A SUMMABILITY METHOD DUE TO LINEAR DIFFERENTIAL EQUATIONS AND A UNIQUENESS PROPERTY OF SOLUTIONS OF SINGULAR DIFFERENTIAL EQUATIONS

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0 Introduction

The purpose of this paper is to expose a method which will match a function $f(z)$ existing in a domain D to a formal series $\sum_0^\infty A_n Z^n$ whose radius of convergence may be zero. This matching process has to be done in a "natural way", and has to be "regular", which means that if a power series $\sum_0^\infty A_n Z^n$ converges absolutely in the circle $E = \{z \mid |z| < r\}$, then the summability function $f(z)$ produced by our method in the domain D and matched to $\sum_0^\infty A_n Z^n$ will coincide with $\sum_0^\infty A_n Z^n$ in the domain $E \cap D$. Euler, in his time, matched the function $f(z) = \int_0^\infty \frac{e^{-w} dw}{(1+zw)}$ to the power series $\sum_0^\infty (-1)^n n! Z^n$. This matching process was justified by the following properties (**1**, p. 26):

(i) $\sum_0^\infty (-1)^n n! Z^n$ is a formal solution of the linear differential equation:

$$Z^2 f' + (Z+1) f = 1.$$

(ii) $f(z) = \int_0^\infty \frac{e^{-w} dw}{1+zw}$ is a solution of the linear differential equation

$$z^2 f' + (z+1) f = 1.$$

(iii) $f(z)$ has $\sum_0^\infty (-1)^n n! Z^n$ as its asymptotic expansion for $z \to 0$ and $\mathrm{Re}\, z > 0$.

Euler's procedure can be converted into a "regular" summability method if proper answers are given to the following questions:

(i) How many linear differential equations does a formal series satisfy?

(ii) How many (if any) functions $f(z)$ satisfying the linear differential equation have this formal series as their asymptotic expansion?

(iii) How may we define in a systematic way the summability functions such that the method will be regular?

The following notation and abbreviations will be used throughout the paper. The capital letters A, B, C will stand for the corresponding formal series $\sum_0^\infty A_nZ^n$, $\sum_0^\infty B_nZ^n$, $\sum_0^\infty C_nZ^n \ldots$. The abbreviations S.C.Z., S.C.P. and L.D.E. will stand for a power series whose radius of convergence is zero, a power series whose radius of convergence is positive and Linear Differential Equation respectively. Finally, $a(z)$, $b(z)$, $c(z)$, ... will denote the power series,

$$\sum_0^\infty a_nz^n, \sum_0^\infty b_nz^n, \sum_0^\infty c_nz^n,$$

which are S.C.P.

From now on we will consider the set of all formal power series $\sum_0^\infty A_nZ^n$ as a vector space over the complex field and define on this the following operations:

(i) $A+B = C$ means C is the series such that for every $n \geqq 0$

$$C_n = A_n+B_n.$$

(ii) $AB = C$ means C is the series such that for every $n \geqq 0$

$$C_n = \sum_{v=0}^{v=n} A_vB_{n-v}.$$

(iii) If $B_0 \neq 0$, $A/B = C$ means C is the series such that for every $n \geqq 0$

$$A_n = \sum_{v=0}^{v=n} B_vC_{n-v}$$

(iv) A', the formal derivative of A, will be defined by $A' = \sum_1^\infty nA_nZ^{n-1}$ and the kth formal derivative of A, $A^{(k)}$, will be defined by

$$A^{(k)} = \sum_k^\infty n(n-1)\ldots(n-k+1)A_nZ^{n-k}.$$

(See **2**, pp. 11-14).

We must keep in mind that addition, multiplication, division and differentiation are merely operations on an infinite set of equations. The rest of this paper may be summarised as follows. In Section 1 we define an equivalence relation $\mathscr{R}$ on the set of all formal series, and find its characteristics and invariants. In Section 2 we define a characteristic domain for every L.D.E. and prove an existence and uniqueness theorem for the summability functions. In Section 3 we prove some functional properties of the summability functions and define our method.

1

Let us be given two formal series A, B, where

$$A = \sum_0^\infty A_nZ^n, \quad B = \sum_0^\infty B_nZ^n.$$

Definition 1. We say that A is in the same class of B, and write $A\mathscr{R}B$ if there exist three power series S.C.P.

$$a(z) = \sum_0^\infty a_n z^n, \quad b(z) = \sum_0^\infty b_n z^n, \quad c(z) = \sum_0^\infty c_n z^n$$

such that $a(z).b(z) \not\equiv 0$ and

$$a(z)A + b(z)B = c(z) \tag{1.1}$$

or

$$\sum_{\nu=0}^{\nu=n} a_\nu A_{n-\nu} + \sum_{\nu=0}^{\nu=n} b_\nu B_{n-\nu} = c_n \quad n \geqq 0. \tag{1.1$'$}$$

Proposition 1. *Definition* 1 *determines on the set of formal series an equivalence relation which separates this set into disjoint classes.*

We omit the trivial proof. If two series A, B, belong to the same equivalence class we may agree that to every ordered couple $\langle A, B\rangle$ where $A\mathscr{R}B$, there exists at least one ordered triplet $\langle a(z), b(z), c(z)\rangle$ such that (1.1) is true. What is interesting is the fact that we can prove a uniqueness theorem for the triples if A, B, are S.C.Z.

Proposition 2. *Let $A\mathscr{R}B$, where A is S.C.Z. If*

$$a(z)A + b(z)B = c(z) \tag{1.2}$$

and also

$$a_1(z)A + b_1(z)B = c_1(z) \tag{1.3}$$

where $a(z)$, $b(z)$, $v(z)$, $a_1(z)$, $b_1(z)$, $c_1(z)$ are S.C.P. and

$$a(z).b(z) \not\equiv 0, \; a_1(z).b_1(z) \not\equiv 0,$$

then

$$\frac{a(z)}{a_1(z)} = \frac{b(z)}{b_1(z)} = \frac{c(z)}{c_1(z)}.$$

Proof. Multiply (1.2) by $b_1(z)$ and (1.3) by $b(z)$ and subtract the multiplied equations to obtain

$$[b_1(z)a(z) - b(z)a_1(z)]A = b_1(z)c(z) - b(z)c_1(z). \tag{1.4}$$

Then $b_1(z)a(z) - b(z)a_1(z)$ must be identically zero. If not, then

$$b_1(z)a(z) - b(z)a_1(z) \equiv z^m d(z) \tag{1.5}$$

where m is a non-negative integer and $d(z)$ is S.C.P. with $d(0) \neq 0$. Substitute (1.5) into (1.4) to obtain

$$z^m A = \frac{b_1(z)c(z) - b(z)c_1(z)}{d(z)}. \tag{1.6}$$

Since A is an S.C.Z., $z^m A$ is also an S.C.Z., which contradicts the right hand of (1.6) which must be an S.C.P. This means that $b_1(z)a(z) - b(z)a_1(z) \equiv 0$, together with $b_1(z)c(z) - b(z)c_1(z) \equiv 0$ and the result follows.

Definition 2. Let U denote an equivalence class relative to $\mathscr{R}$. We say that U is a *summability class* if $A \in U$ implies $A'\mathscr{R}A$.

We remark that if $A'\mathscr{R}A$ and $A\mathscr{R}B$ then $B'\mathscr{R}B$. This is easily verified and we shall not prove it.

Proposition 3. *Let U be a summability class and let $A \in U$, A is an S.C.Z.; then there exists a* unique *L.D.E. of the form*

$$z^s A' + b(z)A = c(z), \tag{1.7}$$

where s is a natural number greater than 1 *and* $b(0) \neq 0$.

Proof. Any relation of the form (1.1) can be made such that $a(z)$ is equal to z^s with s the least integer possible. By a slight modification of (3, p. 22) we know that $s > 1$, since A is S.C.Z. Moreover, if s is the minimal integer, then $b(0) = b_0 \neq 0$, otherwise we would have $c(0) = 0$ in (1.7), and we could arrive at the relation $z^{s-1}A' + (b(z)/z)A = c(z)/z$. Proposition 3, and the result follows. Every L.D.E. of type (1.7), where $s > 1$, and $b_0 \neq 0$ has a characteristic ordered couple $\langle s, b_0 \rangle$. This ordered couple happens to be an invariant characteristic of the whole class U containing the power series A.

Proposition 4. *Let $A \in U$, where A satisfies an L.D.E. of type* (1.7) *with $s > 1$, and $b_0 \neq 0$. If $A\mathscr{R}B$ then B satisfies an L.D.E. of type* (1.7) *with the same ordered couple $\langle s, b_0 \rangle$.*

Proof. We are given the equations $A'\mathscr{R}A$ written as

$$z^s A' + b(z)A + [-c(z)] \,.\, 1 = 0. \tag{1.8}$$

Writing the relation $A\mathscr{R}B$ as

$$0A' + a_1(z)A + [b_1(z)B - c_1(z)] \,.\, 1 = 0 \tag{1.9}$$

and from differentiating (1.3) we have

$$a_1(z)A' + a_1'(z)A + [b_1'(z)B + b_1(z)B' - c_1'(z)] \,.\, 1 = 0. \tag{1.10}$$

We look on (1.8) (1.9) (1.10) as a homogeneous system with A', A,1, as "unknowns". Since the system possesses a non-trivial solution, the determinant of the coefficient matrix must be identically 0. Computing the determinant and rearranging its equation we obtain the final equation

$$z^s B' + \left[b(z) + z^s \frac{b_1'(z)}{b_1(z)} - z^s \frac{a_1'(z)}{a_1(z)} \right] B$$

$$= \frac{z^s[a_1(z)c_1'(z) - a_1'(z)c_1(z)] + b(z)a_1(z)c_1(z) - c(z)a_1^2(z)}{a_1(z)b_1(z)}. \tag{1.11}$$

We remember that $b(z)$, $c(z)$, $a_1(z)$, $b_1(z)$, $c_1(z)$ are S.C.P. such that $a_1(z).b_1(z) \neq 0$, and observe that $z^s(b_1'(z)/b_1(z))$ and $z^s(a_1'(z)/a_1(z))$ are S.C.P.

since $s>1$. This means that the coefficient of B in (1.11) must be an S.C.P. Moreover, because $s>1$ we have

$$\lim_{z\to 0}\,[b(z)+z^s(b_1'(z)/b_1(z))-z^s(a_1'(z)/a_1(z))] = \lim_{z\to 0} b(z) = b_0.$$

Since the left-hand side of (1.11) is a formal power series so also is the right-hand side. Further it is an S.C.P. and our result follows.

Proposition 5. *The class U containing the S.C.Z. $\sum_0^\infty e^{n!}Z^n$ is not a summability class.*

Proof. On the contrary, assume that this series satisfies an L.D.E. of type (1.7), then we have for every $n+1 \geqq s$

$$(n+1-s)e^{(n+1-s)!}+\sum_{v=0}^{v=n} b_{n-v}e^{v!} = c_n.$$

Since $s>1$ we may write

$$b_0 = c_n/e^{n!}-(n+1-s)e^{(n+1-s)!-n!}+\sum_{v=0}^{v=n-1} b_{n-v}e^{v!-n!}. \tag{1.12}$$

Since $a(z)$, $b(z)$ are S.C.P., there exists $\rho>0$, $\rho \neq 1$ such that

$$|\, b_v \,|<M/\rho^v, \quad |\, c_v \,|<M/\rho^v,$$

where $M>0$. By this we have from (1.12)

$$|\, b_0 \,| \leqq \frac{M}{e^{n!}\rho^n}+\frac{(n+1-s)e^{(n+1-s)!}}{e^{n!}}+\frac{Me^{(n-1)!}(1/\rho^n-1)}{e^{n!}\rho(1/\rho-1)}. \tag{1.13}$$

The right-hand side of (1.13) tends to 0, as n tends to infinity, so that $b_0 = 0$, which gives a contradiction.

2

Before finding the functions we match to given power series, we define a characteristic domain of an L.D.E. of type (1.7).

Suppose we are given $r = \min\{r_b, r_c\}$, where r_b, r_c are the corresponding radii of convergence of $b(z)$, $c(z)$ appearing in (1.7). Let $\langle s, b_0\rangle$ be the invariant couple of the class U associated with (1.7).

Definition 2. We call D_U^r the *characteristic domain* of the L.D.E. (1.7), if

$$D_U^r = \bigcup_{k=0}^{k=s-2} D_k^r, \tag{2.1}$$

and D_k^r is one of $(s-1)$ open sectors contained in the circle $|\, z \,|<r$, $\arg z = \theta$ $z\in D_k^r$, $k = 0, 1, \ldots, s-2$ and θ satisfies:

$$-\frac{\pi}{2}-2\pi k<\phi-(s-1)\theta<\frac{\pi}{2}-2\pi k, \quad k = 0, 1, \ldots, s-2, \tag{2.2}$$

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where

$$b_0 = | b_0 | e^{i\phi}. \tag{2.3}$$

The reason for this definition appears later.

Proposition 6. *Given a power series A which satisfies an L.D.E. of type* (1.7), *then:*

(i) *In every sector D_k^r of the characteristic domain D_U^r there exists a holomorphic function $G_A(z)$ whose asymptotic expansion is the series A.*

(ii) *In every D_k^r, $k = 0, 1, \ldots, s-2$, $G_A(z)$ is unique.*

(iii) *If we associate with every point ζ of D_k^r the corresponding solution $g(z, \zeta)$ of* (1.7) *which satisfies $g(\zeta, \zeta) = C_0$, then*

$$G_A(z) = \lim g(z, \zeta), \quad \bar{D}_k^r \ni \zeta \to 0, \quad k = 0, 1, \ldots, s-2, \tag{2.3}$$

for z contained in any closed subdomain of D_k^r for every limit process of (2.3) *(k is held constant) and $\bar{D}_k^r$ is any closed subsector contained in D_k^r and having $z = 0$ in its closure.*

Proof. (i) From a theorem of Wasow (3, p. 57, Theorem 12.1) we conclude that in every open sector of the z-plane contained in some circle $| z | < r_0$, r_0 positive, with vertex at the origin and a positive central angle not exceeding $\pi/(s-1)$, there exists a function $G_A(z)$ which satisfies (1.7) and A is its asymptotic expansion in every proper subsector. If the function $G_A(z)$ exists only in $| z | < r_0 < r$, then we easily can extend its existence in the corresponding sector for $| z | < r$ by (1.7).

(ii) We need two lemmas to prove this statement.

Lemma 1. *Let $s > 1$, $b_0 \neq 0$; then for any fixed real $m > 0$,*

$$\lim | \zeta |^{-m} \exp\left(- \int_\zeta^z \frac{b(t)}{t^s} \, dt \right) = 0, \quad \bar{D}_k^r \ni \zeta \to 0, \quad k = 0, 1, \ldots, s-2, \tag{2.4}$$

uniformly for z contained in any closed subsector of D_k^r and ζ belonging to any closed subsector $\bar{D}_k^r$ which has $\zeta = 0$ in its closure. The path of integration in (2.4) *is contained in D_k^r.*

Define

$$B(z) - B(\zeta) = \int_\zeta^z \sum_{\nu = s}^{\infty} b_\nu t^{\nu - s} dt \tag{2.5}$$

and

$$P(z) - P(\zeta) = - \int_\zeta^z \sum_{\nu = 0}^{\nu = s-1} \frac{b_\nu}{t^{s-\nu}} \, dt \tag{2.6}$$

so that

$$P(z) = \sum_{\nu = 0}^{\nu = s-2} \frac{b_\nu}{(s - \nu - 1) z^{s - \nu - 1}} - b_{s-1} \ln z. \tag{2.7}$$

We take a branch of $\ln z$ with

$$0 \leqq \arg z < 2\pi;$$

then

$$\left| \exp\left(- \int_{\zeta}^{z} \frac{b(t)}{t^s} \right) dt \right| = |\exp(B(\zeta)-B(z)+P(z))|\, |\exp(-P(\zeta))|. \tag{2.8}$$

By assumption the term $\exp(B(\zeta)-B(z)+P(z))$ is uniformly bounded. Substituting $\zeta = \rho e^{i\theta}$, $0<\rho<r$, $0 \leqq \theta<2\pi$, we obtain

$$\operatorname{Re}[-P(\zeta)] = \frac{-|b_0| \cos[\phi-(s-1)\theta]}{(s-1)\rho^{s-1}} - \sum_{v=1}^{v=s-2} \operatorname{Re}\left\{ \frac{b_v e^{v\ -i(s-v-1)}}{(s-v-1)\rho^{s-v-1}} \right\}$$

$$+\operatorname{Re}\{b_{s-1} \ln \rho e^{i\theta}\} \leqq -\frac{|b_0| \cos[\phi-(s-1)\theta]}{(s-1)\rho^{s-1}}$$

$$+\sum_{v=1}^{v=s-2} \frac{|b_v|}{(s-v-1)\rho^{s-1-v}} + |b_{s-1}| \ln \rho + 2\pi |b_{s-1}|. \tag{2.9}$$

The dominating term of $\operatorname{Re}[-P(\zeta)]$ is $-\dfrac{|b_0| \cos[\phi-(s-1)\theta]}{(s-1)\rho^{s-1}}$ and so if θ varies in a closed subsector of D_k^r, $\operatorname{Re}[-P(\zeta)] \to -\infty$ when $\rho \to 0$ and the result follows.

Lemma 2. *We use the same notation and make the same assumptions as in Lemma* 1. *In addition, let Γ be a smooth Jordan arc imbedded in $\bar{D}_k^r$ and having $z = 0$ as one of its end points. Let $G_1(z)$ be a holomorphic solution of* (1.7) *for $z \in \Gamma$, except possibly at $z = 0$. Let m be a positive number such that for $z \in \Gamma$, $z \to 0$, $G_1(z) = O(|z|^{-m})$ then $G_1(z) = G_A(z)$ for $z \in \Gamma$, where $G_A(z)$ exists by* (i).

Proof. Since $G_1(z)$, $G_A(z)$ satisfy (1.7) for $z \in \Gamma$, define

$$\Delta(z) = G_1(z) - G_A(z) \tag{2.10}$$

so that $\Delta(z)$ satisfies

$$z^s \Delta'(z) + b(z)\Delta(z) = 0. \tag{2.11}$$

The function $\Delta(\zeta) \exp\left(- \int_{\zeta}^{z} b(t)dt/t^s \right)$ satisfies (2.11) for every $\zeta \in \Gamma$, $\zeta \neq 0$, so by the existence and uniqueness theorem of differential equations it must coincide with $\Delta(z)$. By (i), $\lim_{\Gamma \ni \zeta \to 0} G_A(\zeta) = A_0$ so that $\Delta(\zeta) = O(|\zeta|^{-m})$. Using Lemma 1 we obtain, for every $z \in \Gamma$, $z \neq 0$, $\Delta(z) = 0$. We remark that the integration path is taken along Γ in the expression of $\Delta(z)$.

(iii) Every function $g(z, \zeta)$ is uniquely determined when restricted to any simply connected domain of existence not including $z = 0$. Our remark is true, in particular, if $z \in D_k^r$, and $\zeta \in \bar{D}_k^r$, $\zeta \neq 0$. By considering the explicit formula for $g(z, \zeta)$ we can show that $g(z, \zeta_n)$, $\zeta_n \to 0$ are Cauchy sequences. It is easily verified that

$$| g(z, \zeta_1) - g(z, \zeta_2) | \leqq | C_0 | \left| \exp\left(-\int_{\zeta^1}^{z} \frac{b(t)}{t^s}\, dt \right| + | C_0 | \left| \exp\left(-\int_{\zeta_2}^{z} \frac{b(t)}{t^s}\, dt \right) \right|$$

$$+ \left| \int_{\zeta_1}^{\zeta_2} \frac{c(t)}{t^s} \left[\exp\left(\int_{z}^{t} \frac{b(\eta)}{\eta^s}\, d\eta \right) \right] dt \right|. \qquad (2.12)$$

The first and second right-hand terms tend to zero by Lemma 1. The paths of integration in the third right-hand term can be taken at our convenience as long as they are imbedded in $\bar{D}_k^r$. In particular, choose the path from ζ_2 to ζ_1 to be a straight segment. Then the third right hand term is dominated by

$$| \zeta_2 - \zeta_1 | \max | c(t) | \left| \exp\left(-\operatorname{Re} \int_{t}^{z} \frac{b(\eta)}{\eta^s}\, d\eta \right) \right| | t |^{-s}.$$

By Lemma 1 our result then follows.

3

We are able now to state some functional results.

Proposition 7.

(i) *Let A and A' satisfy an L.D.E. of type* (1.7); *then*

$$G_A'(z) = G_{A'}(z)$$

in the intersection of the characteristic domains of A and A'.

(ii) *If $A \mathscr{R} B$ by* (1.9), *and A,B satisfy L.D.E.'s of type* (1.7), *then*

$$a_1(z) G_A(z) + b_1(z) G_B(z) = c_1(z) \qquad (3.1)$$

in the intersection of the characteristic domains of A,B and the three circles with centre at the origin where $a_1(z)$, $b_1(z)$, $c_1(z)$ are holomorphic.

Proof. (i) *Case one.* Assume A is an S.C.P. then $A = \sum_{n=0}^{\infty} A_n z^n$ is a holomorphic function. Since A satisfies (1.7), by proposition 6, $G_A(z)$ must coincide with $\sum_0^{\infty} A_n z^n$ in the characteristic domain of A. By a similar argument, $\sum_1^{\infty} n A_n z^{n-1}$ must coincide with the function $G_{A'}(z)$ in the characteristic domain of A', and so the result follows.

Case two. Assume A is S.C.Z., then if A satisfies (1.7), A' must satisfy the unique equation

$$z^s (A')' + \left[b(z) + s z^{s-1} - z^s \frac{b'(z)}{b(z)} \right] (A') = c'(z) - c(z) \frac{b'(z)}{b(z)}. \qquad (3.2)$$

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This follows from Propositions 3 and 4. Moreover, we conclude that the characteristic domain of A' is included in the characteristic domain of A. It is easily verified that, if $G_A(z)$ satisfies (1.7) in the characteristic domain of A, then $G'_A(z)$ satisfies (3.2) in the characteristic domain of A', and by Proposition 6, it must coincide with $G_{A'}(z)$ in the characteristic domain of A'.

(ii) *Case one.* If A is S.C.P., so is B. In this case $A = \sum_0^\infty A_n z^n$ must coincide with $G_A(z)$ (by Proposition 6 (ii)) in the characteristic domain of the L.D.E. satisfied by A, and $G_B(z)$ must coincide with $\sum_0^\infty B_n z^n$ in the characteristic domain of the L.D.E. of B. The expression

$$a_1(z)G_A(z)+b_1(z)G_B(z) = a_1(z)\sum_0^\infty A_n z^n + b_1(z)\sum_0^\infty B_n z^n$$

is defined in the intersection of the characteristic domains of the L.D.E.'s of A and B and the circles with centre at the origin where $a_1(z)$ and $b_1(z)$ are holomorphic and coincides with $c(z)$ in the common domain of existence of $a_1(z)$ and $b_1(z)$. Notice that the intersection domain may be empty.

Case two. If A is S.C.Z., it is easily verified that $a_1(z)A$ is S.C.Z. which satisfies a unique L.D.E.

$$z^s(a_1(z)A)' + \left[b(z) - z^s\frac{a_1'(z)}{a_1(z)}\right][a_1(z)A] = c(z)a_1(z). \tag{3.3}$$

Moreover, $a_1(z)G_A(z)$ satisfies (3.3) in the characteristic domain of (3.3) which is the intersection of the characteristic domain of (1.7) and the circle whose centre is the origin and in which $a_1(z)$ is holomorphic. We conclude that in the above domain

$$G_{a_1(z)A}(z) = a_1(z)G_A(z).$$

In the same manner we observe that $a_1(z) - b_1(z)B$ satisfies a unique L.D.E. whose characteristic domain is the intersection of the characteristic domain of the L.D.E. satisfied by B and the circles whose centres are situated at the origin and where $c_1(z)$ and $b_1(z)$ are holomorphic. In this domain we also observe that the function $a_1(z) - b_1(z)G_A(z)$ satisfies the L.D.E. of $c_1(z) - b_1(z)B$. Using Proposition 6 again, we must have

$$c_1(z) - b_1(z)G_B(z) = G_{c_1(z)-b_1(z)B}(z)$$

$$a_1(z)G_A(z) = G_{a_1(z)A}(z)$$

so that

$$G_{a_1(z)A}(z) = c_1(z) - b_1(z)G_B(z) \tag{3.4}$$

in the intersection of the domains of existence of both sides of (3.4), and the result follows. We are ready now to define our summability method. As a matter of fact, we may adopt one of three methods. We agree to define an L.D.E. of order zero as an algebraic equation in the following manner.

Definition 3. The power series A is said to satisfy an L.D.E. of order zero if $A = \sum_0^\infty a_n z^n$ where $\sum_0^\infty a_n z^n$ is S.C.P. and its characteristic domain will be the circle with centre at the origin and in which $\sum_0^\infty a_n z^n$ is absolutely convergent.

Definition 4. We say that $G_A(z)$ is the *summability function* of the power series A if:

(i) A satisfies an L.D.E. of lowest order (one or zero).

(ii) $G_A(z)$ is a solution of the above L.D.E. in its characteristic domain.

(iii) A is the asymptotic expansion of $G_A(z)$ in the characteristic domain.

Remark. The regularity property of this method is an immediate consequence of the definition.

Definition 5. We say that $G_A(z)$ is the *summability function* of A if:

(i) A satisfies an L.D.E. of type (1.7).

(ii) $G_A(z)$ is a solution of the above L.D.E. in the characteristic domain.

(iii) A is the asymptotic expansion of $G_A(z)$ in the characteristic domain.

Definition 6. We say that $G_A(z)$ is the *summability function* of A if:

(i) A satisfies an L.D.E. of type (1.7).

(ii) $G_A(z) = \lim_{D_k^r \ni \zeta \to 0,\, k = 0, 1, \dots s-2} g(z, \zeta)$

where z varies in the characteristic domain, and ζ varies in any proper closed domain contained in the characteristic domain and having $\zeta = 0$ in its closure, and $g(z, \xi)$ are the family of solutions of the above L.D.E. determined uniquely by the property $g(\zeta, \zeta) = C_0$.

Remarks. 1. Some of these results can be partially extended to L.D.E.'s of order n. It can also be shown that series which are formal solutions of L.D.E.'s of order n must satisfy necessary conditions which are " sharp ", that means that they cannot be improved.

2. Although it seems that these results have to do mainly with the theory of L.D.E. it will be shown elsewhere that there are a lot of matrix regular summability methods which sum series to the same functions obtained by our method, and all of them have common typical features.

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