the space of all continuous real-valued functions on X,
$C_B(X)$ the space of all bounded functions in $C(X)$,
$C_0(X)$ the space of all functions in $C(X)$ that vanish at infinity,
$C_c(X)$ the space of all functions in $C(X)$ with compact support.

A variety of norms and seminorms may be considered on these spaces. The supremum norm is generally available for $C_B(X)$, $C_0(X)$ and $C_c(X)$. Korovkin type approximation theorems have been developed for $C_0(X)$ endowed with the supremum norm, by Bauer and Donner [**2**] and other authors. In the following we shall present a general approach that deals simultaneously with a variety of subspaces of $C(X)$, carrying different locally convex topologies. We shall use the concept of *weighted spaces of functions* as developed by Nachbin. We give a short introduction to the general theory. For details and proofs we globally refer to Nachbin's [**5**] fundamental work and the monograph by Prolla [**6**].

A family $\mathcal{W}$ of non-negative upper semicontinuous functions on X is called a *family of weights* if for all $w_1, w_2 \in \mathcal{W}$ there are $w_3 \in \mathcal{W}$ and $\rho > 0$ such that

$$w_1 \leqslant \rho w_3 \quad \text{and} \quad w_2 \leqslant \rho w_3.$$

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With any family of weights $\mathcal{W}$ we associate the subspace of $C(X)$

$$C_{\mathcal{W}}(X) = \{f \in C(X) \mid wf \text{ vanishes at infinity for all } w \in \mathcal{W}\}.$$

Together with the locally convex topology generated by the seminorms

$$p_w(f) = \sup\{|wf(x)| \mid x \in X\}$$

for $w \in \mathcal{W}$ and $f \in C_{\mathcal{W}}(X)$ we call $C_{\mathcal{W}}(X)$ a *weighted space of functions.* If for every $x \in X$ there is some $w \in W$ such that $w(x) \neq 0$, then the topology of $C_{\mathcal{W}}(X)$ is obviously Hausdorff. We shall mention some examples for weighted spaces of functions.

EXAMPLE 1.1.

(a) If $\mathcal{W}$ consists of the constant function $w \equiv 1$, then $C_{\mathcal{W}}(X) = C_0(X)$, and p_w is the supremeum norm.

(b) If $\mathcal{W} = C_0^+(X) = \{w \in C_0(X) \mid w \geqslant 0\}$, then $C_{\mathcal{W}}(X) = C_B(X)$, but the weighted topology (called the *strict* topology) is generally coarser than the supremum norm topology on $C_B(X)$.

(c) If $\mathcal{W}$ consists of the characteristic functions of all compact subsets of X, then $C_{\mathcal{W}}(X) = C(X)$ with the topology of compact convergence.

(d) If $\mathcal{W}$ consists of the characteristic functions of all finite subsets of X, then $C_{\mathcal{W}}(X) = C(X)$ with the topology of pointwise convergence.

(e) If $\mathcal{W} = C^+(X) = \{w \in C(X) \mid w \geqslant 0\}$, then $C_{\mathcal{W}}(X) = C_c(X)$. The weighted topology is generally finer than the supremum norm but coarser than the inductive limit topology on $C_c(X)$.

Example 1.1(d) shows in particular that weighted spaces of functions need not be complete. Note that $C_c(X)$ is a dense subspace of $C_{\mathcal{W}}(X)$ for any choice of the family of weights $\mathcal{W}$.

The following characterisation of the dual of a weighted space of functions may be found in Prolla's monogragh [**6**]. By $M_B(X)$ we denote the space of all finite regular Borel measures on X, and by $M_B^+(X)$ the positive cone in $M_B(X)$. The norm of $\mu \in M_B(X)$ is defined as $\|\mu\| = |\mu|(X)$. By the Riesz Representation Theorem every bounded linear functional on $C_c(X)$ endowed with the supremum norm, may be uniquely represented as a measure $\mu \in M_B(X)$. For weighted spaces in general, every continuous linear functional on $C_{\mathcal{W}}(X)$ is seen to be continuous with respect to the inductive topology of the subspace $C_c(X)$, hence may be represented for the functions in $C_{\mathcal{W}}(X)$ by a (not necessarily bounded) Borel measure on X.

For all $f \in C_{\mathcal{W}}(X)$ and $w \in \mathcal{W}$, the function wf is integrable with respect to every $\mu \in M_B(X)$. Thus by $w\mu$ we denote the linear functional

$$f \mapsto \mu(wf) : C_{\mathcal{W}}(X) \to \mathbb{R}.$$

As $|w\mu(f)| \leqslant \|\mu\|\, p_w(f)$, the functional $w\mu$ is in the dual of $C_{\mathcal{W}}(X)$. Moreover, it may be seen that the functionals of this type constitute the whole dual of $C_{\mathcal{W}}(X)$. For the proof of this fact we refer to Prolla [**6**, Theorem 5.42]:

Theorem 1.2. *The dual of $C_{\mathcal{W}}(X)$ may be identified with the space of all functionals $w\mu$, where $w \in \mathcal{W}$ and $\mu \in M_B(X)$; more precisely: For $w \in \mathcal{W}$ the polar of the 0-neighbourhood $V = \{f \in C_{\mathcal{W}}(X) \mid p_w(f) \leqslant 1\}$ is given by*

$$V^\circ = \{w\mu \mid \mu \in M_B(X),\ \|\mu\| \leqslant 1\}.$$

Positive linear functionals on $C_{\mathcal{W}}(X)$ are represented by positive measures $\mu \in M_B^+(X)$.

Reviewing some of our Examples 1.1, we realise that in (a) the dual of $C_{\mathcal{W}}(X)$ consists of $M_B(X)$, in (c) of those measures in $M_B(X)$ having compact support, and in (d) of the linear combinations of point evaluations (called *simple* measures).

2. The Main Approximation Theorem

The weighted spaces $C_{\mathcal{W}}(X)$ are endowed with the pointwise order for functions, and a linear operator $T : C_{\mathcal{W}}(X) \to C_{\mathcal{W}}(X)$ is said to be *positive* if $T(f) \geqslant 0$, whenever $f \geqslant 0$. A family $\mathcal{T}$ of linear operators on $C_{\mathcal{W}}(X)$ is *equicontinuous* if for every $w \in \mathcal{W}$ there are $w' \in \mathcal{W}$ and $\rho > 0$ such that for the corresponding seminorms

$$p_w(T(f)) \leqslant \rho\, p_{w'}(f)$$

holds for all $f \in C_{\mathcal{W}}(X)$ and $T \in \mathcal{T}$.

Korovkin type theorems deal with approximation processes modelled by equicontinuous nets $(T_\alpha)_{\alpha\in\mathcal{A}}$ of positive linear operators. We write $T_\alpha(f) \to f$ if the net $(T_\alpha(f))_{\alpha\in\mathcal{A}}$ converges to f in the topology of $C_{\mathcal{W}}(X)$.

For a subset $\mathcal{M}$ of $C_{\mathcal{W}}(X)$, the *Korovkin closure* $\mathcal{K}(\mathcal{M})$ of $\mathcal{M}$ consists of all functions $f \in C_{\mathcal{W}}(X)$ such that $T_\alpha(f) \to f$ whenever $(T_\alpha)_{\alpha\in\mathcal{A}}$ is an equicontinuous net of positive linear operators on $C_{\mathcal{W}}(X)$ and $T_\alpha(g) \to g$ holds for all $g \in \mathcal{M}$. A *Korovkin system* for $C_{\mathcal{W}}(X)$ is a subset $\mathcal{M} \subset C_{\mathcal{W}}(X)$ such that $\mathcal{K}(\mathcal{M}) = C_{\mathcal{W}}(X)$. By $\operatorname{span}(\mathcal{M})$ we denote the linear span of $\mathcal{M}$; that is, the subspace of $C_{\mathcal{W}}(X)$ generated by $\mathcal{M}$.

Now we are ready to formulate our main result. For the special case of $C_0(X)$ with the supremum norm, that is, the case of our Example 1.1(a), it is due to Bauer and Donner [**2**].

Theorem 2.1. *Let X be a locally compact Hausdorff space, and let $\mathcal{W}$ be a family of weight functions on X. Let $\mathcal{M}$ be a subset of $C_{\mathcal{W}}(X)$. For a function $f \in C_{\mathcal{W}}(X)$ the following are equivalent:*

(a) $f \in \mathcal{K}(\mathcal{M})$.

(b) *For every* $x \in X$ *such that* $w(x) > 0$ *for at least one weight function* $w \in \mathcal{W}$,

$$f(x) = \sup_{\substack{w\in\mathcal{W}\\ \varepsilon>0}} \inf\{g(x) \mid g \in \operatorname{span}(\mathcal{M}), \quad wf \leqslant wg + \varepsilon\}$$
$$= \inf_{\substack{w\in\mathcal{W}\\ \varepsilon>0}} \sup\{g(x) \mid g \in \operatorname{span}(\mathcal{M}), \quad wg \leqslant wf + \varepsilon\}.$$

(c) *For every* $x \in X$ *such that* $w(x) > 0$ *for at least one weight function* $w \in \mathcal{W}$, *and for every* $\mu \in M_B^+(X)$ *and* $w \in \mathcal{W}$

$$\mu(wg) = g(x) \quad \text{for all } g \in \mathcal{M} \quad \text{implies} \quad \mu(wf) = f(x).$$

PROOF: (a) $\Rightarrow$ (c): Let us assume that (c) fails for the function $f \in C_{\mathcal{W}}(X)$, the point $x \in X$, the measure $\mu \in M_B^+(X)$ and the weight function $w \in \mathcal{W}$; that is, we have

$$\mu(wg) = g(x) \quad \text{for all } g \in \mathcal{M}, \quad \text{but} \quad \mu(wf) \neq f(x).$$

We shall show that (a) fails as well: Let U_0 be a fixed compact neighbourhood for x and let $\mathcal{U}$ be a basis of open neighbourhoods of x that are all subsets of U_0. By Urysohn's Lemma, for all $U \in \mathcal{U}$ there are functions $\phi_U \in C_c(X)$ such that

$$0 \leqslant \phi_U \leqslant 1, \quad \phi_U(x) = 1, \quad \text{and} \quad \phi_U \equiv 0 \quad \text{on} \quad X \setminus U.$$

Using those functions we define operators T_U on $C_{\mathcal{W}}(X)$ by

$$T_U(h) = h(1 - \phi_U) + \mu(wh)\,\phi_U$$

for $h \in C_{\mathcal{W}}(X)$. Clearly $T_U(h) \in C_{\mathcal{W}}(X)$, and the operators T_U are linear and positive. Equicontinuity is easily checked: Given $w' \in \mathcal{W}$, choose $w'' \in \mathcal{W}$ and $\rho > 0$ such that both

$$w \leqslant \rho w'' \quad \text{and} \quad w' \leqslant \rho w''.$$

With an upper bound $\sigma > 0$ for the upper semicontinuous function w' on U_0, we compute for all $y \in X$ and $U \in \mathcal{U}$

$$\begin{aligned}
|T_U(h)(y)\,w'(y)| &\leqslant \Big(|h(y)|\,\big(1 - \phi_U(y)\big) + |\mu(wh)|\,\phi_U(y)\Big)\,w'(y)\\
&\leqslant |h(y)|\,w'(y) + \sigma\,|\mu(wh)|\\
&\leqslant p_{w'}(h) + \sigma\,\|\mu\|\,p_w(h)\\
&\leqslant \rho\,p_{w''}(h) + \sigma\rho\,\|\mu\|\,p_{w''}(h)\\
&= \rho\,\big(1 + \sigma\,\|\mu\|\big)\,p_{w''}(h).
\end{aligned}$$

Thus $p_{w'}\big(T_U(h)\big) \leqslant \rho\,(1+\sigma\,\|\mu\|)\,p_{w''}(h)$. Ordered by reverse inclusion (that is, $U \leqslant V$ if $U \supset V$), the neighbourhood system $\mathcal{U}$ serves as the index set of the net $(T_U)_{U\in\mathcal{U}}$. For every function $g \in \mathcal{M}$ we observe that

$$T_U(g)(y) = g(y) \quad \text{for all } y \in X \setminus U.$$

For $y \in U \subset U_0$ on the other hand, and for $w' \in \mathcal{W}$ and $\sigma > 0$ an upper bound for w' on U_0, we verify

$$\begin{aligned}\big|T_U(g)(y) - g(y)\big|w'(y) &= \big|\big(g(y)\,(1-\phi_U(y)) + \mu(wg)\,\phi_U(y)\big) - g(y)\big|\,w'(y)\\ &\leqslant\ \phi_U(y)\ |g(x) - g(y)|\ w'(y)\\ &\leqslant\ \sigma\ |g(x) - g(y)|\,.\end{aligned}$$

Thus $p_{w'}\big(T_U(g) - g\big) \leqslant \varepsilon$ for all $U \in \mathcal{U}$ such that $|g(x) - g(y)| \leqslant \varepsilon/\sigma$ for all $y \in U$. This shows that $T_U(g) \to g$ for all $g \in \mathcal{M}$. But for the given function f and the point $x \in X$ we have $T_U(f)(x) = \mu(wf) \neq f(x)$. By assumption (c) there exists at least one weight function $w' \in \mathcal{W}$ that does not vanish at x. This shows $p_{w'}\big(T_U(f) - f\big) \not\to 0$, hence $T_U(f) \not\to f$, contradicting (a).

(c) $\Rightarrow$ (b): Suppose that the function $f \in C_{\mathcal{W}}(X)$ satisfies (c), and let $x \in X$ be such that $w(x) > 0$ for some $w \in \mathcal{W}$. We shall prove that the first equality in (b) holds. The second one may be verified in an analogous way. The inequality

$$\sup_{\substack{w\in\mathcal{W}\\ \varepsilon>0}} \inf\{g(x) \mid g \in \operatorname{span}(\mathcal{M}),\ wf \leqslant wg + \varepsilon\}\ \geqslant f(x)$$

holds in any case, as for $w \in \mathcal{W}$ such that $w(x) > 0$ and $\varepsilon > 0$ we realise that

$$\inf\{g(x) \mid g \in \operatorname{span}(\mathcal{M}),\ \ wf \leqslant wg + \varepsilon\} \geqslant\ f(x) - \varepsilon/w(x).$$

For the converse inequality, it suffices to show that for any $w \in \mathcal{W}$ and $\varepsilon > 0$ we have

$$\inf\{g(x) \mid g \in \operatorname{span}(\mathcal{M}),\ wf \leqslant wg + \varepsilon\} \leqslant f(x).$$

As the infimum on the left hand side of the last inequality increases, if we replace $w \in \mathcal{W}$ by a larger weight function $w' \in \mathcal{W}$, and as $\mathcal{W}$ is directed upward, we may assume that $w(x) > 0$. Next we define a functional p on $C_{\mathcal{W}}(X)$ by

$$p(h) = \inf\{g(x) + 2\delta/w(x) \mid \delta \geqslant 0,\ g \in \operatorname{span}(\mathcal{M}),\ wh \leqslant wg + \delta\}.$$

As $h(x) \leqslant p(h) \leqslant \big(2/w(x)\big)\,p_w(h)$, the functional p is real-valued on $C_{\mathcal{W}}(X)$. It is easily checked to be sublinear, and for $g \in \operatorname{span}(\mathcal{M})$ we realise that $p(g) = g(x)$. Thus, by the Hahn-Banach theorem there is a linear functional Φ on $C_{\mathcal{W}}(X)$ satisfying

$$\Phi(h) \leqslant p(h) \quad \text{for all } h \in C_{\mathcal{W}}(X) \qquad \text{and} \qquad \Phi(f) = p(f)$$

for the given function $f \in C_{\mathcal{W}}(X)$. As $\Phi(h) \leqslant p(h) \leqslant 0$, whenever $h \leqslant 0$, the functional Φ is positive and continuous with respect to the topology of $C_{\mathcal{W}}(X)$, thus may be represented as $w\mu$, where $\mu \in M_B^+(X)$. By the above, this means $\mu(wg) = g(x)$ for all $g \in \mathcal{M}$, and therefore by (c)

$$\begin{aligned} f(x) = \mu(wf) = \Phi(f) = p(f) \\ = \inf\{g(x) + 2\delta/w(x) \mid \delta \geqslant 0,\ g \in \operatorname{span}(\mathcal{M}),\ wf \leqslant wg + \delta\}. \end{aligned}$$

Thus for any choice of $0 < \varepsilon' \leqslant \varepsilon/w(x)$ we may find $\delta \geqslant 0$ and $g \in \operatorname{span}(\mathcal{M})$ such that

$$wf \leqslant wg + \delta \quad \text{and} \quad g(x) + 2\delta/w(x) \leqslant f(x) + \varepsilon' \leqslant g(x) + \delta/w(x) + \varepsilon',$$

whence $\delta/w(x) \leqslant \varepsilon' \leqslant \varepsilon/w(x)$ and $\delta \leqslant \varepsilon$. Summarising, this shows

$$wf \leqslant wg + \varepsilon \quad \text{and} \quad g(x) \leqslant f(x) + \varepsilon',$$

hence

$$\inf\{g(x) \mid g \in \operatorname{span}(\mathcal{M}),\ wf \leqslant wg + \varepsilon\} \leqslant f(x)$$

as desired.

(b) $\Rightarrow$ (a): Let us assume that (b) holds for the function $f \in C_{\mathcal{W}}(X)$, but that (a) is false. Then there is an equicontinuous net $(T_\alpha)_{\alpha \in \mathcal{A}}$ of positive linear operators on $C_{\mathcal{W}}(X)$ such that

$$T_\alpha(g) \to g \quad \text{for all } g \in \mathcal{M}, \qquad \text{but} \qquad T_\alpha(f) \not\to f;$$

that is, $p_w\big(T_\alpha(f) - f\big) \not\to 0$ for some weight function $w \in \mathcal{W}$. There are $\delta > 0$, a subnet $(T_\beta)_{\beta \in \mathcal{B}}$ of $(T_\alpha)_{\alpha \in \mathcal{A}}$ and points $x_\beta \in X$ such that

$$w(x_\beta)\left|T_\beta(f)(x_\beta) - f(x_\beta)\right| \geqslant \delta \quad \text{for all } \beta \in \mathcal{B}.$$

We may assume (after selecting another subnet) that either

$$w(x_\beta)\left(T_\beta(f)(x_\beta) - f(x_\beta)\right) \geqslant \delta \quad \text{or} \quad w(x_\beta)\left(T_\beta(f)(x_\beta) - f(x_\beta)\right) \leqslant -\delta$$

holds for all $\beta \in \mathcal{B}$. We assume the first case, as we continue with our argument. The second case is similar. Let us recall that the operators $(T_\beta)_{\beta \in \mathcal{B}}$ were supposed to be equicontinuous. For the given weight function $w \in \mathcal{W}$ from above there are $w' \in \mathcal{W}$ and $\rho > 0$ such that

$$p_w\big(T_\beta(h)\big) \leqslant \rho\, p_{w'}(h)$$

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for all $h \in C_{\mathcal{W}}(X)$ and $\beta \in \mathcal{B}$. Next we observe that the net $(x_\beta)_{\beta \in B}$ permits the selection of a further subnet $(x_\gamma)_{\gamma \in C}$ that converges to some $x_0 \in X$ or to $x_0 = \infty$. (The latter refers to the one-point compactification on X). We continue our argument with this subnet. Following condition (b), for any choice of $\varepsilon > 0$ and the weight function $w' \in \mathcal{W}$ from above there is $g \in \operatorname{span}(\mathcal{M})$ such that

$$w'f \leqslant w'g + \varepsilon/\rho, \quad \text{and} \quad w(x_0)g(x_0) \leqslant w(x_0)f(x_0) + \varepsilon.$$

(The second condition is void for $x_0 = \infty$ or for $w(x_0) = 0$. But our assumption that (a) fails for the function f implies in particular that $\mathcal{W}$ contains a non-zero weight function and that the equalities in (b) hold for at least one point $x \in X$. Thus we may always find a function $g \in \operatorname{span}(\mathcal{M})$ fulfilling the first condition.) With $h = (f-g)^+ = \sup\{(f-g), 0\} \in C_{\mathcal{W}}(X)$ we have

$$0 \leqslant w'h \leqslant \varepsilon/\rho \quad \text{and} \quad f \leqslant g + h.$$

Thus $p_{w'}(h) \leqslant \varepsilon/\rho$ and $p_w\big(T_\gamma(h)\big) \leqslant \varepsilon$ for all $\gamma \in C$ by the above. This shows

$$T_\gamma(f) \leqslant T_\gamma(g) + T_\gamma(h), \quad \text{hence} \quad w\,T_\gamma(f) \leqslant w\,T_\gamma(g) + \varepsilon.$$

But we have $T_\gamma(g) \to g$, as $g \in \operatorname{span}(\mathcal{M})$. There is $\gamma_0 \in C$ such that $p_w\big(T_\gamma(g) - g\big) \leqslant \varepsilon$ for all $\gamma \geqslant \gamma_0$. Summarising, we obtain for those γ

$$w\,T_\gamma(f) \leqslant w\,T_\gamma(g) + \varepsilon \leqslant wg + 2\varepsilon,$$

and using the selected the points $x_\gamma \in X$,

$$w(x_\gamma)f(x_\gamma) + \delta \leqslant w(x_\gamma)T_\gamma\big(f\big)(x_\gamma) \leqslant w(x_\gamma)g(x_\gamma) + 2\varepsilon.$$

To obtain the desired contradiction, now we have to separate our cases:

If $x_0 = \infty$, then both

$$w(x_\gamma)f(x_\gamma) \to 0 \quad \text{and} \quad w(x_\gamma)g(x_\gamma) \to 0,$$

a contradiction to our last inequality, if we choose $\varepsilon < \delta/2$.

If $x_0 \in X$, but $w(x_0) = 0$, then $w(x_\gamma) \to 0$, as w is non-negatiave and upper semicontinuous. Again, with $\varepsilon < \delta/2$ we obtain a contradiction.

Finally, if $x_0 \in X$, and $w(x_0) > 0$, we may use the second condition for the choice of the function $g \in \operatorname{span}(\mathcal{M})$ and the continuity of f and g in x_0: There is $\gamma_1 \geqslant \gamma_0$ such that for all $\gamma \geqslant \gamma_1$

$$|f(x_\gamma) - f(x_0)| \leqslant \varepsilon \quad \text{and} \quad |g(x_\gamma) - g(x_0)| \leqslant \varepsilon$$

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holds. Furthermore we may assume that $w(x_\gamma) \leqslant \sigma$ for some $\sigma > 0$. (The function w is bounded above on a compact neighbourhood of x_0.) Thus for all such $\gamma \geqslant \gamma_1$

$$\begin{aligned} w(x_\gamma)g(x_\gamma) &\leqslant w(x_\gamma)g(x_0) + \varepsilon\sigma \\ &\leqslant w(x_\gamma)f(x_0) + \varepsilon + \varepsilon\sigma \\ &\leqslant w(x_\gamma)f(x_\gamma) + \varepsilon(1+2\sigma) \\ &\leqslant w(x_\gamma)g(x_\gamma) + \varepsilon(3+2\sigma) - \delta, \end{aligned}$$

and $\delta \leqslant \varepsilon(3+2\sigma)$, a contradiction if we choose ε sufficiently small, thus completing our proof. ∎

An application of Theorem 2.1 yields a Stone-Weierstrass theorem for weighted spaces. The result is well-known for the special case of $C_0(X)$ with the supremum norm (see [**1**, Chapter 4.4]).

COROLLARY 2.2. *Let X be a locally compact Hausdorff space, and let $\mathcal{W}$ be a family of weight functions on X. If $\mathcal{M} \subset C_\mathcal{W}(X)$ is a Korovkin system for $C_\mathcal{W}(X)$, then the vector sublattice generated by $\mathcal{M}$ is dense in $C_\mathcal{W}(X)$.*

PROOF: Let $\mathcal{M}$ be a Korovkin system for $C_\mathcal{W}(X)$. To prove our claim, it suffices to show that every positive function $f \in C_\mathcal{W}(X)$ may be approximated in the topology of $C_\mathcal{W}(X)$ by functions in the vector sublattice generated by $\mathcal{M}$: Let $w \in \mathcal{W}$ be any weight function. Given $\varepsilon > 0$, the set

$$Y = \{y \in X \mid w(y)f(y) \geqslant \varepsilon\}$$

is closed in X because the function wf is upper semicontinuous, hence Y is compact by the definition of $C_\mathcal{W}(X)$. Let $\sigma > 0$ be an upper bound for w on Y. For every $y \in Y$ there is by Theorem 2.1(b) a function $g_y \in \operatorname{span}(\mathcal{M})$ such that

$$wg_y \leqslant wf + \varepsilon \quad \text{and} \quad g_y(y) > f(y) - \varepsilon/\sigma.$$

The latter inequality holds even on an open neighbourhood U_y of y, and by the compactness of Y, finitely many of those neighbourhoods, say $U_{y_1}, \dots, U_{y_n}$ cover all of Y. Now we choose the function

$$g = g_{y_1} \vee \ldots \vee g_{y_n} \vee 0$$

in the vector sublattice generated by $\mathcal{M} \subset C_\mathcal{W}(X)$. Then we have $wg \leqslant wf + \varepsilon$ and $g(y) \geqslant f(y) - \varepsilon/\sigma$ for all $y \in Y$. On multiplying by w, the latter yields $w(y)g(y) \geqslant w(y)f(y) - \varepsilon$ for all $y \in Y$. But for $x \in X \setminus Y$ we observe that $w(x)g(x) \geqslant 0 \geqslant w(x)f(x) - \varepsilon$ holds as well. Summarising, this yields

$$w(x)\,|f(x) - g(x)| \leqslant \varepsilon \quad \text{for all} \quad x \in X,$$

hence $p_w(f-g) \leqslant \varepsilon$. ∎

The following is a useful tool to identify Korovkin systems in $C_\mathcal{W}(X)$:

COROLLARY 2.3. *Let $\mathcal{M}$ be a subset of $C_{\mathcal{W}}(X)$. Suppose that for every $x \in X$ such that $w(x) > 0$ for at least one weight function $w \in \mathcal{W}$,*

(i) *there is $g_x \in \mathcal{M}$ such that $g_x(x) \neq 0$,*

(ii) *for every $x \neq y \in X$ there is $g_{x,y} \in \operatorname{span}(\mathcal{M})$ such that*

$$g_{x,y} \geqslant 0, \quad g_{x,y}(x) = 0 \quad \text{and} \quad g_{x,y}(y) > 0.$$

Then $\mathcal{M}$ is a Korovkin system for $C_{\mathcal{W}}(X)$.

PROOF: It is obvious how the criterion in Corollary 2.2 implies condition (c) of Theorem 2.1 for every function $f \in C_{\mathcal{W}}(X)$: Let $x \in X$ such that $w(x) > 0$ for at least one weight function $w \in \mathcal{W}$ and let $\mu \in M_B^+(X)$ and $w \in \mathcal{W}$ be such that $w\mu(g) = g(x)$ for all $g \in \mathcal{M}$. For any $x \neq y \in X$ there is a function $g_{x,y} \in \operatorname{span}(\mathcal{M})$ as in (ii); that is, $w\mu(g_{x,y}) = 0$, and the point y cannot be contained in the support of the positive Borel measure $w\mu$. This shows that $w\mu$ is in fact a multiple of the point evaluation in x, and the point evaluation itself, as by our assumption (i) there is a function $g_x \in \mathcal{M}$ that does not vanish in x. Thus we have $w\mu(f) = f(x)$ for all $f \in C_{\mathcal{W}}(X)$, indeed. ∎

In their original work, Bauer and Donner [**2**] provide several examples for Korovkin systems in $C_0(X)$. Many more applications for approximation processes modelled by sequences of positive operators can be found in [**1**]. We shall conclude this paper with two elementary examples of Korovkin systems for weighted spaces different from $C_0(X)$.

EXAMPLE 2.4. (a) Let X be a locally compact Hausdorff space. If $C(X)$ contains a one-to-one function f, then for every strictly positive function $g \in C(X)$ the set $\mathcal{M} = \{g, gf, gf^2\}$ fulfils the criterion of Corollary 2.3: Clearly $g \in \mathcal{M}$ satisfies (i), and for a fixed $x \in X$ the function

$$y \longmapsto g(y)\big(f(y) - f(x)\big)^2$$

is contained in $\operatorname{span}(\mathcal{M})$, non-negative, vanishes at x but is positive at all $y \neq x$, hence satisfies (ii). Thus, for any choice of a family of weights $\mathcal{W}$ on X such that $\mathcal{M}$ is contained in $C_{\mathcal{W}}(X)$, the subset $\mathcal{M}$ is a Korovkin system for $C_{\mathcal{W}}(X)$.

(b) Let $X = [0, +\infty)$, and let $\mathcal{W}$ consist of the functions $w_\alpha(x) = e^{-\alpha x}$ for all $\alpha > 0$. Following part (a), the subset

$$\mathcal{M} = \{f_k \mid f_k(x) = x^k \quad \text{for } k = 0, 1, 2\}$$

is a Korovkin system for $C_{\mathcal{W}}(X)$: Let us illustrate this example with an approximation process modelled by a modified version of the classical Bernstein operators. For a function $f \in C_{\mathcal{W}}(X)$ define

$$T_n(f)(x) = \begin{cases} \displaystyle\sum_{k=0}^{n^2} \binom{n^2}{k} f\left(\frac{k}{n}\right) \left(\frac{x}{n}\right)^k \left(1 - \frac{x}{n}\right)^{n^2-k}, & \text{for } x < n \\ f(n), & \text{for } x \geqslant n. \end{cases}$$

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These operators T_n are clearly linear and positive on $C_{\mathcal{W}}(X)$. With some straight forward computations one may check the following:

$$\begin{aligned} T_n(f_0)(x) &= 1 && \text{for all } x \in [0,+\infty), \\ T_n(f_1)(x) &= x && \text{for all } x < n, \\ T_n(f_2)(x) &= \frac{n^2-1}{n^2}x^2 + \frac{1}{n}x && \text{for all } x < n. \end{aligned}$$

This shows in particular that $T_n(f_k) \to f_k$ for $k = 0,1,2$ in the topology of $C_{\mathcal{W}}(X)$. Furthermore, one may check that the sequence $\left(T_n\right)_{n\in\mathbb{N}}$ is indeed equicontinuous, as for any $f \in C_{\mathcal{W}}(X)$ and $\alpha > 0$

$$|f| \leqslant e^{\alpha x} \quad \text{implies that} \quad |T_n(f)| \leqslant e^{(e\alpha)x};$$

that is

$$p_{w_\alpha}\left(T_n(f)\right) \leqslant p_{w_{(\alpha/e)}}(f)$$

for all $f \in C_{\mathcal{W}}(X)$ and $n \in \mathbb{N}$. Thus Theorem 2.1 applies, and we may conclude that $T_n(f) \to f$ for all $f \in C_{\mathcal{W}}(X)$.

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