

## ON THE UNIQUENESS IN CAUCHY'S PROBLEM FOR ELLIPTIC SYSTEMS WITH DOUBLE CHARACTERISTICS

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*Dedicated to Professor Katuzi Ono on his 60th birthday*

1. We consider in the 2 dimensional space with the coordinate  $(x, y)$ . Let  $\Gamma$  be a segment of the  $y$ -axis containing the origin in its interior and let  $\Omega$  be a domain whose boundary contains  $\Gamma$ . We treat the solutions  $u_p$  ( $p = 1, \dots, m$ ) of the elliptic system

$$(1.1) \quad \frac{\partial u_p}{\partial x} + \sum_{q=1}^m a_{p,q}(x, y) \frac{\partial u_q}{\partial y} + \sum_{q=1}^m b_{p,q}(x, y) u_q = 0,$$

where  $a_{p,q} \in C^1(\bar{\Omega})$ ,  $b_{p,q} \in L^\infty(\bar{\Omega})$  and  $u_p \in C^1(\bar{\Omega})$ . The system (1.1) is written in the form

$$(1.2) \quad U_x + AU_y + BU = 0,$$

where  $U = (u_1, \dots, u_p)$ ,  $A = (a_{p,q})$  and  $B = (b_{p,q})$ . The characteristics of this system are said to have multiplicities not greater than two in  $\bar{\Omega}$ , if the following condition is satisfied: There is a non-singular matrix  $T$  whose elements belong to  $C^1(\bar{\Omega})$  such that the matrix

$$A' = T^{-1}AT$$

has the direct sum

$$A' = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_s \end{pmatrix}$$

of one- or two-rowed square blocks of the type

$$\alpha_k = (\lambda_k)$$

or

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$$\alpha_k = \begin{pmatrix} \lambda_k & \mu_k \\ 0 & \lambda_k \end{pmatrix},$$

respectively.

Douglis [3] showed in 1960 that if the characteristics of the system (1. 2) are complex and of multiplicities not greater than two in  $\bar{\Omega}$ , then any solution of (1. 2), which is zero on  $\Gamma$ , is identically zero in  $\Omega$ .

If the direct sum  $A'$  consists of only one-rowed blocks, that is, the characteristics are distinct, then this theorem was proved by Carleman [2]. On the other hand uniqueness for elliptic equations, in any number of dimensions, whose characteristics are at most double was shown by several mathematicians (see c.f. [5], [6], [10], [11], [12], [13],).

In this note we shall try to prove uniqueness in Cauchy's problem for the elliptic system (1. 2) under weaker assumptions. That is the following

**MAIN THEOREM.** *Assume that the characteristics of the system (1. 2) are complex (elliptic) and of multiplicities not greater than two in  $\bar{\Omega}$ . Then there is a positive constant  $\delta$  such that if the solutions  $u_p$  of (1. 2) are in  $C^1(\bar{\Omega})$  and satisfy*

$$u_p = o(\exp(-y^{-2\delta})) \quad (y \rightarrow 0, \quad p = 1, \dots, m)$$

along  $\Gamma$ , then  $u \equiv 0$  in  $\Omega$ .

For single elliptic equations of second order with real coefficients this theorem was proved in any dimension by Mergelyan [9], Landis [7] and Lavrentév [8]. When characteristics of (1. 2) are distinct, this statement was shown by the author [4]. Thus we proceed as in [4]. The method used in this note consists in establishing an energy integral estimates developed by Calderón [1] and Mizohata [10].

2. In this section we assume that  $\Omega$  is a domain which contains the origin 0. We consider in  $\Omega$  the first order elliptic system

$$u_{1x} + \lambda u_{1y} + \mu u_{2y} = f_1, \tag{2. 1}$$

$$u_{2x} + \lambda u_{2y} = f_2,$$

where  $u, \lambda \in C^1(\bar{\Omega})$ . We set

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix}.$$

Then the system (2. 1) is written in the form

$$(2. 2) \quad LU \equiv U_x + \Lambda U_y = F.$$

We put  $|U| = |u_1| + |u_2|$ . Let us prepare a mean value property for solutions of (2. 2).

**PROPOSITION 1.** *For solutions of (2. 2), it holds that if  $1 < p < 2$  and  $0 < R < 1$ ,*

$$(2. 3) \quad |U(0)| \leq CR^{(2/p)-2} \left\{ \left( \iint_{r \leq R} |U|^2 dx dy \right)^{1-\frac{1}{p}} + \left( \iint_{r \leq R} |F|^2 dx dy \right)^{1-\frac{1}{p}} \right\},$$

where  $C$  is a constant independent of  $R$  and depends only on the values of  $U$ ,  $\Lambda$  and  $F$  in  $\Omega$ .

*Proof.* We denote simply by  $C$  the constants independent of  $R$ . We take a  $C^\infty$  function such that

$$\phi(r) = \begin{cases} 1 & \text{in } r \leq R/2 \\ 0 & \text{in } r > R \end{cases}$$

and  $|\phi_x|, |\phi_y| \leq CR^{-1}$ . Set  $V = \phi U$ . Then we see

$$(2. 4) \quad LV = L\phi U + \phi F,$$

where

$$L\phi = \phi_x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \phi_y \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix}.$$

Let  $E(x, y)$  be the fundamental solutions of the following elliptic system with constant coefficients

$$L_0 = \frac{\partial}{\partial x} + \Lambda(0) \frac{\partial}{\partial y}.$$

It is well known that

$$(2. 5) \quad E(x, y) = O(r^{-1}) \quad (r \rightarrow 0, \quad r = (x^2 + y^2)^{1/2}).$$

Since  $V$  has compact carrier and  $V_x + \Lambda(0)V_y = LV + (\Lambda(0) - \Lambda) \cdot V_y$ , we have from the property of  $E$

$$(2. 6) \quad U(0) = \iint_{r \leq R} E(x, y) \{LV + (\Lambda(0) - \Lambda)V_y\} dx dy.$$

We see by (2.4) and (2.5)

$$(2.7) \quad \left| \iint_{r \leq R} E(x, y) LV \, dx dy \right| \\ \leq CR^{-1} \iint_{r \leq R} r^{-1} (|U| + |F|) dx dy.$$

By Green's formula we have

$$\iint_{r \leq R} E(x, y) (\Lambda(0) - \Lambda) V_y \, dx dy \\ = - \iint_{r \leq R} E(x, y) (\Lambda(0) - \Lambda)_y V \, dx dy \\ - \iint_{r \leq R} E_y(x, y) (\Lambda(0) - \Lambda) V \, dx dy.$$

Hence we get from (2.5)

$$(2.8) \quad \left| \iint_{r \leq R} E(x, y) (\Lambda(0) - \Lambda) V_y \, dx dy \right| \\ \leq C \iint_{r \leq R} r^{-1} |U| \, dx dy.$$

Combining (2.6), (2.7) and (2.8), we obtain

$$(2.9) \quad |U(0)| \leq C R^{-1} \iint_{r \leq R} r^{-1} (|U| + |F|) dx dy.$$

Put  $m = \max_{\Omega} |U|$ . Then we see by Hölder's inequality

$$\iint_{r \leq R} r^{-1} m^{-1} |U| \, dx dy \leq \left( \iint_{r \leq R} r^{-p} \, dx dy \right)^{1/p} \\ \left( \iint_{r \leq R} (m^{-1} |U|)^q \, dx dy \right)^{1/q},$$

where  $p^{-1} + q^{-1} = 1$ . Since  $q > 2$  and  $m^{-1} |U| \leq 1$ , it holds

$$\iint_{r \leq R} r^{-1} m^{-1} |U| \, dx dy \\ \leq C R^{(2-p)/p} \left( \iint_{r \leq R} (m^{-1} |U|)^2 \, dx dy \right)^{1/2}.$$

Thus we get

$$(2.10) \quad \iint_{r \leq R} r^{-1} |U| \, dx dy \\ \leq C R^{\frac{2}{p}-1} \left( \iint_{r \leq R} |U|^2 \, dx dy \right)^{1-\frac{1}{p}}.$$

Similarly we have

$$\iint_{r \leq R} r^{-1} |F| dx dy \leq C R^{\frac{2}{p}-1} \left( \iint_{r \leq R} |F|^2 dx dy \right)^{1-\frac{1}{p}}.$$

Combining (2. 9), (2. 10) and (2. 11) we have obtained the inequality (2. 3).

3. Let us denote by  $S_d$  an open disk with the center  $(d/2, 0)$  and with the radius  $d/2$ . We put  $\Omega_h = \{0 < x < h\} \cap S_1$ ,  $\Gamma_h = \{0 \leq x \leq h\} \cap \partial S_1$ ,  $l_h = \{x = h\} \cap S_1$  and  $\|U(x, \cdot)\|^2 = \int_{l_h} |U(x, y)|^2 dy$ . In this section we see how the local behavior of the solutions of (1. 2) are controlled by the Cauchy data. We shall apply the method developed by Mizohata [10].

LEMMA 1 ([4]). Let  $u \in C^1(\bar{\Omega}_a)$  and  $u = o(\exp(-r^{-2\delta-\epsilon}))$  ( $r \rightarrow 0$ ) along  $\Gamma_a$  for some positive numbers  $\delta, \epsilon$ . Then there is a function  $v$  such that

(3. 1)  $v \in C^0(\bar{\Omega}_a) \cap C^1(\bar{\Omega}_a - \{0\})$  and  $v = u$  on  $\Gamma_a$ ,

(3. 2)  $\|v(h)\|^2, \|v_x(h)\|^2$  and  $\|v_y(h)\|^2 = o(\exp(-h^{-\delta}))$  ( $h \rightarrow 0$ ).

The details of the proof are omitted (see [4]). Here we show only how the function is constructed. Let  $\varphi$  be a  $C^\infty$  function on real line such that

$$\int_{-\infty}^{\infty} \varphi(x) dx = 1 \text{ and the carrier of } \varphi \subset \{|x| \leq 1\}.$$

Then we define  $v$  in the form

$$v(x, y) = \begin{cases} u(x, \sqrt{x-x^2}) f_{k(x)/3}(y-\sqrt{x-x^2}) & \text{for } y \geq 0 \\ u(x, -\sqrt{x-x^2}) f_{k(x)/3}(y+\sqrt{x-x^2}) & \text{for } y < 0, \end{cases}$$

where  $k(x) = \exp(-x^{-\delta-(\epsilon/3)})$  and  $f_s(x) = \int_{-2s}^{2s} \varphi((x-y)/s) s^{-1} ds$ .

LEMMA 2 (Mizohata [10]). Let  $\lambda \in C^1(\bar{\Omega}_a)$  and  $w \in C^0(\bar{\Omega}_a) \cap C^1(\bar{\Omega}_a - \{0\})$ . We assume that the imaginary part of  $\lambda \neq 0$  in  $\bar{\Omega}_a$  and  $w = 0$  on  $\Gamma_a$  and  $\|w(\epsilon)\|, \|w_x(\epsilon)\|, \|w_y(\epsilon)\| \rightarrow 0$  ( $\epsilon \rightarrow 0$ ). Then there are positive constants  $h_0, n_0$  and  $c$  depending only on  $\lambda$  and  $w$  such that if  $0 < h < h_0, n > n_0$ , it holds

$$\int_0^h \varphi_n^2 \|w_x + \lambda w_y\|^2 dx + c \varphi_n^2(h) \geq \frac{1}{4n} \left( \int_0^h \varphi_n'^2 \|w\|^2 dx + \int_0^h \varphi_n^2 \|\lambda_2 w_y\|^2 dx \right),$$

where  $\varphi_n(x) = (x + n^{-1})^{-n}$  and  $\lambda = \lambda_1 + i\lambda_2$ .

The proof is omitted (see [10]).

Now we consider in  $\Omega_a$  the nonlinear elliptic system

$$(3.3) \quad \begin{aligned} u_{1x} + \lambda u_{1y} + \mu u_{2y} &= F_1(x, y, u_1, u_2), \\ u_{2x} + \lambda u_{2y} &= F_2(x, y, u_1, u_2), \end{aligned}$$

where  $\lambda, \mu \in C^1(\bar{\Omega}_a)$  and the imaginary part of  $\lambda \neq 0$  in  $\bar{\Omega}_a$ . We assume that

$$(3.4) \quad |F_i(x, y, u_1, u_2)| \leq C(|u_1| + |u_2|) \quad (i = 1, 2).$$

We prepare the following

**PROPOSITION 2.** *Let  $u_1, u_2$  be in  $C^1(\bar{\Omega}_a)$  and solutions of (3.3) in  $\Omega_a$ . If for some  $\varepsilon > 0, \delta > 1$ ,*

$$u_1, u_2 = o(\exp(-r^{-2\delta-\varepsilon})) \quad (r \rightarrow 0) \quad \text{on } \Gamma_a,$$

then we have

$$\int_0^h \|u_i\|^2 dx = o(\exp(-h^{-\delta})) \quad (h \rightarrow 0, i = 1, 2).$$

*Proof.* From Lemma 1 there are functions  $v_i$  ( $i = 1, 2$ ) such that

$$(3.5) \quad v_i \in C^0(\bar{\Omega}_a) \cap C^1(\bar{\Omega}_a - \{0\}) \text{ and } v_i = u_i \text{ on } \Gamma_a,$$

$$(3.6) \quad \|v_i(h)\|^2, \|v_{ix}(h)\|^2 \text{ and } \|v_{iy}(h)\|^2 \\ = o(\exp(-h^{-h-\frac{\varepsilon}{3}})) \quad (h \rightarrow 0).$$

We put  $w_i = u_i - v_i$ . Then the equations (3.3) are reduced to

$$\begin{aligned} w_{1x} + \lambda w_{1y} + \mu w_{2y} &= G_1(x, y, u_1, u_2, v_{1x}, v_{2x}, v_{2y}), \\ w_{2x} + \lambda w_{2y} &= G_2(x, y, u_1, u_2, v_{2x}, v_{2y}). \end{aligned}$$

We easily see

$$(3.7) \quad |G_i| \leq C(|u_1| + |u_2| + |v_{1x}| + |v_{2x}| + |v_{2y}|) \quad i = 1, 2.$$

From now on we denote simply by  $c$  the constants independent of  $n$  and  $h$ . We have by Lemma 2 for  $h < h_0, n > n_0$ ,

$$(3.8) \quad \int_0^h \varphi_n^2 \|G_1 - \mu w_{2y}\|^2 dx + c\varphi_n^2(h) \\ \geq \frac{1}{4n} \int_0^h \varphi_n^2 \|w_1\|^2 dx$$

and

$$(3.9) \quad \int_0^h \varphi_n^2 \|G_2\|^2 dx + c\varphi_n^2(h) \\ \cong \frac{1}{4n} \left( \int_0^h \varphi_n'^2 \|w_2\|^2 dx + \int_0^h \varphi_n^2 \|\lambda_2 w_{2y}\|^2 dx \right).$$

Multiplying both sides of (3.9) by  $4nM$  for large  $M$ , we add (3.9) to (3.8). Then it holds

$$\int_0^h \varphi_n^2 (2\|G_1\|^2 + 4nM\|G_2\|^2) dx + nc\varphi_n^2(h) \\ \cong M \int_0^h \varphi_n'^2 \|w_2\|^2 dx + \frac{1}{4n} \int_0^h \varphi_n'^2 \|w_1\|^2 dx \\ + (M - c) \int_0^h \varphi_n^2 \|\lambda_2 w_{2y}\|^2 dx.$$

Let us fix  $M$  such that  $M - c > 0$ . Then we obtain

$$nc \left\{ \int_0^h \varphi_n^2 (\|G_1\|^2 + \|G_2\|^2) dx + \varphi_n^2(h) \right\} \\ \cong \frac{1}{n} \int_0^h \varphi_n'^2 (\|w_1\|^2 + \|w_2\|^2) dx.$$

We substitute  $w_i = u_i - v_i$  and (3.7) into this inequality. Then we have

$$nc \left\{ \int_0^h \varphi_n^2 (\|u_1\|^2 + \|u_2\|^2 + \|v_{1x}\|^2 + \|v_{2x}\|^2 + \|v_{2y}\|^2) dx + \varphi_n^2(h) \right\} \\ + \frac{c}{n} \int_0^h \varphi_n'^2 (\|v_1\|^2 + \|v_2\|^2) dx \\ \cong \frac{1}{n} \int_0^h \varphi_n'^2 (\|u_1\|^2 + \|u_2\|^2) dx.$$

If  $h + \frac{1}{n}$  is sufficiently small, we see

$$nc \left\{ \int_0^h \varphi_n^2 (\|v_{1x}\|^2 + \|v_{2x}\|^2 + \|v_{2y}\|^2) dx + \varphi_n^2(h) \right\} \\ + \frac{c}{n} \int_0^h \varphi_n'^2 (\|v_1\|^2 + \|v_2\|^2) dx \\ \cong \frac{1}{n} \int_0^h \varphi_n'^2 (\|u_1\|^2 + \|u_2\|^2) dx.$$

Combining (3.6) and (3.10), we obtain

$$n^2 c \left( \frac{h}{2} + \frac{1}{n} \right)^{2n+2} \left\{ n^{2n} \exp(-h^{-\delta-(\varepsilon/3)}) + \left( h + \frac{1}{n} \right)^{-2n} \right\} \\ \cong \int_0^{h/2} (\|u_1\|^2 + \|u_2\|^2) dx.$$

Let us take  $h + \frac{1}{n}$  sufficiently small and  $nh$  sufficiently large. As an easy computation shows, in order to prove

$$(3.11) \quad \int_0^{h/2} \|u_i\|^2 dx = o\left(\exp\left(-\left(\frac{h}{2}\right)^{-\delta}\right)\right) \quad (h \rightarrow 0),$$

it is sufficient to show that we can choose  $n$  in such a way that

$$(3.12) \quad n^{2n+2} \leq \exp\left(\left(\frac{h}{2}\right)^{-\delta-\varepsilon_1}\right)$$

and

$$(3.13) \quad n^2 \left( \frac{hn+2}{2hn+2} \right)^{2n} \leq \exp\left(-\left(\frac{h}{2}\right)^{-\delta-\varepsilon'}\right) \quad \varepsilon_1, \varepsilon' > 0$$

where  $\varepsilon_1$  is a given number and  $\varepsilon'$  will be determined later. If there is a positive number  $\bar{\varepsilon}$  such that

$$(3.14) \quad n^{1+\bar{\varepsilon}} \leq \left(\frac{h}{2}\right)^{-\delta-\varepsilon_1},$$

then (3.12) holds. Since  $nh$  is sufficiently large, if we show that

$$(3.15) \quad \left(\frac{3}{5}\right)^{2n} \leq \exp\left(-\left(\frac{h}{2}\right)^{-\delta-\varepsilon'}\right),$$

then (3.13) holds. Let us take positive numbers  $\varepsilon'$ ,  $\bar{\varepsilon}$  such that

$$\delta + \varepsilon' < (\delta + \varepsilon_1)/1 + \bar{\varepsilon}.$$

Noting that  $1 < 2 \log\left(\frac{5}{3}\right)$ , we can take  $n$  in such a way that

$$(3.16) \quad \frac{1}{2 \log \frac{5}{3}} \left(\frac{2}{h}\right)^{\delta+\varepsilon} < n < \left(\frac{2}{h}\right)^{(\delta+\varepsilon_1)/(1+\bar{\varepsilon})}$$

It is easily seen that the inequality (3.16) implies (3.14), (3.15) and that  $nh \rightarrow \infty$ . Thus we have proved (3.11).

We consider the following elliptic system in  $\Omega_a$ .

$$(3.17) \quad U_x + AU_y = F(x, y, U),$$

where  $U = (u_1, \dots, u_m)$ ,  $F = (F_1, \dots, F_m)$  and

$$A = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_s \end{pmatrix}.$$

We assume that  $\alpha_k \in C^1(\bar{\Omega}_a)$  and  $\alpha_k$  is one- or two-rowed square blocks of the type

$$\alpha_k = (\lambda_k)$$

or

$$\alpha_k = \begin{pmatrix} \lambda_k & \mu_k \\ 0 & \lambda_k \end{pmatrix},$$

respectively. Further let us assume that

$$(3.18) \quad |F_k(x, y, u_1, \dots, u_m)| \leq C(|u_1| + \dots + |u_m|).$$

Then we can prove the following in a quite similar manner as in Proposition 2.

**COROLLARY 1.** *Let  $U$  be in  $C^1(\bar{\Omega}_a)$  and a solution of (3.17) in  $\Omega_a$ . If for some  $\varepsilon > 0$ ,  $\delta > 1$ ,*

$$U = o(\exp(-r^{-2\delta-\varepsilon})) \quad (r \rightarrow 0) \quad \text{on } \Gamma_a,$$

*then we have*

$$\int_0^h \|u_i\|^2 dx = o(\exp(-h^{-\delta})) \quad (h \rightarrow 0, \quad i = 1, \dots, m).$$

Now we can prove the following

**THEOREM 1.** *Let  $U$  be a solution of the elliptic system (3.17) in  $\Omega_a$  and  $U$  be in  $C^1(\bar{\Omega}_a)$ . Then if for some  $\varepsilon > 0$ ,  $\delta > 1$ ,*

$$U = o(\exp(-r^{-2\delta-\varepsilon})) \quad (r \rightarrow 0) \quad \text{on } \Gamma_a,$$

*we have*

$$U = o(\exp(-r^{-\delta})) \quad (r \rightarrow 0) \quad \text{in } S_{1/2} \cap \Omega_a.$$

*Proof.* We set  $-2\delta - \varepsilon = -2\left(\delta + \frac{\varepsilon}{3}\right) - \frac{\varepsilon}{3}$ . We regard  $\delta + \frac{\varepsilon}{3}$  as new  $\delta$  and  $\frac{\varepsilon}{3}$  as  $\varepsilon$  in Theorem 1. Then by Corollary 1 we have

$$(3.19) \quad \int_0^h \|u_i\|^2 dx = o\left(\exp\left(-h^{-\delta - \frac{\epsilon}{3}}\right)\right) \quad (h \rightarrow 0, \quad i = 1, \dots, m)$$

For the point  $(x^{(0)}, y^{(0)})$  in  $S_{1/2}$  we denote by  $r_1(x^{(0)}, y^{(0)})$  the radius of a circle tangent to  $S$  whose center is  $(x^{(0)}, y^{(0)})$ . It is easily seen that

$$(3.20) \quad r_1(x^{(0)}, y^{(0)}) \sim x^{(0)} \quad (x^{(0)} \rightarrow 0).$$

Let us apply Proposition 1 for the disk with center  $(x^{(0)}, y^{(0)})$  and with radius  $r_1(x^{(0)}, y^{(0)})$ . Putting  $p = 3/2$  in (2.3) we have for  $(x^{(0)}, y^{(0)}) \in S_{1/2}$

$$\begin{aligned} & |u_i(x^{(0)}, y^{(0)})| \\ & \leq Cr_1(x^{(0)}, y^{(0)})^{-2/3} \left\{ \left( \iint_{r_0 \leq r_1(x^{(0)}, y^{(0)})} \left( \sum_{i=1}^m |u_i|^2 \right) dx dy \right)^{1/3} \right. \\ & \quad \left. + \left( \iint_{r_0 \leq r_1(x^{(0)}, y^{(0)})} \left( \sum_{i=1}^m |F_i|^2 \right) dx dy \right)^{1/3} \right\}, \end{aligned}$$

where  $r_0 = \sqrt{(x - x^{(0)})^2 + (y - y^{(0)})^2}$  and  $C$  is a constant independent of  $(x^{(0)}, y^{(0)})$ . Combining (3.18) and (3.21), we see

$$\begin{aligned} & |u_i(x^{(0)}, y^{(0)})| \\ & \leq C r_1(x^{(0)}, y^{(0)})^{-2/3} \left\{ \int_0^{x_0 + r_1(x^{(0)}, y^{(0)})} \left( \sum_{i=1}^m |u_i|^2 \right) dx dy \right\}^{1/3}. \end{aligned}$$

By Corollary 1 and (3.20), we obtain

$$|u_i(x^{(0)}, y^{(0)})| \leq C x_0^{-2/3} \exp\left(-x_0^{-\delta - \frac{\epsilon}{3}}\right).$$

Thus we have proved the theorem.

4. We consider the next transformation from  $(x, y)$ -plane to  $(\theta, \rho)$ -plane as in [4].

$$(4.1) \quad \rho = r^2/x, \quad \theta = \tan^{-1}(y/x).$$

Put  $R_{1/2} = \{(\theta, \rho) \mid |\theta| < \pi/2, 0 < \rho < 1/2\}$ . We eliminate the part  $\rho = 1/2$  from the boundary of  $R_{1/2}$  and denote the remainder by  $\partial R_{1/2}$ . And let us put  $\bar{R}_d = R_d + \partial R_d$ . Then the transformation (4.1) maps  $S_{1/2}$  onto  $R_{1/2}$  in one-to-one way. And we see that this transformation and its inverse are  $C^\infty$ . we have

$$\begin{pmatrix} \theta_x & \theta_y \\ \rho_x & \rho_y \end{pmatrix} = \begin{pmatrix} -\frac{1}{\rho} \tan \theta & 1/\rho \\ 1 - \tan^2 \theta & 2 \tan \theta \end{pmatrix}.$$

For the function  $u(x, y)$  in  $\bar{S}_{1/2}$  we define a function  $\tilde{u}(\theta, \rho)$  in  $\bar{R}_{1/2}$  by

$$\tilde{u}(\theta, \rho) = \begin{cases} u(\theta, \rho) & \text{for } (\theta, \rho) \in R_{1/2} \\ u(0, 0) & \text{for } (\theta, \rho) \in \partial R_{1/2}. \end{cases}$$

From now on we denote  $\tilde{u}(\theta, \rho)$  simply by  $u$ . It is easily seen that if  $u(x, y) \in C^1(\bar{S}_{1/2})$  then  $u(\theta, \rho) \in C^1(\bar{R}_{1/2})$ .

We consider the next equation in  $S_{1/2}$

$$(4.3) \quad u_x + \lambda u_y \equiv H,$$

where  $u \in C^1(\bar{S}_{1/2})$ ,  $\lambda \in C^1(\bar{S}_{1/2})$  and the imaginary part of  $\lambda \neq 0$  in  $\bar{S}_{1/2}$ . We set  $f(\theta, \rho) = |\rho_x + \lambda \rho_y|^2 \cos^4 \theta$  and  $\lambda = \lambda_1 + i\lambda_2$ . Then by (4.2) the equation (4.3) is transformed into

$$(4.4) \quad u_\rho + \frac{1}{\rho} (Q + iP)u_\theta \equiv \tilde{H},$$

where

$$\begin{aligned} \tilde{H} &= f^{-1} \cos^4 \theta (\rho_x + \bar{\lambda} \rho_y) H, \\ P(\theta, \rho) &= \lambda_2 f^{-1} \cos^2 \theta, \end{aligned}$$

and

$$\begin{aligned} Q(\theta, \rho) &= f^{-1} \cos \theta \{ \sin \theta (\sin^2 \theta - \cos^2 \theta) \\ &\quad + \lambda_1 (\cos^3 \theta - 3 \cos \theta \sin^2 \theta) + 2|\lambda|^2 \cos^2 \theta \sin^2 \theta \}. \end{aligned}$$

LEMMA 3 ([4]). *The function  $f(\theta, \rho)$  is in  $C^1(\bar{R}_{1/2})$  and there is a positive constant  $m$  such that  $f > m$  in  $\bar{R}_{1/2}$ . And  $QP_\theta/P$  is continuous in  $\bar{R}_{1/2}$ .*

The proof is omitted (see [4]).

From now on we denote by  $\| \cdot \|$  a  $L^2$  norm with respect to  $\theta$  ( $|\theta| < \pi/2$ ). And we put

$$\phi_n(\rho) = \exp(n\rho^{-\delta}) \quad (\delta > 0, n > 0).$$

We denote  $\phi_n$  simply by  $\phi$ . Then we have

PROPOSITION 4 ([4]). *If  $u \in C^1(\bar{S}_{1/2})$  and satisfies for some positive numbers  $\delta, \varepsilon (\delta > 1)$ ,*

$$u = o(\exp(-r^{-\delta-\varepsilon})) \quad (r \rightarrow 0) \text{ in } S_{1/2},$$

*then we have*

$$\begin{aligned}
 & \int_0^h \phi^2 \|u_\rho + \frac{1}{\rho} (Q + iP) u_\theta\|^2 d\rho \\
 (4.6) \quad & \geq n\delta(\delta + 1 - M) \int_0^h \frac{\phi^2}{\rho^{\delta+2}} \|u\|^2 d\rho \\
 & - cn\delta \int_0^h \frac{\phi^2}{\rho^{\delta+1}} \|u\|^2 d\rho - \frac{c}{h} \phi^2(h) \\
 & + \frac{1}{2} \int_0^h \left\| \frac{1}{\rho} \phi iP u_\theta - \phi' u \right\|^2 d\rho,
 \end{aligned}$$

where  $M = \max_{R_{1/2}} \left| Q_\theta + 1 - \frac{P_\theta Q}{P} \right|$  and  $c$  is a constant independent of  $n, h$  and  $\delta$ .

Since the proposition was shown in detail in [4], we omit the proof.

We consider the elliptic system (3.3) in  $S_{1/2}$ . Then we have the following

PROPOSITION 5. *Let  $u_1, u_2$  be in  $C^1(\bar{S}_{1/2})$  and solutions of the elliptic system (3.3) in  $S_{1/2}$ . If it holds for  $\delta > \max(2, M - 1)$  ( $M$  is the constant in (4.6))*

$$u_i = o(\exp(-r^{-\delta})) \quad (r \rightarrow 0, i = 1, 2) \text{ in } S_{1/2},$$

*then  $u_i$  vanish identically in a neighborhood of the origin.*

*Proof.* We denote simply by  $c$  the positive constant independent of  $n$ . We assume  $u \neq 0$  in  $\rho < h$ . And we shall show that  $u = 0$  in  $\rho < h/2$ . Then we see by (4.6)

$$\begin{aligned}
 & c \left\{ \frac{\phi^2(h)}{h} + \int_0^h \phi^2 \left\| u + \frac{1}{\rho} (Q + iP) u_\theta \right\|^2 d\rho \right\} \\
 (4.8) \quad & \geq n \int_0^h \frac{\phi^2}{\rho^{\delta+2}} \|u\|^2 d\rho \\
 & + \int_0^h \left\| \frac{1}{\rho} \phi iP u_\theta - \phi' u \right\|^2 d\rho \\
 & \geq n \int_0^h \frac{\phi^2}{\rho^{\delta+2}} \|u\|^2 d\rho
 \end{aligned}$$

$$+ \int_0^h \|\rho^{\frac{\delta}{2}-1} \phi i P u_\theta - \rho^{\frac{\delta}{2}} \phi' u\|^2 d\rho.$$

We set

$$I_n^2 = \int_0^h \|\rho^{\frac{\delta}{2}} \phi' u\|^2 d\rho$$

$$p_n^2 I_n^2 = \int_0^h \|\rho^{\frac{\delta}{2}-1} \phi i P u_\theta\|^2 d\rho.$$

Then (4. 8) becomes

$$c \left\{ \frac{\phi^2(h)}{h} + \int_0^h \phi^2 \left\| u_\rho + \frac{1}{\rho} (Q + iP) u_\theta \right\|^2 d\rho \right\}$$

$$\cong (p_n - 1)^2 I_n^2 + \frac{1}{n} I_n^2.$$

Hence we have

$$(4. 9) \quad cn \left\{ \frac{\phi^2(h)}{h} + \int_0^h \phi^2 \left\| u_\rho + \frac{1}{\rho} (Q + iP) u_\theta \right\|^2 d\rho \right\}$$

$$\cong p_n^2 I_n^2 + I_n^2.$$

We consider the first equation of the elliptic system. That is

$$(4. 10) \quad u_{1x} + \lambda u_{1y} + \mu u_{2y} = F_1$$

Then this equation is transformed into

$$|\rho_x + \lambda \rho_y|^2 (\rho^{\frac{\delta}{2}} u_1)_\rho + (\theta_x + \lambda \theta_y) (\rho_x + \bar{\lambda} \rho_y) (\rho^{\frac{\delta}{2}} u_1)_\theta$$

$$= (\rho_x + \bar{\lambda} \rho_y) \rho^{\frac{\delta}{2}} F_1 - \mu (\rho_x + \bar{\lambda} \rho_y) \cdot$$

$$\rho^{\frac{\delta}{2}} (u_{2\rho} \rho_y + u_{2\theta} \theta_y) + \frac{\delta}{2} |\rho_x + \lambda \rho_y|^2 \cdot \rho^{\frac{\delta}{2}-1} u_1.$$

By Lemma 3 we see

$$|\rho_x + \lambda \rho_y|^2 = \cos^{-4} \theta f(\theta, \rho) \quad (f > m > 0 \text{ in } R_{1/2})$$

and

$$|\rho_x + \lambda \rho_y| \leq \text{const. } \cos^{-2} \theta.$$

Thus (4. 10) becomes

$$\begin{aligned}
& |(\rho^{\frac{\delta}{2}} u_1)_\rho + \frac{1}{\rho} (Q + iP)(\rho^{\frac{\delta}{2}} u_1)_\theta| \\
(4.11) \quad & \leq c\{|F_1| + |u_1| + \rho^{\frac{\delta}{2}} \cos^2\theta \cdot \\
& \quad \left(\frac{1}{\rho} |u_{2\theta}| + |\tan\theta| |u_{2\rho}|\right)\},
\end{aligned}$$

where  $P$  and  $Q$  are of the type (4.5). Let us regard  $\rho^{\frac{\delta}{2}} u_1$  in (4.11) as  $u$  in (4.9). Then we have from (4.9)

$$\begin{aligned}
(4.12) \quad & c \left\{ \frac{1}{h} \phi^2(h) + \int_0^h \phi^2 \|F_1\|^2 d\rho + \int_0^h \phi^2 \|u_1\|^2 d\rho \right. \\
& + \int_0^h \phi^2 \|\rho^{\frac{\delta}{2}-1} \cos^2\theta u_{2\theta}\|^2 d\rho \\
& \left. + \int_0^h \phi^2 \|\rho^{\frac{\delta}{2}} \cos\theta \sin\theta u_{2\rho}\|^2 d\rho \right\} \\
& \cong n \int_0^h \frac{\phi^2}{\rho^{\delta+2}} \|\rho^{\frac{\delta}{2}} u_1\|^2 d\rho.
\end{aligned}$$

Now we consider the second equation of the elliptic system. Then we have from (4.4) and (4.5)

$$(4.13) \quad |u_{2\rho}| \leq |F_2| + c\rho^{-1} \cos\theta |u_{2\theta}|.$$

By (4.12) and (4.13), we see

$$\begin{aligned}
(4.14) \quad & c \left\{ \frac{1}{h} \phi^2(h) + \int_0^h \phi^2 (\|F_1\|^2 + \|F_2\|^2) d\rho \right. \\
& \left. + \int_0^h \phi^2 \|\rho^{\frac{\delta}{2}-1} \cos^2\theta u_{2\theta}\|^2 d\rho \right. \\
& \left. \cong n \int_0^h \frac{\phi^2}{\rho^{\delta+2}} \|\rho^{\frac{\delta}{2}} u_1\|^2 d\rho. \right.
\end{aligned}$$

On the other hand we have from (3.3), (4.5) and (4.9)

$$\begin{aligned}
(4.15) \quad & cn \left\{ \frac{1}{h} \phi^2(h) + \int_0^h \phi^2 \|F_2\|^2 d\rho \right\} \\
& \cong \int_0^h \|\rho^{\frac{\delta}{2}-1} \phi iP u_{2\theta}\|^2 d\rho + \int_0^h \|\rho^{\frac{\delta}{2}} \phi' u_2\|^2 d\rho.
\end{aligned}$$

Let us note that  $|P| \geq c \cos^2\theta (c > 0)$ . Then (4.15) becomes

$$(4.16) \quad cn \left\{ \frac{1}{h} \phi^2(h) + \int_0^h \phi^2 \|F_2\|^2 d\rho \right\}$$

$$\geq \int_0^h \phi^2 \|\rho^{\frac{\delta}{2}-1} \cos^2 \theta u_{2\theta}\|^2 d\rho + n^2 \int_0^h \frac{\phi^2}{\rho^{\delta+2}} \|u_2\|^2 d\rho.$$

Multiplying both sides of (4. 16) by a large constant, we add (4. 16) to (4. 14). Then we obtain

$$(4. 17) \quad c \left\{ \frac{1}{h} \phi^2(h) + \int_0^h \phi^2 (\|F_1\|^2 + \|F_2\|^2) d\rho \right\} \\ \geq \int_0^h \frac{\phi^2}{\rho^2} (\|u_1\|^2 + \|u_2\|^2) d\rho.$$

Combining (3. 4) and (4. 17), we have for sufficiently small  $h$

$$\frac{c}{h} \phi^2(h) \geq \int_0^h \phi^2 (\|u_1\|^2 + \|u_2\|^2) d\rho.$$

Hence

$$\frac{c}{h} \phi^2(h) \phi^{-2}\left(\frac{h}{2}\right) \geq \int_0^{h/2} (\|u_1\|^2 + \|u_2\|^2) d\rho.$$

Let  $n$  tend to zero. Then  $u_1 = u_2 = 0$  in  $\rho < h/2$ . Thus we have completed the proof.

We consider the elliptic system (3. 17) in  $S_{1/2}$ . Then we can prove the following in a quite similar manner as in Proposition 5

**COROLLARY 2.** *Let  $U$  be in  $C^1(\bar{S}_{1/2})$  and a solution of the elliptic system (3. 17) in  $S_{1/2}$ . If it holds for  $\delta > \max(2, M - 1)$*

$$U = o(\exp(-r^{-\delta})) \quad (r \rightarrow 0) \text{ in } S_{1/2},$$

*then  $U$  vanish identically.*

Combining Theorem 1 and Corollary 2, we obtain

**THEOREM 2.** *Let  $U$  be a solution of the elliptic system (3. 17) in  $\Omega_a$  and  $U$  be in  $C^1(\bar{\Omega}_a)$ . Then there is a positive number  $\delta$  such that if for  $\delta' > \delta$*

$$U = o(\exp(-r^{-\delta'})) \quad (r \rightarrow 0) \text{ on } \Gamma_a,$$

*then  $U = 0$  in a neighborhood of the origin.*

Theorem 2 means our Main Theorem by an adequate coordinate transformation.

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