A CIRCULAR PROPERTY OF THE OCCURRENCE OF SEQUENCE PATTERNS IN THE FAIR COIN-TOSSED PROCESS

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Let \( \Omega = \{0, 1\} \) and for each integer \( n \geq 1 \) let \( \Omega_n = \Omega \times \cdots \times \Omega \) (n-tuple) and \( \Omega_n^k = \{(a_1, a_2, \ldots, a_n) \mid (a_1, a_2, \ldots, a_n) \in \Omega_n \text{ and } \sum_{i=1}^n a_i = k\} \) for all \( k = 0, 1, \ldots, n \). Let \( \{X_m\}_{m=1}^\infty \) be a sequence of i.i.d. random variables such that \( P(X_1 = 0) = P(X_1 = 1) = \frac{1}{2} \). Any element \( A \) in \( \Omega_n \) is called a sequence pattern and for each sequence pattern \( A \), let \( T_A \) be the first occurrence time of \( A \) (with respect to the process \( \{X_m\}_{m=1}^\infty \) which is defined by \( T_A(X_1, X_2, \cdots) = \inf \{m \mid (X_{m-n+1}, \ldots, X_m) = A\} \). For any two distinct sequence patterns \( A \) and \( B \), \( A \) occurs (stochastically) after \( B \) (denoted by \( B \prec A \)) if \( P(T_A > T_B) > \frac{1}{2} \). In this paper, we prove that if \( n \geq 4, ~k = 1, 2, 3(n - 3, n - 2, n - 1), \) and \( n \neq 2k \), then there is an arrangement \( \{B_1, B_2, \ldots, B_{(\frac{n}{2})}\} \) of \( \Omega_n^k \) such that \( B_1 \prec B_2 \cdots \prec B_{(\frac{n}{2})} \prec B_1 \). Albeit we are not able to prove the statement for any \( k \neq n/2 \), from our proof for special cases we strongly believe that the statement is true for any \( n \geq 4 \) and \( k \neq n/2 \). This new result reveals a circular property of the first occurrence among the sequence patterns in \( \Omega_n^k \) and also provides us with an understanding of the regularity of the fair coin-tossing process. We start with the following notation and lemmas.

For each sequence pattern \( A = (a_1, a_2, \ldots, a_n) \) and \( B = (b_1, b_2, \ldots, b_n) \) (not necessarily distinct) in \( \Omega_n \), we define \( A \cdot B = \sum_{j=1}^n 2^j e_j \), where for each \( j = 1, 2, \ldots, n, \) \( e_j = 1 \) or 0 (according to whether \( b_{j-1}, b_j, \ldots, b_1 \) or not).

**Lemma 1.** If \( A \) and \( B \) are two distinct sequence patterns in \( \Omega_n \), then \( P(T_A < T_B)/P(T_B < T_A) = (B \cdot B - B \cdot A)/(A \cdot A - A \cdot B) \).

**Lemma 2.** If \( 1 \leq k \leq n - 1, ~k \neq n/2, ~n \geq 4, \) and \( A = (a_1, a_2, \ldots, a_n) \) is in \( \Omega_n^k \), then \( B \prec A \), where \( B = (a_n, a_1, a_2, \ldots, a_{n-1}) \).

**Theorem 1.** If \( n \geq 4 \), then there is an arrangement \( \{B_1, B_2, \ldots, B_{n}\} \) of \( \Omega_n^1 \) (or \( \Omega_n^{n-1} \)) such that \( B_1 \prec B_2 \cdots \prec B_n \prec B_1 \).

For each \( A = (a_1, a_2, \ldots, a_n) \) in \( \Omega_n \), let \( A^1 = A \) and \( A^{i+1} = (a_{n-i+1}, \ldots, a_n, a_1, \ldots, a_{n-1}) \) for all \( j = 1, 2, \ldots, n - 1 \). For each \( i = 0, 1, \ldots, n - 2, \) let \( A_i = (1, 0, \cdots, 0, 1, 0, \cdots, 0) \).

**Lemma 3.** If \( n = 2m + 1 \) and \( i \leq n - 3 \), then \( A_{i+1} \prec A^m_{i} \) if \( m \geq 3 \) or \( m = 2 \) but \( i \neq 1 \).

**Lemma 4.** If \( n = 2m + 1 \) and \( A = (1, 1, 0, \cdots, 0) \), then \( A \prec A^m_{n} \) for all \( m \geq 3 \) such that \( B_1 \prec B_2 \cdots \prec B_{(\frac{n}{2})} \prec B_1 \).

**Proof.** Let \( A_i \) and \( A_i^{*} \) be as defined just above Lemma 3 for all \( j = 1, 2, \ldots, n \) and \( i = 0, 1, \ldots, m - 1 \). By Lemma 3, \( A_{i+1} \prec A_i \) for all \( j = 1, 2, \ldots, n - 1 \) and all \( i = 0, 1, \ldots, m - 1 \). By Lemma 3, \( A_{i+1}^{*} \prec A_i^{*} \) for all \( i = 0, 1, \cdots, m - 1 \). By Lemma 3, \( A_{i+1} \prec A_i^{*} \) for all \( i = 0, 1, \cdots, m - 1 \). By Lemma 4, \( A_{i} \prec A_{i+1} \). Therefore, \( A_i^{*} \prec A_{i+1} \prec A_{i+2} \prec \cdots \prec A_{m-1} \prec A_{m-2} \prec \cdots \prec A_{1}^{*} \prec A_{0}^{*} \prec A_{0}^{\prime} \). Since \( n = 2m + 1 \) and \( j \leq n \) and \( 0 \leq i \leq m - 1 \), the proof of Lemma 5 is now complete.

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Lemma 6. If \( n = 2m \geq 6 \), then \( A^{i+1}_i \ll A^m_i \) for all \( i = 0, 1, \ldots, m - 1 \).

Lemma 7. If \( n = 2m \geq 6 \), then there is an arrangement \( \{B_1, B_2, \ldots, B_{m}\} \) of \( \Omega^2_n \) (or \( \Omega^{n-2}_n \)) such that \( B_1 \ll B_2 \ll \cdots \ll B_{m} \).

Proof. By Lemma 2, \( A^{i+1}_i \ll A^m_i \) for all \( i = 1, 2, \ldots, n - 2 \) and all \( i = 0, 1, \ldots, m - 2 \). Now by a direct computation, \( A^0_i \ll A^m_i = 2^{2m}, A^0_i \ll A^m_i = 2^{m - 1}, A^{m-1}_i \ll A^{m-1}_i = 2^{2m} + 2^m, \) and \( A^{m-1}_i \ll A^m_i = 2 \). Hence \( A^m_i \ll A^{m-1}_i \). Therefore, \( A^m_i \ll A^{m-1}_i \ll \cdots \ll A^0_i \ll A^m_i \ll \cdots \ll A^0_i \ll A^m_i \). Since \( n = 2m \geq 6 \), \( \{A^m_i | 0 \leq i \leq m - 2 \text{ and } 1 \leq j \leq n \} \cup \{A^{m-1}_i, \ldots, A^{m-1}_i\} \) is the proof of Lemma 7 is now complete.

Theorem 2. If \( n \geq 5 \), then there is an arrangement \( \{B_1, B_2, \ldots, B_{m}\} \) of \( \Omega^2_n \) (or \( \Omega^{n-2}_n \)) such that \( B_1 \ll B_2 \ll \cdots \ll B_{m} \).

Proof. If \( n = 5 \), Theorem 2 can be proved by a direct computation and Lemma 1. If \( n > 5 \), Theorem 2 is proved by Lemmas 5 and 7.

For each \( i = 0, 1, \ldots, n - 3 \), and \( j = 0, 1, \ldots, n - 3 - i \), let \( A_{ij} = (1, 0, 0, \ldots, 0, 1, 0, \ldots, 0) \) be a sequence pattern in \( \Omega^2_n \).

The following lemmas are essential to Theorem 3 below. However, the proofs of these lemmas are omitted.

Lemma 8. In \( n \geq 8 \), then \( A^{i+1}_{i,0} \ll A^{i+1}_{0,i-3} \) for all \( i = 0, 1, \ldots, n - 4 \).

Lemma 9. If \( n \geq 8 \), then \( A^{i+1}_i \ll A^{i+3}_i \).

Lemma 10. If \( n = 3m + 3 \) and \( m \geq 2 \), then \( A^{m+1}_{2m+1} \ll A^{m+1}_{m+1} \).

Lemma 11. If \( n = 3m + 3 \) and \( m = 1 \), then \( A^{m+1}_{2m+1} \ll A^{m+1}_{m+1} \).

Lemma 12. If \( n = 3m + 3 \) and \( m = 2 \), then \( A^{m+1}_{2m+1} \ll A^{m+1}_{m+1} \).

Lemma 13. If \( 1 \leq i \leq n - 2i - 7 \), then \( A^{i+1}_{i+1,i+1} \ll A^{i+1}_{n-2i-4,i+1} \).

Lemma 14. If \( 1 \leq j \leq i \) and \( 1 \leq n - i - j - 3 \), then \( A^{j+1}_{n-3,j} \ll A^{j+1}_{j+1,j} \).

Lemma 15. If \( 1 \leq j \leq i \) and \( i + 2 \leq n - i - j - 3 \), then \( A^{i+1}_{i+1} \ll A^{i+1}_{j+1,j-3} \).

Theorem 3. If \( n \geq 7 \), then there is an arrangement \( \{B_1, B_2, \ldots, B_{m}\} \) of \( \Omega^2_n \) (or \( \Omega^{n-3}_n \)) such that \( B_1 \ll B_2 \ll \cdots \ll B_{m} \).

Proof. Since the case that \( \{B_1, B_2, \ldots, B_{m}\} = \Omega^{n-3}_n \) can be proved by interchanging 0 and 1, we prove only the case that \( \{B_1, B_2, \ldots, B_{m}\} = \Omega^{n-3}_n \). If \( n = 7 \), then by a direct computation and Lemma 2, the sequence patterns in \( \Omega^2_n \) can be arranged as follows:

\[
\begin{align*}
(1110000) & \ll (0101001) \ll (1010100) \ll (0010101) \ll (1001010) \ll (0010101) \\
& \ll (0101010) \ll (1010100) \ll (0001011) \ll (0010110) \ll (1010100) \ll (1010000) \\
& \ll (0110001) \ll (1101001) \ll (1001101) \ll (1000110) \ll (0101100) \ll (1100100) \ll (0011001) \ll (0001101) \\
& \ll (0110010) \ll (1110010) \ll (1011000) \ll (1101000) \ll (1001100) \ll (0100110) \ll (1100011) \\
& \ll (1000011) \ll (0001101) \ll (0011100) \ll (0111100) \ll (1110000)
\end{align*}
\]

Now we assume that \( k = 3 \) and \( n \geq 8 \). When \( n \neq 3m + 3 \) for some positive integer \( m \), Theorem 3 is proved by combining Lemmas 2, 8, 9, 13, 14, and 15. When \( n = 3m + 3 \) for some positive integer \( m \), Theorem 3 is proved by combining Lemmas 2, 8, 13, 14, 15, and at certain steps, Lemmas 10, 11, 12 will be used to avoid the cyclic behavior. Since the detailed proof is very lengthy, we omit it.
From our constructive proofs of Theorems 2 and 3, we strongly believe that Theorem 3 holds in general, i.e., if \( n \geq 4 \), \( k = 1, 2, \ldots, n-1 \), and \( k \neq n/2 \), then there is an arrangement \( \{B_1, B_2, \ldots, B(\Omega)\} \) of \( \Omega_n^k \) such that \( B_1 \ll B_2 \ll \cdots \ll B(\Omega) \ll B_1 \).

Chen and Lin (1984) proved that if \( n \geq 4 \) and \( n = 2k \), then \( P(T_A < T_A^i) = \frac{1}{2} \) for \( i = 1, 2 \) and \( P(T_A < T_D) < \frac{1}{2} \) for all \( D \in \Omega_n^{2k} - \{A, A_1, A_2\} \); here \( A = (0, 1, 0, 1, \ldots, 0, 1) \), \( A_1 = (1, 0, 1, 0, \ldots, 1, 0) \), and \( A_2 = (0, 1, 0, 1, \ldots, 0, 1, 1, 0) \). Hence there does not exist an arrangement \( \{B_1, B_2, \ldots, B(\Omega)\} \) of \( \Omega_n^{2k} \) such that \( B_1 \ll B_2 \ll \cdots \ll B(\Omega) \ll B_1 \).

Under some mild conditions, the conjecture and the results in this paper can presumably be extended to the situation in which \( \Omega = \{1, 2, \ldots, r\} \) and \( P(X_1 = i) = 1/r \) for all \( i = 1, 2, \ldots, r \); here \( r \) is a positive integer \( \geq 3 \). Chen and Zame (1979) also briefly discussed this case.

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References