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CARDAN MOTION IN ELLIPTIC GEOMETRY

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1. Introduction. Cardan motion in Euclidean geometry may be defined as the motion of a plane Γ_1 with respect to a coinciding plane Γ such that two points A_1, A_2 of Γ_1 move along two orthogonal lines a_1, a_2 of Γ. The properties of this classical motion are well-known: the path of a point of Γ_1 is in general an ellipse with its center at the intersection o of a_1 and a_2; there are ∞^1 points of Γ_1 (their locus being the circle c_1 with $A_1A_2 = 2d$ as diameter) the paths of which are line segments. The moving polhode is the circle c_1, the fixed polhode is the circle $(o; 2d)$. We investigate here Cardan motion-defined in the same way-in the elliptic plane.

2. The fundamental relation. In Γ we introduce a coordinate system (x, y, z) such that the equation of the absolute Ω reads

$$x^2 + y^2 + z^2 = 0. \tag{2.1}$$

o is chosen as $(0, 0, 1)$ and the equations of a_1, a_2 are $y = 0, x = 0$ respectively. The distance A_1A_2 is equal to α.

If $\alpha = \pi/2$ the only possible positions of Γ_1 are: A_1 at the point $B_1 = (1, 0, 0)$ and A_2 anywhere on a_2, or A_2 at $B_2 = (0, 1, 0)$ and A_1 anywhere on a_1; the motion is degenerated into the rotations about B_1 and about B_2, the path of any point consists of two circles. We exclude this special case from now on and may suppose without loss of generality that $0 < \alpha < \pi/2$.

Let A_1 be at $(u_1, 0, 1)$, A_2 at $(0, u_2, 1)$. A point Q on the line $l = A_1A_2$ is $(u_1, \lambda u_2, 1 + \lambda)$ and the intersections S_1, S_2 of l and Ω are given by $\lambda_{1,2} = (-1 \pm iW)(1 + u_2^2)^{-1}$, with $W^2 = u_1^2u_2^2 + u_1^2 + u_2^2$.

As $2i\alpha = \ln(\lambda_1, \lambda_2, 0, \infty)$ we obtain

$$u_1^2u_2^2 + u_1^2 + u_2^2 = \tan^2\alpha \tag{2.2}$$

as the fundamental relation between the coordinates of A_1, A_2. It corresponds to a well-known trigonometric formula for the right-angled triangle oA_1A_2. If in (2.2) we put $u_1^2 = \tan^2\alpha(1 - s^2)$ we get $u_2^2 = s^2\sin^2\alpha/(1 - s^2\sin^2\alpha)$. Therefore if $s = snt$, the Jacobian elliptic function with modulus $k = \sin\alpha$, the relation is expressed parametrically by

$$u_1 = \tan\alpha \cdot cnt,\ u_2 = \sin\alpha \cdot snt/dnt, \tag{2.3}$$

which proves already that the Cardan motion in elliptic geometry will appear to be of genus *one*. (In Euclidean geometry the analogue of (2.2) would be $u_1^2 + u_2^2 = 4d^2$ and the motion is rational.)

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3. The path of a point on l. If A_1 is on a_1, A_2 on a_2 and we reflect the plane Γ_1 into the coordinate axes $x = 0$, $y = 0$ or $z = 0$ (or what is the same thing, into the points B_1, B_2 or o, A_1 remains on a_1 and A_2 on a_2. This implies that the path of any point of Γ_1 is invariant for these reflections. Hence o, B_1, B_2 are centers of any path and its equation will be a function of x^2, y^2 and z^2. We consider first the path of a point $Q(\lambda)$ on l. A_1 and A_2 correspond to $\lambda = 0$ and $\lambda = \infty$ respectively, for S_1, S_2 we have in view of (2.2): $\lambda_{1,2} = (-1 \pm i \tan \alpha). (1 + u_2^2)^{-1}$.

Hence the two midpoints M_1, M_2 of A_1A_2, harmonically separated by A_1, A_2 and by S_1, S_2 as well, correspond to

(3.1) $\quad \lambda = \mp a$, with $a = \{\cos \alpha \cdot (1 + u_2^2)\}^{-1}$.

We introduce on l a new parameter μ by

(3.2) $\quad \mu = \tan \frac{1}{2}\alpha(\lambda - a)/(\lambda + a)$,

from which it follows that M_1, M_2, S_1, S_2, A_1, A_2 correspond to $\mu = \infty$, 0, $-i$, i, $-\tan \frac{1}{2}\alpha$, $\tan \frac{1}{2}\alpha$ respectively. Hence for $Q(\mu)$ we have $\mu = \tan M_1Q$, which implies that μ does not change if l moves.

(2.3) may be written as

$$\cos^2 \alpha(1 + u_1^2)(1 + u_2^2) = 1$$

or (by means of (3.1)) as

(3.3) $\quad \cos \alpha(1 + u_1^2) = a, \quad \cos \alpha(1 + u_2^2) = a^{-1}$,

from which it follows that

(3.4) $\quad u_1^2 = (a - \cos \alpha)/\cos \alpha,\ u_2^2 = (1 - a \cos \alpha)/a \cos \alpha$.

The inverse of (3.2) reads

(3.5) $\quad \lambda = -a(\mu + \tan \frac{1}{2}\alpha)/(\mu - \tan \frac{1}{2}\alpha)$.

The coordinates of Q on l are therefore

(3.6) $\quad x = (\mu - \tan \frac{1}{2}\alpha)u_1,\ y = -a(\mu + \tan \frac{1}{2}\alpha)u_2,\ z = -a(\mu + \tan \frac{1}{2}\alpha) + (\mu - \tan \frac{1}{2}\alpha)$,

from which it follows by means of (3.4),

$$x^2 = (\mu - \tan \tfrac{1}{2}\alpha)^2(a - \cos \alpha),\ y^2 = (\mu + \tan \tfrac{1}{2}\alpha)^2(1 - \cos \alpha)a,$$

(3.7) $\quad z^2 = \{a(\mu + \tan \frac{1}{2}\alpha) - (\mu - \tan \frac{1}{2}\alpha)\}^2 \cos \alpha$.

Hence x^2, y^2, z^2 are quadratic functions of a which acts now as a parameter. This implies that the path p of Q is a quartic curve. Its equation is quadratic in x^2, y^2, z^2 as we could expect.

From (3.4) it follows that for real positions the inequalities

(3.8) $\quad \cos \alpha \leqq a \leqq \sec \alpha$

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hold. Any value of a corresponds to four positions of the moving plane. The equation of the path p will be obtained if we eliminate a in (3.7). This may be done as follows. We have

$$(3.9)\quad \begin{aligned} P_1 &\equiv x^2 + y^2 + z^2 = 4a(\mu^2 + 1)\sin^2 \tfrac{1}{2}\alpha, \\ P_2 &\equiv (\mu + \tan \tfrac{1}{2}\alpha)^2 x^2 + (\mu - \tan \tfrac{1}{2}\alpha)^2 y^2 = (\mu^2 - \tan^2 \tfrac{1}{2}\alpha)^2 \\ &\qquad \times (-a^2 \cos \alpha + 2a - \cos \alpha), \\ P_3 &\equiv x^2y^2 = (\mu^2 - \tan^2 \tfrac{1}{2}\alpha)^2\, a(-\cos \alpha + a + a\cos^2 \alpha - a^2 \cos \alpha). \end{aligned}$$

Hence a $P_2 - P_3 = (\mu^2 - \tan^2 \frac{1}{2}\alpha)^2\, a^2 \sin^2 \frac{1}{2}\alpha$ and *we obtain for the equation of the path* $p(\mu)$:

$$(3.10)\quad \begin{aligned} &(\mu^2 - \tan^2 \tfrac{1}{2}\alpha)^2 \cos^2 \tfrac{1}{2}\alpha (x^2 + y^2 + z^2)^2 - (\mu^2 + 1) \\ &\quad \times \{(\mu + \tan \tfrac{1}{2}\alpha)^2 x^2 + (\mu - \tan \tfrac{1}{2}\alpha)^2 y^2\} \times (x^2 + y^2 + z^2) \\ &\qquad + 4(\mu^2 + 1)^2 \sin^2 \tfrac{1}{2}\alpha \cdot x^2y^2 = 0. \end{aligned}$$

For $\mu = \pm i$ we obtain the paths of the points S_1, S_2 of l, which are seen to be the absolute Ω counted twice. The path of A_1, with $\mu = -\tan \frac{1}{2}\alpha$ is given by $y^2(y^2 + z^2) = 0$ and consists therefore of a_1, counted twice, and the two isotripic lines through B_1 (corresponding to singular Ω-displacements). If the intersections of a_i and Ω are denoted by C_{i1}, $C_{i2} (i = 1, 2)$ we see from (3.10) that the path is tangent to Ω at these four points.

By means of (3.6), (3.3) and (2.3) the coordinates of the points of the path p may be expressed as functions of the parameter t. We have $a \cos \alpha = dn^2 t$, and

$$(3.11)\quad \begin{aligned} &x = (\mu - \tan \tfrac{1}{2}\alpha) \sin\alpha\, cn t,\ y = -(\mu + \tan \tfrac{1}{2}\alpha) \sin\alpha \cdot sn t\, dn t, \\ &z = -(\mu + \tan \tfrac{1}{2}\alpha)\, dn^2 t + (\mu - \tan \tfrac{1}{2}\alpha) \cos \alpha. \end{aligned}$$

Summing up we have: *the path of an arbitrary point of the line l during the Cardan motion is a quartic curve of genus one, with three centers and four times tangent to the absolute.*

4. The double points of the path. A quartic curve of genus one has two double points. In view of the symmetries of the path they can only be situated on a coordinate axis, and as a_1, a_2 are equivalent, this must be $z = 0$.

The intersections of this line and the path, according to (3.7), correspond to $a = (\mu - \tan \frac{1}{2}\alpha)/(\mu + \tan \frac{1}{2}\alpha)$ and this gives us for the coordinates of the double points D_1 and D_2:

$$(4.1)\quad x^2 : y^2 = (\mu - \tan \tfrac{1}{2}\alpha)(\mu - \cot \tfrac{1}{2}\alpha) : (\mu + \tan \tfrac{1}{2}\alpha)(\mu + \cot \tfrac{1}{2}\alpha).$$

Obviously the points $Q(\mu)$ and $Q(\mu^{-1})$ have the same double points. As α has been supposed to be an acute angle it follows from (4.1) that D_1, D_2 are real points if either $\mu \leqq -\cot \frac{1}{2}\alpha$, $-\tan \frac{1}{2}\alpha \leqq \mu \leqq \tan \frac{1}{2}\alpha$, or $\mu \geqq \cot \frac{1}{2}\alpha$.

Furthermore D_1, D_2 correspond to real positions of the plane, according to (3.8), if

$$(4.2)\quad \cos \alpha \leqq (\mu - \tan \tfrac{1}{2}\alpha)/(\mu + \tan \tfrac{1}{2}\alpha) \leqq \cos^{-1} \alpha,$$

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which is equivalent to

(4.3) $\quad \mu \leqq -\cot \frac{1}{2}\alpha, \quad \text{or} \quad \mu \geqq \cot \frac{1}{2}\alpha.$

Hence: *the double points* D_1, D_2 *of the path of* $Q(\mu)$ *are nodes if* $\mu^2 > \cot^2 \frac{1}{2}\alpha$, *cusps if* $\mu^2 = \cot^2 \frac{1}{2}\alpha$, *(real) isolated points* if $\mu^2 \leqq \tan^2 \frac{1}{2}\alpha$ *and conjugate imaginary points in the remaining cases.*

The double points of the paths of $Q(\mu)$ and $Q(\mu^{-1})$ are the same, but if they are real they are of different type.

5. The equations of the Cardan motion. So far we have only considered the path of a point Q on l. We investigate now that of an arbitrary point R of Γ_1. To that end we introduce in Γ_1 a coordinate system (x_1, y_1, z_1) with the vertices M_1, M_2 and the point M_3, the pole of l with respect to Ω; $M_1M_2M_3$ is a self-polar triangle of the absolute.

We have $\mu = x_1/y_1$. The equation of l in the (x, y, z)-system is $u_2x + u_1y - u_1u_2z = 0$, hence $M_3 = (u_2, u_1, -u_1u_2)$, which implies that the path of M_3 in Γ is given by

(5.1) $\quad x = \cos\alpha \cdot snt, \quad y = cnt, dnt, \quad z = -\sin\alpha \cdot snt \cdot cnt.$

Eliminating t, this path is seen to be represented by the equation

(5.2) $\quad (x^2 + y^2 + z^2)z^2 - \tan^2\alpha \cdot x^2y^2 = 0.$

Hence the *path* $p(M_3)$ *is a quartic curve of genus one, with nodes at* B_1, B_2 *and tangent to* Ω *at the four points* $C_{ij}(i, j = \ , 2)$.

The path of $M_1(\mu = \infty)$ follows from (3.11):

(5.3) $\quad x = -\sin\alpha \cdot cnt, \quad y = \sin\alpha \cdot snt\, dnt, \quad z = dn^2t - \cos\alpha,$

and that of $M_2(\mu = 0)$ is

(5.4) $\quad x = \sin\alpha \cdot cnt, \quad y = \sin\alpha \cdot snt\, dnt, \quad z = dn^2t + \cos\alpha.$

We know now the paths of the vertices of the coordinate triangle $M_1M_2M_3$. From these follows the path of an arbitrary point $R(x_1, y_1, z_1)$ if proportionality factors are chosen in a suitable way. The values of the expression $x^2 + y^2 + z^2$ at M_1, M_2 and M_3 (see (5.3), (5.4), (5.1)) are $4 \sin^2 \frac{1}{2}\alpha \cdot dn^2t$, $4 \cos^2 \frac{1}{2}\alpha \cdot dn^2t$ and dn^2t respectively. *Hence for normalized coordinates the equations*

$$x = -\cos \tfrac{1}{2}\alpha \cdot \frac{cnt}{dnt} x_1 + \sin \tfrac{1}{2}\alpha \cdot \frac{cnt}{dnt} y_1 + \cos\alpha \cdot \frac{snt}{dnt} z_1,$$

(5.5) $$y = \cos \tfrac{1}{2}\alpha \cdot snt \cdot x_1 + \sin \tfrac{1}{2}\alpha \cdot snt \cdot y_1 + cnt \cdot z_1,$$

$$z = \frac{\cos \frac{1}{2}\alpha}{\sin\alpha} \cdot \left(dnt - \frac{\cos\alpha}{dnt}\right) x_1 + \frac{\sin \frac{1}{2}\alpha}{\sin\alpha}\left(dnt + \frac{\cos\alpha}{dnt}\right) y_1 - \sin\alpha \frac{snt\, cnt}{dnt} z_1$$

give us the relation between the plane $\Gamma_1(x_1, y_1, z_1)$ *and the plane* $\Gamma(x, y, z)$ *if the former has Cardan motion with respect to the latter* (with t as the parameter).

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It is easy to verify that the matrix of the linear transformation (5.5) is indeed, for all values of t, an orthogonal matrix with determinant *one*; the absolute Ω is invariant for (5.5), which represents therefore a set of elliptic displacements. Furthermore, for $A_1 = (-\sin \frac{1}{2}\alpha, \cos \frac{1}{2}\alpha, 0)$ we have $y = 0$ and for $A_2 = (\sin \frac{1}{2}\alpha, \cos \frac{1}{2}\alpha, 0)$, $x = 0$. For $Q = (x_1, y_1, 0)$ the formulas give us the path of a point on l, in accordance with (3.11).

If x_1, y_1, z_1 are arbitrary (5.5) represents the path in Γ of the general point R of Γ_1. After multiplication by *dnt*, the coordinates are seen to be quadratic functions of *snt*, *cnt* and *dnt*. In their common parallelogram of periods in the complex plane (with sides $4K$ and $4iK'$ in the usual notation) they take any value *four* times. Hence a linear function of x, y, z has eight zeros. The conclusion is: *the path of an arbitrary point of the moving plane is a curve of order eight and genus one.*

We knew already that it is invariant for reflections into the three coordinate axes. From the properties of the Jacobian functions it follows that the three reflections correspond to the parameter transformations $t' = t + 2K$, $t' = -t + 2iK'$ and $t' = -t + 2K + 2iK'$. Furthermore (5.5) is singular either if $dnt = 0$ or if dnt (and snt and cnt) has a pole. This implies that for the intersections of Ω and the path we have either $x = 0$ or $y = 0$, which means that the path has a four-fold contact with the absolute at the four points C_{ij}.

The general path is of order eight but that for a point on l and for M_3 is only four. This exceptional behavior could have been expected. Indeed, to a given position of l correspond two positions of Γ_1, one following from the other by reflection into l (or into M_3). This implies that during the complete motion any point on l and the point M_3 pass twice through any point of their path, which is therefore a quartic curve counted twice. In other words: the points (x_1, y_1, z_1) and $(x_1, y_1, -z_1)$ describe the same path. This, again, may be verified by means of the properties of *snt*, *cnt* and *dnt*: the first point is at the moment t at the same point of Γ as the second at the moment $t' = t + 2K + 2iK'$.

6. The inverse motion. The motion of Γ with respect to Γ_1 follows from (5.5) if we interchange the roles of (x, y, z) and (x_1, y_1, z_1). As the inverse of an orthogonal matrix is identical with its transpose the expressions for x_1, y_1, z_1 in terms of x, y, z are found by reflecting the matrix in (5.5) into the principal diagonal. The conclusion is: *a point of* Γ *describes in* Γ_1 *a curve which is in general of order eight and genus one*; it is symmetric with respect to $z_1 = 0$. Furthermore: the four points (x, y, z), $(-x, y, z)$, $(x, -y, z)$ and $x, y, -z)$ describe the same path in Γ_1. In particular: *the path of a point on* $x = 0$, *or* $y = 0$, *or on* $z = 0$ *is in general a quartic curve of genus one.* Still more special are the paths of the vertices o, B_1, B_2, that of $o = (0, 0, 1)$, being given by

$$(6.1) \qquad x_1 = \cos \tfrac{1}{2}\alpha(dn^2t - \cos\alpha),\ y_1 = \sin \tfrac{1}{2}\alpha(dn^2t + \cos\alpha),$$
$$z_1 = -\sin^2\alpha \cdot snt\, cnt$$

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is the conic

$$(6.2) \qquad x_1^2 \cos^2 \tfrac{1}{2}\alpha - y_1^2 \sin^2 \tfrac{1}{2}\alpha + z_1^2 \cos \alpha = 0,$$

passing through A_1, A_2 and with centers at M_1, M_2, M_3. This could be expected: the path is the locus of the vertex of a right angle the sides of which pass through two fixed points, a well-known configuration in elliptic geometry. The path of $B_1 = (1, 0, 0)$ is *the line* $\sin \tfrac{1}{2}\alpha \cdot x_1 + \cos \tfrac{1}{2}\alpha \cdot y_1 = 0$, that of $B_2 = (0, 1, 0)$ is *the line* $\sin \tfrac{1}{2}\alpha \cdot x_1 - \cos \tfrac{1}{2}\alpha \cdot y_1 = 0$; both pass through M_3 and their angle is α; they are the polar lines of A_2 and A_1 with respect to Ω. There is the following analogy between our Cardan motion and its inverse: at the former two points with distance α move along two lines with angle $\pi/2$, at the latter two points with distance $\pi/2$ move along lines with angle α.

7. The polhodes. The instantaneous center of rotation is the intersection of the perpendicular at A_1 on a_1 and that at A_2 on a_2. Its coordinates in Γ are $(u_1, u_2, 1)$. The fixed polhode is therefore (in view of (2.2)).

$$(7.1) \qquad x^2y^2 + (x^2 + y^2)z^2 - \tan^2 \alpha \cdot z^4 = 0$$

or by making use of (2.3),

$$(7.2) \qquad x = \tan \alpha \cdot cnt\, dnt,\ y = \sin \alpha \cdot snt,\ z = dnt.$$

The fixed polhode of the Cardan motion is a quartic curve of genus one; it has (isolated) double points at B_1 and B_2.

The parametric representation of the moving polhode follows from (7.2) if we transform the coordinates by means of the transpose of the matrix of (5.5). We obtain after some algebra

$$(7.3) \qquad x_1 = \cos \tfrac{1}{2}\alpha(\cos \alpha - dn^2t),\ y_1 = \sin \tfrac{1}{2}\alpha(\cos \alpha + dn^2t),$$
$$z_1 = \sin^2\alpha \cos \alpha \cdot snt\, cnt,$$

or after eliminating t,

$$(7.4) \qquad \cos \alpha(x_1^2 \cos^2 \tfrac{1}{2}\alpha - y_1^2 \sin^2 \tfrac{1}{2}\alpha) + z_1^2 = 0.$$

Hence *the moving polhode is a conic* passing through A_1, A_2.

We could have determined the polhodes also by differentiating (5.5) with respect to t, making use of the derivatives of the elliptic functions. Then (7.3) follows from the conditions $\dot{x} = \dot{y} = \dot{z} = 0$.

8. Final remarks. Cardan motion in *hyperbolic* geometry may be treated in a similar way. If $\Omega \equiv x^2 + y^2 - z^2 = 0$ and $A_1A_2 = d$, the fundamental relation (2.2) changes into

$$(8.1) \qquad -u_1^2u_2^2 + u_1^2 + u_2^2 = \tanh^2 d,$$

or, if we introduce Jacobian functions with modulus $k = \tanh d$,

$$(8.2) \qquad u_1 = k\, snt, \quad u_2 = k\, cnt/dnt.$$

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Kinematics in the elliptic plane, as considered above, has of course much similarity with (Euclidean) spherical kinematics. If a three dimensional space moves about a fixed point o in such a way that two lines through o (with angle α) move in two orthogonal fixed planes we have a complete analogue with the Cardan motion treated in this paper.

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