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Cartan decomposition for $\mathrm{Sp}(1,n)$ and some applications to group dynamics

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Abstract. **In this article, we present an elementary proof of the Cartan decomposition theorem for the group $\mathrm{Sp}(1,n)$. As an application, we determine the largest regular domain for discrete quaternionic hyperbolic groups acting on $\mathbb{HP}^n$. Furthermore, we demonstrate that Bers' simultaneous uniformization and Köebe's retrosection theorem fail to hold for higher-dimensional quaternionic Kleinian groups.**

1 Introduction

Quaternionic linear algebra plays a fundamental role across multiple disciplines due to its rich geometric structure and computational advantages over real and complex systems. In mathematics, quaternionic matrices arise in differential geometry and representation theory (see [12, 19]). In physics, they provide elegant formulations for string theory, twistors, and gauge theories, where their non-commutativity captures essential symmetries (e.g., [18]). Meanwhile, in computer science and engineering, quaternions enable efficient representations of 3D rotations, making them indispensable in robotic kinematics and real-time graphics (cf. [11, 27]).

Despite its interdisciplinary utility, quaternionic linear algebra remains underdeveloped compared to its real and complex counterparts. Critical limitations—such as the absence of a determinant-like function and the lack of a comprehensive spectral theory (see [18])—hinder progress in both theoretical and applied contexts. These gaps are particularly evident in the study of discrete group dynamics on hyperbolic spaces, where tools from classical linear algebra often fall short. To address these challenges, we turn to the symplectic subgroup $\mathrm{Sp}(1,n)$, whose geometric structure naturally circumvents these limitations through its action on quaternionic hyperbolic spaces.

In this article, we focus on $\mathrm{Sp}(1,n)$ and its dynamical action on the unitary quaternionic ball $\mathbf{H}^n_{\mathbb{H}}$. Our primary contribution is an elementary geometric proof of the Cartan decomposition theorem for $\mathrm{Sp}(1,n)$ (Theorem 1.1); this decomposition serves as the cornerstone for our subsequent analysis of limit sets. Specifically, we show that the action of $\mathrm{PSp}(1,n)$ on $\mathbb{HP}^n$ undergoes a radical structural change as we change from dimension one to the higher-dimensional setting (Theorem 1.2).

To contextualize our results, we first recall the indefinite symplectic group $\mathrm{Sp}(1,n)$, defined as the set of invertible quaternionic matrices preserving the Hermitian form H

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of signature $(1, n)$. Explicitly,

$$\mathrm{Sp}(1,n) = \left\{g \in \mathrm{GL}(n+1,\mathbb{H}) \mid gHg^* = H\right\},$$

where $\mathrm{GL}(n+1,\mathbb{H})$ denotes the general linear group of $(n+1)\times(n+1)$ invertible matrices over the quaternions $\mathbb{H}$; g^* is the conjugate transpose of g (for a quaternion $q = a + bi + cj + dk$, its conjugate is $\overline{q} = a - bi - cj - dk$); and H is the Hermitian matrix defining the bilinear form, given by

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

With this in mind, our first result is the following:

Theorem* 1.1 *(Cartan Decomposition) *For each $g \in \mathrm{Sp}(1,n)$ there are elements $k_+, k_- \in \mathrm{Sp}(n+1)\cap\mathrm{Sp}(1,n)$, and $\lambda(g) \geq 0$ such that $g = k_+\mu(g)k_-$, and $\mu(g) \in \mathrm{Sp}(1,n)$, where*

$$\mu(g) = \begin{pmatrix} e^{\lambda(g)} & & \\ & I_{n-1} & \\ & & e^{-\lambda(g)} \end{pmatrix}$$

A particularly elegant connection emerges in modeling real hyperbolic 3-space using quaternions. The upper half-space model of $\mathbb{H}^3_+$ can be described using Hamiltonian quaternions $\mathbb{H}$, identified as:

$$\mathbb{H}^3_+ = \{q \in \mathbb{H} \mid \mathrm{Im}(q) > 0\},$$

where the imaginary part is defined in the quaternionic sense. This framework naturally accommodates Möbius transformations through fractional linear maps:

$$q \mapsto (aq+b)(cq+d)^{-1}, \quad a,b,c,d \in \mathbb{C},\ ad - bc \neq 0,$$

as demonstrated by Ahlfors and Abikoff (see [1, 2]). This approach not only establishes a quaternionic analytic foundation for studying discrete isometry groups (cf. [5, 23]) but also interprets Möbius transformation groups as projective transformations acting on $\mathbb{HP}^n$. Our next result is a direct application of the Cartan decomposition, and provides a description of limit set for the subgroups in $\mathrm{PSp}(1,n)$.

***Theorem* 1.2** *For any non-elementary discrete subgroup $G \subset \mathrm{PSp}(1,n)$, the set*

$$\Omega(G) = \mathbb{HP}^n - \bigcup_{p\in\Lambda_{CG}(G)} p^{\perp},$$

where $\Lambda_{CG}(G)$ is the set of accumulation points of an orbit and $p^{\perp} = \{[q] \in \mathbb{HP}^n : pQq^* = 0\}$, coincides with G's equicontinuity set and represents the largest open domain where G acts properly discontinuously.

This interplay between quaternionic geometry and group dynamics invites exploration of classical theorems in higher dimensions. Foundational results like Koebe's Retrosection Theorem and Bers' Simultaneous Uniformization Theorem—cornerstones of

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Riemann surface theory and Kleinian groups—raise a natural question: Can these be extended to quaternionic projective spaces? Before addressing this, we briefly recall their classical formulations.

Koebe's Theorem states that every compact Riemann surface of genus $g \geq 1$ can be represented as Ω/G, where G is a *Schottky group* (a finitely generated, discrete, free group of loxodromic Möbius transformations) and $\Omega \subset \widehat{\mathbb{C}}$ is a domain with a Cantor-set boundary (see [22, 24, 26]). Geometrically, such groups are characterized by disjoint disks D_i, D'_i in $\widehat{\mathbb{C}}$ mapped onto each other by their generators (cf. [24]). However, no analogous characterization exists in higher-dimensional projective spaces (see [9, 6]), prompting two potential generalizations:

- **Algebraic Approach:** Define the uniformizing group G as a finitely generated, discrete group which is purely loxodromic.
- **Geometric Approach:** Require the group G to admit:
  - A finite symmetric generating set Σ.
  - A family $\{A_a\}_{a\in\Sigma}$ of compact, pairwise disjoint subsets in $\mathbb{HP}^n$ satisfying:

$$\bigcup_{b\in\Sigma\setminus\{a^{-1}\}} a(A_b) \subsetneq A_a.$$

Similarly, Bers' Theorem (see [3]) asserts that for any closed orientable surface S of genus $g \geq 2$ with two complex structures X and Y, there exists a discrete group $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$ and Γ-invariant Jordan domains $\Omega_1, \Omega_2 \subset \widehat{\mathbb{C}}$ such that:

(1) $X \cong \Omega_1/\Gamma$ and $Y \cong \Omega_2/\Gamma$,
(2) $\Omega_1 \cup \Omega_2$ is dense in $\widehat{\mathbb{C}}$.

Contrasting these classical frameworks with the quaternionic setting, our main results reveal fundamental limitations:

***Theorem* 1.3** *Neither the algebraic nor geometric versions of Koebe's Retrosection Theorem extend to* $\mathrm{PGL}(n+1, \mathbb{H})$*-groups acting on* $\mathbb{HP}^n$ *for* $n \geq 2$.

***Theorem* 1.4** *No analog of Bers' Simultaneous Uniformization Theorem exists for* $\mathrm{PGL}(n+1, \mathbb{H})$*-groups acting on* $\mathbb{HP}^n$ *with* $n \geq 2$.

The argument of both proofs hinges on two key results:

- **Structural Rigidity** (Klingler and Mok-Yeung, Theorem 5.1): Complete finite-volume quaternionic hyperbolic manifolds admit a unique hyperbolic structure.
- **Limit Set Characterization** (Theorem 1.2).

For instance, assume (for contradiction) that a Bers-type uniformization exists for a compact manifold of the form $M \sqcup M$, where $M = \mathbf{H}^n_{\mathbb{H}}/\Gamma$ and $\Gamma \subset \mathrm{PSp}(1, n)$ is a discrete group. By rigidity, the uniformizing group $G \subset \mathrm{PSp}(1, n)$ must satisfy $G = \Gamma$ (up to conjugation) and have $\Omega(G) = \mathbf{H}^n_{\mathbb{H}}$ as its only connected component. This leads to the contradiction $M \sqcup M = M$. An analogous argument applies to Koebe's case.

This paper is structured as follows: Section 2 begins by reviewing essential results in quaternion linear algebra and quaternionic hyperbolic geometry, while also establishing

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the notation used in subsequent sections. Building on this foundation, Section 3 presents an elementary proof of the Cartan decomposition for the group $\mathrm{PSp}(1,n)$. Section 4 then examines the structure of the limit set for quaternionic hyperbolic groups, offering a detailed characterization. Finally, Section 5 concludes the paper by demonstrating that Koebe's and Bers' theorems do not hold in the $\mathrm{PSp}(1,n)$ setting, providing a complete proof of this result.

2 Preliminaries

2.1 Quaternionic Linear Algebra

The content of this subsection is based on [4, 27, 28]. In what follows, $\mathbb{H}$ will denote the **quaternions**. It is well known that every element $q \in \mathbb{H}$ can be expressed as

$$q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k},$$

where $q_0, q_1, q_2, q_3 \in \mathbb{R}$, and the following relations hold:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

The **conjugate** of q is the quaternion

$$q^* = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}.$$

Equipped with the norm $|q| = \sqrt{qq^*}$, $\mathbb{H}^n$ becomes a normed $\mathbb{H}$-module.

A **quaternionic linear subspace** of $\mathbb{H}^n$ is defined as follows:

Definition 2.1 A subset $W \subseteq \mathbb{H}^n$ is called a **quaternionic linear subspace** if it satisfies the following conditions:

(1) W contains the zero vector, i.e., $\mathbf{0} \in W$.
(2) For any $\mathbf{w}_1, \mathbf{w}_2 \in W$, their sum satisfies $\mathbf{w}_1 + \mathbf{w}_2 \in W$.
(3) For any $\mathbf{w} \in W$ and any quaternion $q \in \mathbb{H}$, the scalar multiple $\mathbf{w}q \in W$.

To define the dimension of a quaternionic linear subspace, we introduce the following preliminary concepts:

(1) A set of vectors $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ in W is **linearly independent** if the only solution to

$$\mathbf{w}_1 q_1 + \mathbf{w}_2 q_2 + \cdots + \mathbf{w}_k q_k = \mathbf{0}$$

(where $q_i \in \mathbb{H}$) is $q_1 = q_2 = \cdots = q_k = 0$.
(2) A **basis** for W is a linearly independent set that spans W under right scalar multiplication. The **dimension** of W, denoted $\dim_{\mathbb{H}} W$, is the number of vectors in such a basis.

Given two quaternionic vector spaces V and W, a function $T : V \to W$ is called a **quaternionic linear transformation** if, for all vectors $\mathbf{u}, \mathbf{v} \in V$ and all scalars $p, q \in \mathbb{H}$, it satisfies

$$T(\mathbf{u}p + \mathbf{v}q) = T(\mathbf{u})p + T(\mathbf{v})q.$$

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As in the real or complex case, if $V = \mathbb{H}^n$ and $W = \mathbb{H}^m$, a right $\mathbb{H}$-linear transformation T can be represented by an $m \times n$ quaternionic matrix A acting on column vectors via right multiplication:

$$T(\mathbf{v}) = A\mathbf{v}.$$

Unlike complex linear maps, quaternionic linear maps do not naturally form a quaternionic vector space due to non-commutativity. Instead, they constitute a real vector space. We introduce the following notation:

- $\mathrm{M}(n, \mathbb{H})$ denotes the space of $\mathbb{H}$-linear transformations from $\mathbb{H}^n$ to $\mathbb{H}^n$.
- $\mathrm{GL}(n, \mathbb{H})$ denotes the Lie group of $\mathbb{H}$-linear isomorphisms of $\mathbb{H}^n$.

A **quaternionic Hermitian matrix** is a square matrix A with quaternion entries that satisfies $A = A^*$, where A^* is the **quaternionic conjugate transpose** of A. A Hermitian matrix $A \in \mathrm{M}(n, \mathbb{H})$ is **positive semi-definite** if, for every $v \in \mathbb{H}^n \setminus \{0\}$,

$$v(Av)^* \in \mathbb{R}^+.$$

Theorem 2.1 *(See Proposition 5.2 and Corollary 5.3 in [15])* *Let* $A \in \mathrm{M}(n, \mathbb{H})$. *Then:*

(1) The matrix A^*A *is positive semi-definite.*

(2) If $A \in \mathrm{M}(n, \mathbb{H})$ *is a positive semi-definite invertible matrix, then there exists a unitary matrix* U *(i.e.,* $UU^* = U^*U = \mathrm{Id}_n$*) such that*

$$U^*AU = D$$

is a diagonal matrix with positive real entries.

2.2 Quaternionic Projective geometry

2.2.1 Quaternionic Projective Space and Subspaces

The content of this subsection is based on [10].The **quaternionic projective space** $\mathbb{HP}^n$ is defined as:

$$\mathbb{HP}^n = (\mathbb{H}^{n+1} \setminus \{0\})/\mathbb{H}^*,$$

where $\mathbb{H}^*$ acts by right scalar multiplication. If $[\cdot]\colon \mathbb{H}^{n+1} \setminus \{0\} \to \mathbb{HP}^n$ is the quotient map, a non-empty subset $\mathcal{L} \subset \mathbb{HP}^n$ is called a **quaternionic projective subspace** of dimension k if there exists an $\mathbb{H}$-linear subspace $\widetilde{\mathcal{L}} \subset \mathbb{H}^{n+1}$ of dimension $k+1$ such that

$$[\widetilde{\mathcal{L}} \setminus \{0\}] = \mathcal{L}.$$

Quaternionic projective subspaces of dimension $n-1$ are called **hyperplanes**, and those of dimension 1 are called **quaternionic lines**.

As in the classical case, $e_1, \ldots, e_{n+1}$ will denote the standard basis of $\mathbb{H}^{n+1}$. Now, for a set of points $P \subset \mathbb{HP}^n$, we define:

$$\langle\langle P \rangle\rangle = \bigcap \{\mathcal{L} \subset \mathbb{HP}^n \mid \mathcal{L} \text{ is a quaternionic projective subspace containing } P\}.$$

By definition, $\langle\langle P \rangle\rangle$ is itself a quaternionic projective subspace.

The **Grassmannian of quaternionic lines**, $\mathrm{Gr}(1, n)$, is the space of all 1-dimensional quaternionic projective subspaces of $\mathbb{HP}^n$, endowed with the Hausdorff

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topology. It can be embedded into the quaternionic projective space $\mathbb{P}(\bigwedge^2 \mathbb{H}^{n+1})$ via the **Plücker embedding**:

$$\iota\colon \mathrm{Gr}(1,n) \to \mathbb{P}\left(\bigwedge^2 \mathbb{H}^{n+1}\right), \quad \iota(V) = [v_1 \wedge v_2],$$

where $\langle\langle v_1, v_2\rangle\rangle = V$.

2.2.2 Projective and Pseudo-projective Transformations

A **quaternionic projective transformation g** is an invertible map on $\mathbb{HP}^n$ of the form

$$\mathbf{g}[w] = [g(w)],$$

where g is an $\mathbb{H}$-linear isomorphism of $\mathbb{H}^{n+1}$. Here, g is called a **lift** of **g**. This definition is well-defined because the matrix group acts on the left while the projection acts on the right.

The induced homomorphism $[[\,\cdot\,]]\colon \mathrm{GL}(n+1,\mathbb{H}) \longrightarrow \mathrm{Aut}(\mathbb{HP}^n)$ is surjective, with kernel consisting of non-zero real scalar multiples of the identity. Thus, the group of quaternionic projective transformations of $\mathbb{HP}^n$ is isomorphic to:

$$\mathrm{PSL}(n+1,\mathbb{H}) := \mathrm{GL}(n+1,\mathbb{H})/\{\alpha \mathrm{Id} \mid \alpha \in \mathbb{R}^*\}.$$

For a non-zero linear transformation $M\colon \mathbb{H}^{n+1} \longrightarrow \mathbb{H}^{n+1}$, let $\mathrm{Ker}(M)$ denote its kernel and $\mathrm{Ker}([[M]])$ its projectivization. Then M induces a well-defined map:

$$[[M]]\colon \mathbb{HP}^n \setminus \mathrm{Ker}([[M]]) \to \mathbb{HP}^n, \quad [[M]]([v]) = [M(v)],$$

called a **quaternionic pseudo-projective transformation**. The space of such transformations forms a **compactification** of $\mathrm{PGL}(n+1,\mathbb{H})$, as shown below.

Lemma 2.2 *(Compactness Properties)*

(1) Given a sequence $(g_m) \subset \mathrm{PGL}(n+1,\mathbb{H})$, there exists a subsequence (h_m) and a pseudo-projective transformation h such that $h_m \longrightarrow h$ uniformly on compact subsets of $\mathbb{HP}^n \setminus \mathrm{Ker}(h)$.

(2) Let $(h_m) \subset \mathrm{GL}(n+1,\mathbb{H})$ be a sequence converging pointwise to $h \neq 0$. Then $([[h_m]])$ converges to $[[h]]$.

Proof For each $m \in \mathbb{N}$, let $g_m \in \mathrm{GL}(n+1,\mathbb{H})$ be a lift of $\mathbf{g}_m$. Define

$$|g_m| = \max\{|g_{ij}^{(m)}| \mid i,j = 1,\ldots,n+1\}.$$

Since $|g_m|^{-1}g_m$ is also a lift of $\mathbf{g}_m$ and bounded sequences in $\mathrm{M}(n+1,\mathbb{H})$ admit convergent subsequences, we may assume that $(|g_m|^{-1}g_m)$ converges to a non-zero matrix $g \in \mathrm{M}(n+1,\mathbb{H})$. Thus,

$$\lim_{m\to\infty} |g_m|^{-1}g_m = g$$

in the compact-open topology.

Now, let $K \subset \mathbb{HP}^n \setminus \mathrm{Ker}([[g]])$ be compact, and define

$$\mathbf{K} = \{k \in \mathbb{H}^{n+1} \mid [k] \in K\} \cap \mathbb{S}^n_{\mathbb{H}},$$

where $\mathbb{S}^n_{\mathbb{H}}$ is the unit sphere in $\mathbb{H}^{n+1}$. Then

$$\lim_{m\to\infty} g_m|_{\mathbf{K}} = g|_{\mathbf{K}}$$

in the compact-open topology, proving the claim. ∎

In the complex case, the space of pseudo-projective transformations has proven useful for precisely describing the equicontinuity region of families of projective transformations. For a detailed treatment, see [7].

2.3 Quaternionic Hyperbolic Groups

All the material in this subsection has been adapted from [20]. Consider $\mathbb{H}^{n,1}$, the quaternionic vector space of quaternionic dimension $n + 1$ endowed with the quaternionic Hermitian form:

$$\langle z, w\rangle = w^* H z,$$

where

$$H = \begin{pmatrix} & & 1 \\ & I_{n-1} & \\ 1 & & \end{pmatrix}.$$

We define the following subspaces of $\mathbb{H}^{n,1}$:

$$\begin{cases} V_- = \{z \in \mathbb{H}^{n,1} : \langle z, z\rangle < 0\}, \\ V_0 = \{z \in \mathbb{H}^{n,1} \setminus \{0\} : \langle z, z\rangle = 0\}, \\ V_+ = \{z \in \mathbb{H}^{n,1} : \langle z, z\rangle > 0\}. \end{cases}$$

The **quaternionic hyperbolic** n**-space** is defined as $\mathbf{H}^n_{\mathbb{H}} = [V_-] \subset \mathbb{HP}^n$. This space is projectively equivalent to the quaternionic unitary ball, and its boundary is $[V_0]$. In analogy with the complex case, we call this model the **Siegel domain**.

As in the real and complex cases, given a projective subspace $P \subset \mathbb{HP}^n$, we define its orthogonal complement as:

$$P^\perp = \{[w] \in \mathbb{HP}^n \mid \langle w, v\rangle = 0 \text{ for all } [v] \in P\}.$$

For a point $p \in \mathbb{HP}^n$, the orthogonal complement $p^\perp$ is a quaternionic hyperplane that is:

- osculating to $\partial\mathbf{H}^n_{\mathbb{H}}$ at p if $p \in [V_0]$,
- secant to $\mathbf{H}^n_{\mathbb{H}}$ if $p \in [V_-]$,
- disjoint from $\mathbf{H}^n_{\mathbb{H}}$ if $p \in [V_+]$.

The metric on $\mathbf{H}^n_{\mathbb{H}}$ is given by the distance function ρ, where:

$$\cosh^2\left(\frac{\rho([z], [w])}{2}\right) = \frac{\langle z, w\rangle\langle w, z\rangle}{\langle z, z\rangle\langle w, w\rangle}.$$

The quaternionic indefinite groups of signature $(1, n)$, in symbols Sp$(1, n)$, is the subgroup of GL$(n + 1, \mathbb{H})$ that preserves the Hermitian form $\langle z, w\rangle$ under left action.

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Thus, $g \in \mathrm{Sp}(1,n)$ satisfies:

$$g^* H g = H.$$

We define $\mathrm{PSp}(1,n)$ as the quotient of $\mathrm{Sp}(1,n)$ by its center, which consists of real scalar matrices. Since $\det(g)$ has unit norm, the only such scalars are ± 1, so:

$$\mathrm{PSp}(1,n) = \mathrm{Sp}(1,n)/\{\pm I\}.$$

The group $\mathrm{PSp}(1,n)$ is the group of isometries of $\mathbf{H}^n_{\mathbb{H}}$. Following the complex case, we refer to its subgroups as **quaternionic hyperbolic groups**.

As in real and complex hyperbolic geometry, non-trivial elements of $\mathrm{PSp}(1,n)$ are classified as:

(1) **Loxodromic** if it fixes exactly two points in $\partial \mathbf{H}^n_{\mathbb{H}}$.
(2) **Parabolic** is it fixes exactly one point in $\partial \mathbf{H}^n_{\mathbb{H}}$.
(3) **Elliptic** if it fixes at least one point in $\mathbf{H}^n_{\mathbb{H}}$.

The following notion of a **limit set** for quaternionic hyperbolic groups was introduced by Chen and Greenberg [8]:

Definition 2.2 For a subgroup $G \subset \mathrm{PSp}(1,n)$, the **Chen-Greenberg limit set** $\Lambda_{CG}(G)$ is the set of accumulation points in $\partial \mathbf{H}^n_{\mathbb{H}}$ of the orbit Gx for any $x \in \mathbf{H}^n_{\mathbb{H}}$.

As with Fuchsian groups, $\Lambda_{CG}(G)$ is independent of the choice of x and contains either **1, 2, or infinitely many points**. A group G is called **non-elementary** if $\Lambda_{CG}(G)$ is infinite.

Finally, we define the **compact symplectic group** and its projectivization, which will be used throughout the text:

$$\begin{aligned} \mathrm{Sp}(n) &= \{g \in \mathrm{GL}(n,\mathbb{H}) \mid gg^* = I_n\}, \\ \mathrm{PSp}(n) &= \{[g] \in \mathrm{PGL}(n,\mathbb{H}) \mid gg^* = I_n\}. \end{aligned}$$

3 Cartan's Decomposition

In this section, we present a geometric proof of the Cartan decomposition theorem for $\mathrm{Sp}(1,n)$. Succinctly, this decomposition is achieved by first changing from the Siegel domain model to the unitary quaternionic ball $\mathbb{B}^n_{\mathbb{H}}$. Then, we construct an operator $S(z)$ that maps the origin 0 to a point $z \in \mathbb{B}^n_{\mathbb{H}}$. Now, by setting $h = g\,S(z)^{-1}$, we ensure that h fixes 0 and thus belongs to $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$. Next, by taking the logarithm of $S(z)$ and diagonalizing it, we rewrite the hyperbolic component in exponential form $S(z) = \exp(X)k$, where X is in the Lie algebra of hyperbolic translations and $k \in \mathrm{Sp}(n)\times\mathrm{Sp}(1)$, thereby clearly separating the hyperbolic translation from the compact rotations.

Consider the quadratic form on $\mathbb{H}^{n,1}$ of signature $(1,n)$ induced by the matrix

$$Q = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix}.$$

We define

$$\mathrm{Sp}_Q(1,n) = \{g \in \mathrm{GL}(n,\mathbb{H}) \mid gQg^* = Q\}.$$

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The groups $\mathrm{Sp}_Q(1, n)$ and $\mathrm{Sp}(1, n)$ are conjugate via the matrix

$$\begin{pmatrix} 2^{-1/2} & 2^{-1/2} & 0 \\ 0 & 0 & I_{n-1} \\ -2^{-1/2} & 2^{-1/2} & 0 \end{pmatrix}.$$

For simplicity in calculations, we present Cartan's Decomposition Theorem for $\mathrm{Sp}_Q(1, n)$. In what follows, we consider the following model of quaternionic hyperbolic space:

$$\mathbb{B}^n_{\mathbb{H}} = \{z \in \mathbb{H}^n \mid z^* z < 1\}.$$

Note that for $g \in \mathrm{Sp}_Q(1, n)$, the relation $gQg^* = Q$ implies g can be written in block form as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a \in \mathbb{H}$, $b \in \mathbb{H}^n$, c is an n-dimensional row vector, and $d \in M(n, \mathbb{H})$ satisfy, *c.f.* [20]:

$$\begin{aligned} aa^* - bb^* &= 1 = a^*a - c^*c, \\ -ac^* + bd^* &= 0 = -a^*b + c^*d, \\ -cc^* + dd^* &= I_n = -b^*b + d^*d. \end{aligned}$$

***Theorem* 3.1** *Given* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_Q(1, n)$, *the map* $z \mapsto (a + zc)^{-1}(b + zd)$ *defines a right group action on* $\mathbb{B}^n_{\mathbb{H}}$.

Proof For $g, h \in \mathrm{Sp}(1, n)$, direct substitution shows $((z)g)h = (z)(gh)$, confirming the action property. ∎

***Lemma* 3.2** *For* $z \in \mathbb{B}^n_{\mathbb{H}}$ *written as a row vector:*

(1) $I_n - z^*z \in \mathrm{GL}(n, \mathbb{H})$ *is a positive-definite quaternionic Hermitian matrix.*
(2) $(1 - zz^*)^{1/2} z = z(I_n - z^*z)^{1/2}$.
(3) $(1 - zz^*)^{1/4} z = z(I_n - z^*z)^{1/4}$.

Proof Positive definiteness follows from $v^*(I_n - z^*z)v = \|v\|^2 - \|zv\|^2 > 0$. The equalities hold due to the spectral properties of analytic functions. ∎

***Definition* 3.1** For $z \in \mathbb{B}^n_{\mathbb{H}}$, define:

$$S(z) = \begin{pmatrix} (1 - zz^*)^{-1/2} & (1 - zz^*)^{-1/2} z \\ z^*(1 - zz^*)^{-1/2} & (I_n - z^*z)^{-1/2} \end{pmatrix}.$$

***Theorem* 3.3 (*Properties of* $S(z)$)** *The matrix* $S(z)$ *satisfies:*

(1) $S(z) \in \mathrm{Sp}_Q(1, n)$,
(2) $(0)S(z) = z$,
(3) $S(z)$ *is positive semi-definite.*

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Proof To prove (1), observe:

$$\begin{aligned}
-1 &= (1-zz^*)^{-1/2}(-1+zz^*)(1-zz^*)^{-1/2}\\
&= -(1-zz^*)^{-1} + (1-zz^*)^{-1/2} zz^* (1-zz^*)^{-1/2},\\
0 &= -(1-zz^*)^{-1} z + (1-zz^*)^{-1/2} z (I_n - z^* z)^{-1/2},\\
I_n &= -z^*(1-zz^*)^{-1} z + (I_n - z^* z)^{-1}.
\end{aligned}$$

Thus, $S(z)$ satisfies the defining relations of $\mathrm{Sp}_Q(1,n)$.

For (2), compute $(0)S(z) = (1-zz^*)^{1/2}(1-zz^*)^{-1/2} z = z$.

For (3), let $A = \begin{pmatrix} (1-zz^*)^{-1/4} & 0 \\ 0 & (I_n - z^* z)^{-1/4} \end{pmatrix}$. For $v \in \mathbb{B}^n_{\mathbb{H}} \setminus \{\mathbf{0}\}$, let $w = A^{-1} v$. Then:

$$v^* S(z) v = w^* \begin{pmatrix} (1-zz^*)^{1/2} & z \\ 0 & I_n \end{pmatrix} \begin{pmatrix} (1-zz^*)^{1/2} & 0 \\ z^* & I_n \end{pmatrix} w \geq 0.$$

■

Lemma 3.4 *If $g \in \mathrm{Sp}(1,n)$ fixes 0, then $g \in \mathrm{Sp}(n+1)$.*

Proof If $(0)g = 0$, then $c = 0$ in the block matrix, reducing g to $\mathrm{Sp}(n+1)$. ■

Theorem 3.5 *Every $g \in \mathrm{Sp}(1,n)$ decomposes uniquely as $g = hS(z)$ with $h \in \mathrm{Sp}(n+1)$.*

Proof Let $z = (0)g$. Then $h = gS(z)^{-1}$ fixes 0, hence $h \in \mathrm{Sp}(n+1)$. ■

Theorem 3.6 *For each $z \in \mathbb{B}^n_{\mathbb{H}}$, there exists $\beta \in \mathbb{H}^n$ such that*

$$\ln(S(z)) = \begin{pmatrix} 0 & \beta^* \\ \beta & 0 \end{pmatrix}.$$

Moreover, if $\beta \neq 0$, there exists $A \in \mathrm{Sp}(n)$ such that:

$$S(z) = \begin{pmatrix} 1 & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} \cosh(|\beta|) & \sinh(|\beta|) & 0 \\ \sinh(|\beta|) & \cosh(|\beta|) & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

Proof Since $S(z)$ is positive semi-definite, diagonalize it as $S(z) = U^* D U$ with U unitary and D diagonal. Define $\ln(S(z)) = U^* \ln(D) U$. The identity:

$$Q(\ln(S(z)))^* Q = -\ln(S(z))$$

implies that $\ln(S(z))$ has the stated form. If $\beta \neq 0$, apply Gram-Schmidt (Theorem 4.3 in [15]) to obtain an orthonormal basis $\{v_0, \dots, v_{n-1}\}$ with $v_0 = \beta|\beta|^{-1}$. Let A have columns v_i; then $A \in \mathrm{Sp}(n)$ and:

$$\begin{pmatrix} 1 & 0 \\ 0 & A^{-1} \end{pmatrix} \ln(S(z)) \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} 0 & e_1^*|\beta| \\ e_1|\beta| & 0 \end{pmatrix}.$$

Exponentiating yields the decomposition. ■

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Proof of Theorem 1.1

From Theorem 3.6, every $g \in \mathrm{Sp}_Q(1,n)$ decomposes as:

$$g = k_1 \begin{pmatrix} \cosh(t) & \sinh(t) & 0 \\ \sinh(t) & \cosh(t) & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix} k_2,$$

where $k_1, k_2 \in \mathrm{Sp}(n+1) \cap \mathrm{Sp}_Q(1,n)$ and $t \geq 0$. Diagonalizing the hyperbolic component yields the exponential form $e^{A(g)}$.

4 Quaternionic hyperbolic groups

In this section, we prove Theorem 1.2. The proof relies on the Cartan decomposition theorem and the λ-lemma (see Lemma 4.2 below), which together provide rigorous control over the dynamics of quaternionic lines. To begin, we introduce the following definition, establishing a conceptual framework for controlling the orbits of compact sets.

***Definition* 4.1** (cf. [16]) Let $(g_m) \subset \mathrm{PSp}(1,n)$ be a sequence of distinct elements, and let $x \in \mathbb{HP}^n$. Then:

(1) We say that (g_m) **tends simply to infinity** if:

- The sequences of compact factors in the Cartan decomposition converge.
- $(\lambda(g_m))$ converges to infinity.

(2) For a sequence $(g_m) \subset \mathrm{PSp}(1,n)$ tending simply to infinity, we define:

$$\mathcal{D}_{(g_m)}(x) = \bigcup \{\text{accumulation points of } (g_m(x_m))\},$$

where the union is taken over all sequences (x_m) converging to x.

The following lemma is straightforward.

***Lemma* 4.1** *For $p \in \partial \mathbf{H}^n_{\mathbb{H}}$, define:*

- $\pi_\beta(p) = \{\ell \in Gr(1,n) : p \in \ell \subset p^\perp\}$.
- *The metric $d_p : \pi_\beta(p) \times \pi_\beta(p) \to \mathbb{R}_+$ by:*

$$d_p(\ell_1, \ell_2) = \arccos\left(\sqrt{\frac{\langle q_1, q_2\rangle\langle q_2, q_1\rangle}{\langle q_1, q_1\rangle\langle q_2, q_2\rangle}}\right),$$

where $q_1, q_2 \in \mathbb{H}^{n+1} - \{\mathbf{0}\}$ satisfy $\ell_i = \langle\langle p, [q_i]\rangle\rangle$.

Then, $(\pi_\beta(p), d_p)$ is a metric space isometric to $(\mathbb{HP}^{n-1}, d_{n-1})$, where d_{n-1} is the standard Fubini-Study metric.

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This lemma naturally induces a metric on concurrent lines lying in an osculating plane to $\partial\mathbf{H}^n_{\mathbb{H}}$. Observe that this metric reproduces the construction of both the Fubini metric and the intrinsic metric of quaternionic hyperbolic space.

In what follows, $\mathrm{Isom}(p,q)$ denotes the set of isometries from $(\pi_\beta(p), d_p)$ to $(\pi_\beta(q), d_q)$, where $p, q \in \mathbf{H}^n_{\mathbb{H}}$. The next lemma is crucial for this paper.

***Lemma* 4.2** (*λ-Lemma, cf. [16]*) *Let $G \subset \mathrm{PSp}(1,n)$ be a discrete group, and let $(g_m) \subset G$ be a sequence tending simply to infinity. Then, there exist quaternionic pseudo-projective transformations g_+ and g_- such that:*

(1) $g_m^{\pm 1}$ converges to $g_\pm$.
(2) The sets $Im(g_+)$ and $Im(g_-)$ are points in the Chen-Greenberg limit set of G. Moreover:

- *$Im(g_+)^\perp = Ker(g_-)$,*
- *$Im(g_-)^\perp = Ker(g_+)$.*

Additionally, there exists a projective equivalence $\phi \in \mathrm{Isom}(Im(g_-), Im(g_+))$ such that for $\ell \in \pi_\beta(Im(g_\mp))$ and $w \in \ell - Im(g_\mp)$, we have:

$$\mathcal{D}_{(g_m^{\pm 1})}(w) = \phi^{\pm 1}(\ell).$$

Proof We first prove part (1). By Cartan's decomposition theorem, there exist sequences $(\alpha_m) \in \mathbb{R}_+$, (κ_m^+), $(\kappa_m^-) \subset K = \mathrm{Sp}(n+1) \cap \mathrm{PSp}(1,n)$ such that:

$$g_m = \left[\kappa_m^+ \begin{pmatrix} e^{\alpha_m} & & \\ & I_{n-2} & \\ & & e^{-\alpha_m} \end{pmatrix} \kappa_m^-\right].$$

Assume $\kappa_m^\pm$ converges to $\kappa_\pm \in K$. Then:

$$\lim_{m\to\infty} g_m = g_+ = \left[\kappa_+ \begin{pmatrix} 1 & \\ & \mathbf{0} \end{pmatrix} \kappa_-\right], \quad \lim_{m\to\infty} g_m^{-1} = g_- = \left[\kappa_-^{-1} \begin{pmatrix} \mathbf{0} & \\ & 1 \end{pmatrix} \kappa_+^{-1}\right].$$

Consequently:

- $Im(g_+) = \kappa_+([e_1])$, $Im(g_-) = \kappa_-^{-1}([e_{n+1}]) \in \Lambda_{CG}(G)$,
- $Ker(g_+) = Im(g_-)^\perp$,
- $Ker(g_-) = Im(g_+)^\perp$.

Now, define the isometry $\phi : \pi_\beta(Im(g_-)) \longrightarrow \pi_\beta(Im(g_+))$ by $\phi(\ell) = \kappa_+(H(\kappa_-(\ell)))$.

For $\ell \in \pi_\beta(Im(g_-))$, $w \in \ell - Im(g_-)$, and $(w_m) \subset \mathbb{HP}^n$ converging to w, we have $[\kappa_-(w_m)] \to [\kappa_-(w)] \in \langle\langle [e_2], \ldots, [e_{n+1}]\rangle\rangle - \{[e_{n+1}]\}$.

Assuming $[\kappa_-(w_m)] = [w_{1m}, \ldots, w_{n+1,m}]$ and $[\kappa_-(w)] = [0, w_2, \ldots, w_{n+1}]$ with $\sum_{j=2}^n |w_j| \neq 0$, we compute:

$$\begin{pmatrix} e^{\alpha_m} & & \\ & I_{n-2} & \\ & & e^{-\alpha_m} \end{pmatrix} \begin{pmatrix} w_{1m} \\ v_m \\ w_{n+1,m} \end{pmatrix} = \begin{pmatrix} e^{\alpha_m} w_{1m} \\ v_m \\ e^{-\alpha_m} w_{n+1,m} \end{pmatrix},$$

where $v_m = (v_{2m}, \ldots, v_{nm})$.

Thus, the accumulation points of $(g_m(w_m))$ lie on the line $\phi(\ell)$. The result follows from this computation. ■

In the following, $Im(g_+)$ (resp. $Im(g_-)$) will be called the **attracting point** (resp. **repelling point** associated with (g_m). The map ϕ wil be called the **transition map** associated to $Im(g_+)$, $Im(g_-)$ and (g_m).

The λ-lemma serves as a fundamental tool for analyzing the dynamics of compact sets, with particular emphasis on line dynamics. Its key dynamical implication can be stated as follows:

Proposition Given a discontinuity region Ω for a discrete group $G \subset \mathrm{PSp}(1, n)$ and a sequence of distinct elements $\{g_n\}_{n\in\mathbb{N}} \subset G$, then (after possibly passing to a subsequence) there exist supporting hyperplanes T_1 and T_2 tangent to $\partial\mathbf{H}^n_{\mathbb{H}}$ such that for any osculating line ℓ of $\partial\mathbf{H}^n_{\mathbb{H}}$ contained in T_1 with $\ell \not\subset \mathbb{HP}^n \setminus \Omega$, there exists a corresponding osculating line $\phi(\ell)$ at $\partial\mathbf{H}^n_{\mathbb{H}}$ satisfying

$$\phi(\ell) \subset T_2 \cap (\mathbb{HP}^n \setminus \Omega),$$

moreover, the map ϕ is induced by an element of $\mathrm{PSp}(n)$. ■

With this in mind, it is not difficult to convince oneself that if we could guarantee the existence of a sequence of distinct elements for which the hyperplanes T_1 and T_2 coincide and the transition map ϕ is the identity, then Theorem 1.2 would be proven. However, in order to achieve this, it is necessary to consider limiting processes in which there is no control over the transition maps. For this reason, it is necessary to extend the maps to global transformations that allow us to keep precise control over the dynamics of the lines. The following definitions are intended to capture these ideas:

***Definition* 4.2** Let $G \subset \mathrm{PSp}(1, n)$ be a discrete and non elementary group and $x, y \in \Lambda_{CG}(G)$, then:

$$G(y, x) = \{\gamma : y^\perp(y) \longrightarrow x^\perp(x) \mid \gamma \text{ is a transition map asociated to } x, y, (\gamma_m)\}$$

Now it is immediate that.

Corollary Given a discrete group $G \subset \mathrm{Psp}(1, n)$ and $x, y \in \Lambda_{CG}(G)$, we have that $G(y, x)$ is non-emtpty. ■

***Definition* 4.3** Let $G \subset \mathrm{PSp}(1, n)$ be a discrete group, $x, y \in \Lambda_{CG}(\Gamma)$, $\mu : y^\perp(y) \longrightarrow x^\perp(x)$ a transition map asociated to x, y, and (γ_m); and T a projective transformation of $P(\bigwedge^2 \mathbb{H}^{n+1})$. We say that T is a Cartan extension for μ with respect to (γ_m). If

(1) The Cartan decomposition of γ_m is $k_m^+ A_m k_m^-$.
(2) We have $k_m^\pm \xrightarrow[m\to\infty]{} k^\pm$, where $k^+, k^- \in K = \mathrm{PSp}(n+1) \cap \mathrm{PSp}(1, n)$.
(3) Also $T = (k^+Hk^-) \wedge (k^+Hk^-)$.

The proof of the following result is straightforward, so we omit it here.

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***Lemma* 4.5** *Let $G \subset \mathrm{PSp}(1,n)$ be a discrete group, and let:*

- *$x_+, x_- \in \Lambda_{CG}(G)$ be the attracting and repelling points, respectively,*
- *ϕ be the transition map associated with x, y and $(g_m) \subset G$.*
- *T be the Cartan extension of ϕ with respect to $(g_m) \subset G$.*

Then, $T(\ell) = \phi(\ell)$ for each $\ell \in \pi_\beta(x_-)$.

***Definition* 4.4** Let $G \subset \mathrm{PSp}(1,n)$ be a non-elementary discrete subgroup, $x^+ \in \Lambda_{CG}(G)$, and $\phi \in G(x^+, x^+)$. We say ϕ is a **limit transition map** if:

(1) There exists a sequence $(x_m^-) \subset \Lambda_{CG}(G)$ of distinct elements converging to x^+.
(2) For each $m \in \mathbb{N}$, there is a sequence $(g_{mj})_j \subset G$ tending simply to infinity with:

- Attracting point x^+,
- Repelling point x_m^-.

(3) $\lambda(g_{mj}) \longrightarrow \infty$ as $|(m,j)| \longrightarrow \infty$.
(4) There exist sequences $(\kappa_{mj}^+)_j$, $(\kappa_{mj}^-)_j$, $(\kappa_m^+)_m$, $(\kappa_m^-)_m \subset \mathrm{PSp}(1,n) \cap \mathrm{PSp}(n+1)$ such that:

- The Cartan decomposition of g_{mj} is $\kappa_{mj}^+ \mu(g_{mj}) \kappa_{mj}^-$.
- $\kappa_{mj}^\pm \longrightarrow \kappa_m^\pm$ as $m \longrightarrow \infty$.
- $(\kappa_m^\pm)$ converges to $\kappa^\pm$.

(5) For each $\ell \in \pi_\beta(x^+)$, $\phi(\ell) = (\kappa^+ \wedge \kappa^+)(H \wedge H)(\kappa^- \wedge \kappa^-)(\ell)$.

***Theorem* 4.6** *Let $G \subset \mathrm{PSp}(1,n)$ be a non-elementary discrete group and $x^+ \in \Lambda_{CG}(G)$. Then:*

$$G(x^+) = \{\phi : \pi_\beta(x^+) \longrightarrow \pi_\beta(x^+) \mid \phi \textit{ is a limit transition map}\}$$

is a non-empty closed Lie subgroup of $\mathrm{Isom}(x^+, x^+)$.

Proof We establish the desired properties of $G(x^+)$ through four sequential steps:

Stage 1: Non-emptiness of $G(x^+)$.
Let $(x_m^-)_{m\in\mathbb{N}} \subset \Lambda_{CG}(G)$ be a sequence of distinct points converging to x^+. By Corollary 4.4, for each m, there exists a transition map $\phi_m \in G(x_m^-, x^+)$. For every m, select a sequence $(g_{mj})_{j\in\mathbb{N}} \subset G$ tending *simply to infinity* (i.e., escaping compact subsets) and associated with ϕ_m. Using the Cartan decomposition, write

$$g_{mj} = \kappa_{mj}^+ \lambda(g_{mj}) \kappa_{mj}^-, \quad \text{where } d_\infty(\kappa_{mj}^\pm, \kappa_m^\pm) < 2^{-j-m}.$$

By compactness of $\mathrm{PSp}(n+1)$, there exist limiting elements $\kappa^+, \kappa^- \in \mathrm{PSp}(n+1) \cap \mathrm{PSp}(1,n)$ such that $d(\kappa_m^\pm, \kappa^\pm) < 2^{-m}$ for all m. The diagonal subsequence $(g_{mm})_m$ tends simply to infinity, and its transition map—being a limit of the ϕ_m—lies in $G(x^+)$. Hence, $G(x^+) \neq \emptyset$.

Stage 2: Closedness in $\mathrm{Isom}(x^+, x^+)$.
Let $(\phi_m)_m \subset G(x^+)$ converge uniformly to some $\phi \in \mathrm{Isom}(x^+, x^+)$. For each m, choose sequences $(g_{mjl})_{l\in\mathbb{N}} \subset G$ and $(\kappa_{mjl}^\pm)_{l\in\mathbb{N}}$ associated with ϕ_m as in Definition 4.4. By

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Lemma 4.5, the action of ϕ_m on any line ℓ is given by

$$\phi_m(\ell) = (\kappa_m^+ \wedge \kappa_m^+)(H \wedge H)(\kappa_m^- \wedge \kappa_m^-)(\ell),$$

where H is the scaling component of the Cartan decomposition. As $m \longrightarrow \infty$, the convergence $\kappa_m^\pm \to \kappa^\pm$ ensures

$$\phi_m(\ell) \longrightarrow (\kappa^+ \wedge \kappa^+)(H \wedge H)(\kappa^- \wedge \kappa^-)(\ell) = \phi(\ell).$$

Thus, $\phi \in G(x^+)$, proving $G(x^+)$ is closed.

Stage 3: Monoid Structure Under Composition.
Let $\phi_1, \phi_2 \in G(x^+)$. For each j, construct sequences $(g_{jm1})_m$ and $(g_{jm2})_m$ associated with ϕ_1 and ϕ_2, respectively. Define the composed sequence $h_{jm} := g_{jm1}g_{jm2}$, which also tends simply to infinity. The transition map ν of (h_{jm}) satisfies $\nu = \phi_1 \circ \phi_2$. Since $\nu \in G(x^+)$, the set $G(x^+)$ is closed under composition, forming a monoid.

Stage 4: Lie Group Structure .
To show $G(x^+)$ is a group, observe that for any $a \in G(x^+)$, the closure $\overline{\langle a \rangle}$ of the cyclic semigroup generated by a is a compact Abelian semigroup in $\mathrm{Isom}(x^+, x^+) \cong \mathrm{PSp}(n-1)$. By Green's theorem [17], every compact Abelian semigroup is a group; hence, $\overline{\langle a \rangle}$ is a compact Abelian Lie group. In particular, $a^{-1} \in G(x^+)$. Since $G(x^+)$ is a closed subgroup of the Lie group $\mathrm{Isom}(x^+, x^+)$, Cartan's closed subgroup theorem ensures $G(x^+)$ is itself a Lie group.

This concludes the proof. ∎

Proof of Theorem 1.2

Assume for contradiction that there exists $y \in \Lambda_{CG}(G)$ and $w \in \Omega(G)$. Let Id_y be the identity of $\mathrm{Isom}(y, y)$. By Lemma 4.6, there exists a sequence $(g_m) \subset G$ tending simply to infinity associated with Id_y. Then, $\langle\langle w, y \rangle\rangle = \mathcal{D}_{(\gamma_m)}(w) \subset \mathbb{HP}^n - \Omega(G)$, a contradiction.

5 Uniformization Theorems

A **quaternionic hyperbolic manifold** is defined as the quotient space $\mathbf{H}^n_{\mathbb{H}}/\Gamma$, where Γ is a discrete subgroup of $\mathrm{PSp}(1, n)$. For finite-volume manifolds endowed with quaternionic hyperbolic structures, Mok and Yeung [25] and Klingler [21] independently established the following rigidity theorem:

***Theorem* 5.1** *([21, 25]) For any discrete subgroup $G \subset \mathrm{PSp}(1, n)$ with $\mathbf{H}^n_{\mathbb{H}}/G$ of finite volume, the quotient manifold admits a unique projective structure compatible with its natural quaternionic structure.*

We now proceed to prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3 (Algebraic Version)

Let $M = \mathbf{H}^n_{\mathbb{H}}/G$ be a compact manifold and assume, by way of contradiction, that:

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(1) There exists a discrete, purely loxodromic free subgroup $H \subset \mathrm{PGL}(n+1, \mathbb{H})$, and
(2) There exists a G-invariant open set $U \subset \mathbb{HP}^n$ with $M \cong U/H$.

Theorem 5.1 implies that G and H are conjugate. Examining the Cayley graph $\Delta(G)$, we observe:

- $\Delta(G)$ is quasi-isometric to $\mathbf{H}^n_{\mathbb{H}}$ by [13], and
- Their Gromov boundaries, see [13], are consequently homeomorphic.

This leads to a contradiction because:

- For free groups, $\partial\Delta(G)$ is a Cantor set, while
- $\partial\mathbf{H}^n_{\mathbb{H}}$ is a topological sphere.

The geometric version follows immediately from these arguments. □

Proof of Theorem 1.4

We argue by contradiction. Let $M = \mathbf{H}^n_{\mathbb{H}}/G$ be a compact quaternionic manifold with $G \subset \mathrm{PSp}(n, 1)$ discrete. Suppose the double $\widetilde{M} = M \sqcup M$ admits a realization as U/H, where:

- $H \subset \mathrm{PGL}(n+1, \mathbb{H})$ is a transformation group, and
- $U \subset \mathbb{HP}^n$ is an H-invariant domain.

Theorem 5.1 establishes that G and H are projectively conjugate. However, this contradicts Theorem 1.2, as $\mathbf{H}^n_{\mathbb{H}}$ is maximal among domains where G acts properly discontinuously. □

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