

Subadditive mean ergodic theorems

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(Received 24 November 1980)

Abstract. The authors investigate which results of the classical mean ergodic theory for bounded linear operators in Banach spaces have analogues for subadditive sequences (F_n) in a Banach lattice B . A sequence (F_n) is subadditive for a positive contraction T in B if $F_{n+k} \leq F_n + T^n F_k$ ($n, k \geq 1$). For example, von Neumann's mean ergodic theorem fails to extend to the general subadditive case, but it extends to the non-negative subadditive case. It is shown that the existence of a weak cluster point $f = Tf$ for $(n^{-1}F_n)$ implies $\|(n^{-1}F_n - f)^+\| \rightarrow 0$. In L_p ($1 \leq p < \infty$) the existence of a weak cluster point for non-negative $(n^{-1}F_n)$ is equivalent with norm convergence. If T is an isometry in L_p ($1 < p < \infty$) and $\sup \{n^{-1}\|F_n\|_p\} < \infty$ then $n^{-1}F_n$ converges weakly. If T in L_1 has a strictly positive fixed point and $\sup \{n^{-1}\|F_n\|_1\} < \infty$ then $n^{-1}F_n$ converges strongly. Most results are proved even in the d -parameter case.

1. Introduction

In this article we attempt to analyse which results of the classical mean ergodic theory for bounded linear operators in Banach spaces have analogues for subadditive sequences F_n in Banach lattices B . A sequence $F_n \in B$ is called subadditive for a positive linear operator T in B if $F_{n+k} \leq F_n + T^n F_k$. We study the norm convergence of $n^{-1}F_n$ and its connection with weak convergence. Kingman's subadditive ergodic theorem states pointwise and L_1 -norm convergence for $n^{-1}F_n$ under the condition $\sup \{n^{-1}\|F_n\|_1: n \geq 1\} < \infty$, when T arises from a measure preserving transformation in a probability space.

Our study was mainly motivated by the following two questions. If, in Kingman's statement, the assumption $\sup \{n^{-1}\|F_n\|_2: n \geq 1\} < \infty$ is added, does convergence in the L_2 -norm hold? If T arises from a uniquely ergodic homeomorphism of a compact metric space and if the F_n are continuous and non-negative, does uniform convergence of $n^{-1}F_n$ hold (as in the additive case)? More generally, one can ask under which conditions on B and T norm convergence of $n^{-1}F_n$ holds when it holds for averages.

Although it turns out that the answers to both preceding questions are negative, as is shown by precise examples in § 6, several positive results can be obtained.

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Under a special condition on B which is satisfied by L_p -spaces ($1 \leq p < \infty$) but not by $C(X)$ or L_∞ , it is possible to extend the mean ergodic theorem to non-negative subadditive sequences when T is a positive contraction: that is the existence of a weak cluster point for $n^{-1}F_n$ is equivalent to the norm convergence. This is done in § 4. To achieve this we prove first in § 3 that for power-bounded T in a general Banach lattice the existence of a fixed weak cluster point f for $(n^{-1}F_n)$ implies $\lim \|(n^{-1}F_n - f)^+\| = 0$, (F_n) being any subadditive sequence.

In § 5, under the condition $\sup \{n^{-1}\|F_n\|\} < \infty$ we prove that $n^{-1}F_n$ converges weakly, when T is an isometry of L_p ($1 < p < \infty$), and that $n^{-1}F_n$ converges in norm, when $B = L_1$ and T is a positive contraction having a strictly positive fixed element.

Whenever possible our results are stated and proved directly for discrete d -parameter abelian semigroups of positive operators. A few results on a.e.-convergence are added, but in this direction several interesting questions remain unsolved.

2. Preliminaries

Let $d \geq 1$ be a fixed integer and let $\mathfrak{S} = \{0, 1, 2, \dots\}^d$ be the additive semigroup of d -dimensional vectors with non-negative integer coordinates. For $a = (a_i)$, $b = (b_i) \in \mathfrak{S}$ we write $a \leq b$ if $a_i \leq b_i$ ($i = 1, \dots, d$) and $a < b$ if $a_i < b_i$ ($i = 1, \dots, d$). \mathfrak{I} denotes the class of all non-empty intervals $[a, b) = \{u \in \mathfrak{S} : a \leq u < b\}$. For $A \subset \mathfrak{S}$ $|A|$ shall be the cardinality of A .

A linear operator T in a Banach lattice B with positive cone B^+ is called positive if $TB^+ \subset B^+$. Throughout $\mathfrak{T} = \{T_u, u \in \mathfrak{S}\}$ will be a semigroup of positive bounded linear operators in B ; in particular, we have $T_{u+v} = T_u \circ T_v$ ($u, v \in \mathfrak{S}$) and $T_0 = \text{identity}$, where $0 = (0, 0, \dots, 0)$ is the neutral element in \mathfrak{S} . T is called a contraction if $\|T\| \leq 1$.

A set function $F: \mathfrak{I} \ni I \rightarrow F_I \in B$ will be called a superadditive process (for \mathfrak{I}) if the following conditions are satisfied:

(2.1) $T_u F_I = F_{I+u}$ whenever $I \in \mathfrak{I}$ and $u \in \mathfrak{S}$.

(2.2) If I_1, I_2 are disjoint sets in \mathfrak{I} and if $I_1 \cup I_2$ is also in \mathfrak{I} then $F_{I_1 \cup I_2} \geq F_{I_1} + F_{I_2}$.

For short we write $F = \{F_I\}$. This definition does not contain the usual boundedness condition (see, for example, condition (1.1.3) in [13]) which is adapted to the case $B = L_1$. We shall impose boundedness conditions separately. In [2] the condition (2.2) was imposed for finitely many sets I_1, \dots, I_k since this was crucial for proofs of almost sure convergence, but here the slightly weaker formulation due to Smythe [15] is sufficient.

F is called subadditive if $\{-F_I\}$ is superadditive, i.e. (2.2) holds with \leq . F is called additive if it is both subadditive and superadditive.

For $n = (n_1, \dots, n_d) \in \mathfrak{S}$ let $\pi(n) = |[0, n)| = \prod_{\nu=1}^d n_\nu$ and $F_n = F_{[0, n)}$. $n \rightarrow \infty$ shall mean that n_ν tends to infinity for $\nu = 1, \dots, d$.

3. Partial convergence for general subadditive processes

An element $f \in B$ is called a fixed point for $\mathfrak{T} = \{T_u : u \in \mathfrak{S}\}$ or \mathfrak{T} -invariant if $T_u f = f$ holds for all $u \in \mathfrak{S}$. \mathfrak{T} is called bounded if $\sup \{\|T_u\| : u \in \mathfrak{S}\} < \infty$.

THEOREM 3.1. *If $F = \{F_I\}$ is a subadditive process for a bounded d -parameter semigroup \mathfrak{X} of positive linear operators in a Banach lattice B and the closed convex hull of $\{|I|^{-1}F_I : I \in \mathfrak{I}\}$ contains a fixed point f for \mathfrak{X} , then $\|(\pi(n)^{-1}F_n - f)^+\|$ tends to zero as $n \rightarrow \infty$.*

Proof. For any $\varepsilon > 0$ there exist real numbers $\alpha_1, \alpha_2, \dots, \alpha_N \geq 0$ with $\sum_{\nu=1}^N \alpha_\nu = 1$ and intervals $I(\nu) = [a(\nu), b(\nu)) \in \mathfrak{I}$ such that

$$\left\| \sum_{\nu=1}^N \alpha_\nu |I(\nu)|^{-1} F_{I(\nu)} - f \right\| < \varepsilon.$$

Put

$$\mathfrak{S}(\nu) = \{t = (t_1, \dots, t_d) \in \mathfrak{S} : t_i \in (b_i(\nu) - a_i(\nu)) \cdot \mathbb{Z}^+\}.$$

If f_n, g_n are vectors in B ($n \in \mathfrak{S}$) then $f_n \leq g_n + o(n)$ shall mean that there exist vectors $h_n \in B$ such that $f_n \leq g_n + h_n$ and $\pi(n)^{-1} \|h_n\| \rightarrow 0$ as $n \rightarrow \infty$. The subadditivity together with the boundedness of \mathfrak{X} yield for each $k \in [0, b(\nu) - a(\nu))$ and for each $\nu = 1, \dots, N$ an estimate

$$F_n \leq T_k \left(\sum_{\substack{u \in \mathfrak{S}(\nu) \\ u < n - k - b(\nu)}} T_u F_{I(\nu)} \right) + o(n).$$

Adding over all such k we obtain

$$|I(\nu)| F_n \leq \sum_{\substack{u \in \mathfrak{S} \\ u < n - b(\nu)}} T_u F_{I(\nu)} + o(n) = \sum_{\substack{u \in \mathfrak{S} \\ u < n}} T_u F_{I(\nu)} + o(n).$$

It follows that

$$F_n - \pi(n)f \leq \sum_{\substack{u \in \mathfrak{S} \\ u < n}} T_u \left(\sum_{\nu=1}^N \alpha_\nu |I(\nu)|^{-1} F_{I(\nu)} - f \right) + o(n),$$

or that

$$(\pi(n)^{-1}F_n - f)^+ \leq \pi(n)^{-1} \sum_{\substack{u \in \mathfrak{S}_1 \\ u < n}} T_u \left(\sum_{\nu=1}^N \alpha_\nu |I(\nu)|^{-1} F_{I(\nu)} - f \right)^+ + \pi(n)^{-1} o(n).$$

Let $M = \sup \{\|T_u\| : u \in \mathfrak{S}_1\}$. As the sum on the right-hand side has $\pi(n)$ terms and each of them has norm at most $M\varepsilon$, we obtain that, for large n ,

$$\|(\pi(n)^{-1}F_n - f)^+\| \leq M\varepsilon + \varepsilon.$$

As ε was arbitrary the assertion follows. □

Note that, by Mazur's theorem, the assumption on f in theorem 3.1 is satisfied if f is a weak cluster point of $\{|I|^{-1}F_I : I \in \mathfrak{I}\}$.

4. Non-negative subadditive processes

We now discuss the convergence of $\pi(n)^{-1}F_n$ for non-negative subadditive processes. We shall need some assumptions on the structure of the Banach lattice B . B is called countably order complete (COC) if, for any non-empty countable subset $E \subset B$ which is majorized by an element of B , $\sup E$ exists. We say that B has an order continuous norm (OCN) if B is COC and any decreasing sequence in B^+ is

norm-convergent. It has been pointed out by Ghossoub & Steele [11, prop. 1] that a COC Banach lattice B has an order continuous norm if and only if, for every sequence $f_n \in B^+$ ($n \in \mathbb{N}$) satisfying $f_{n+k} \leq f_n + f_k$ the sequence, $n^{-1}f_n$ norm-converges to $\inf n^{-1}f_n$.

THEOREM 4.1. *Let B be a Banach lattice with order continuous norm and let $\mathfrak{T} = \{T_u, u \in \mathfrak{S}\}$ be a bounded d -parameter semigroup of positive linear operators in B . If $F = \{F_I\}$ is subadditive and $F_I \geq 0$, then the weak convergence of $\pi(n)^{-1}F_n$ ($n \rightarrow \infty$) to a fixed point f implies the strong convergence.*

Proof. For any h in the dual B^* of B , $h(\pi(n)^{-1}F_n - f)$ tends to zero. By theorem 3.1 $h((\pi(n)^{-1}F_n - f)^+)$ also tends to zero. Thus $(\pi(n)^{-1}F_n - f)^-$ must tend to zero weakly. We have $0 \leq (\pi(n)^{-1}F_n - f)^- \leq f$. By a lemma of Heinich [12] (see also [11]) a Banach lattice B with order continuous norm has the following property: if $0 \leq x_n \leq x \in B$ and x_n converges weakly to x , then x_n converges strongly to x . We apply this lemma with $x_n = f - (\pi(n)^{-1}F_n - F)^-$ and $x = f$. □

In the additive case even the weak convergence to f of a subsequence of $\pi(n)^{-1}F_n$ implies the strong convergence of $\pi(n)^{-1}F_n$; we shall prove below that this can be obtained here if B satisfies a stronger property. The following example shows that the order-continuity of the norm is not sufficient to yield such a result:

Example 4.2. The space $B = C(X)$ with $X = \{0, 1\}$ has an order continuous norm. The operator T defined by $Tf(0) = f(0)$, $Tf(1) = f(0)$ is a positive contraction. For any sequence ε_i ($i = 0, 1, \dots$) of zeros and ones the family F_n ($n \in \mathbb{N}$) defined by $F_n(0) = n$, $F_n(1) = \sum_{i=0}^{n-1} \varepsilon_i$ determines a non-negative subadditive process for $\mathfrak{T} = \{T^n, n \geq 0\}$.

It is easy to choose the ε_i s in such a way that $n^{-1}F_n$ converges to the (T -invariant) constant 1 along a subsequence but not along the entire sequence $n = 1, 2, \dots$.

We shall now prove the desired stronger form of theorem 4.1 and several related interesting propositions for Banach lattices with uniformly monotone norm. Following the terminology of Birkhoff [5] the norm of a Banach lattice B is called uniformly monotone if

for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $f, g \in B$ the properties $0 \leq f \leq g$, $\|f\| = 1$, and $\|g\| \leq 1 + \delta$ imply $\|g - f\| < \varepsilon$.

A Banach lattice having a uniformly monotone norm is called a UMB-lattice. The class of UMB-lattices contains the spaces L_p ($1 \leq p < \infty$) and also the class of Banach lattices having uniformly convex norms. It is an exercise to show that UMB-lattices have an order continuous norm. In the sequel we shall use the following equivalent formulation of the definition of uniform monotonicity:

(4.1) for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $f, g \in B$ the properties $0 \leq f \leq g$, $1 - \delta \leq \|f\|$ and $\|g\| \leq 1 + \delta$ imply $\|f - g\| < \varepsilon$.

We shall need

THEOREM 4.3. *Let $\mathfrak{T} = \{T_u : u \in \mathfrak{S}\}$ be a d -parameter semigroup of positive contractions acting on a UMB-lattice B . Let $\{F_I\}$ be a non-negative subadditive process for \mathfrak{T} .*

For any $w \in \mathfrak{S}$ we have

$$\lim_{n \rightarrow \infty} \pi(n)^{-1} \|F_{n+w} - F_n\| = 0$$

and

$$\lim_{n \rightarrow \infty} \pi(n)^{-1} \|F_n - T_w F_n\| = 0.$$

Proof. It is clear that it is enough to prove the assertions for $w = e(\nu) = (e_1(\nu), e_2(\nu), \dots)$ with $e_i(\nu) = \delta_{i\nu}$ ($1 \leq i, \nu \leq d$) where $\delta_{i\nu}$ is the Kronecker symbol. To prove the first assertion we observe that any non-negative subadditive process satisfies

$$F_{n+e(\nu)} \leq F_n + \sum_{\substack{v < n+e(\nu) \\ \nu_v = n_\nu}} T_v F_e$$

with $e = (1, 1, \dots, 1)$. Using the contraction property of the operators T_u we can conclude

$$\begin{aligned} \pi(n)^{-1} \|(F_{n+e(\nu)} - F_n)^+\| &\leq \pi(n)^{-1} \left\| \sum_{\substack{v < n+e(\nu) \\ \nu_v = n_\nu}} T_v F_e \right\| \\ &\leq \pi(n)^{-1} \cdot (n_1 \cdot n_2 \cdots n_{\nu-1} \cdot n_{\nu+1} \cdots n_d) \|F_e\| \rightarrow 0. \end{aligned}$$

Applying the identity $a = a \wedge b + (a - b)^+$ we see that

$$(4.2) \quad F_{n+e(\nu)} = (F_{n+e(\nu)} \wedge F_n) + (F_{n+e(\nu)} - F_n)^+$$

and

$$\|F_{n+e(\nu)} - F_n\| \leq \|F_n - (F_{n+e(\nu)} \wedge F_n)\| + \|(F_{n+e(\nu)} - F_n)^+\|,$$

so that it is enough to prove $\pi(n)^{-1} \|F_n - (F_{n+e(\nu)} \wedge F_n)\| \rightarrow 0$.

As the operators T_u are contractions, the function $I \rightarrow \|F_I\|$ is subadditive. Hence $\lim \pi(n)^{-1} \|F_n\| = \inf \{\pi(n)^{-1} \|F_n\| : n \in \mathfrak{S}\} = \gamma \geq 0$. If $\gamma = 0$ the theorem is obvious. If not we may and do assume that $\gamma = 1$. Given $\varepsilon > 0$ find $\delta > 0$ from (4.1). From $1 \leq \pi(n + e(\nu))\pi(n)^{-1} \rightarrow 1$ and (4.2) it follows that

$$1 \leq \pi(n + e(\nu))^{-1} \|F_{n+e(\nu)}\| \leq \pi(n)^{-1} \|F_{n+e(\nu)} \wedge F_n\| + \pi(n)^{-1} \|(F_{n+e(\nu)} - F_n)^+\|.$$

Therefore, taking n large enough, we shall have

$$1 - \delta \leq \pi(n)^{-1} \|F_{n+e(\nu)} \wedge F_n\|$$

and

$$1 \leq \pi(n)^{-1} \|F_n\| \leq 1 + \delta.$$

By the choice of δ this yields $\pi(n)^{-1} \|F_n - (F_{n+e(\nu)} \wedge F_n)\| < \varepsilon$ and the first assertion is proved.

The proof of the second assertion is similar. We have

$$F_{n+e(\nu)} \leq T_{e(\nu)} F_n + \sum_{\substack{v < n+e(\nu) \\ \nu_v = 0}} T_v F_e$$

which implies $\pi(n)^{-1} \|(F_{n+e(\nu)} - T_{e(\nu)} F_n)^+\| \rightarrow 0$. By the same argument as above it is enough to prove that

$$\pi(n)^{-1} \|F_{n+e(\nu)} - (F_{n+e(\nu)} \wedge T_{e(\nu)} F_n)\| \rightarrow 0.$$

By

$$1 \leq \pi(n + e(\nu))^{-1} \|F_{n+e(\nu)}\|$$

$$\leq \pi(n + e(\nu))^{-1} \|F_{n+e(\nu)} \wedge T_{e(\nu)} F_n\| + \pi(n + e(\nu))^{-1} \|(F_{n+e(\nu)} - T_{e(\nu)} F_n)^+\|$$

we shall obtain

$$1 - \delta \leq \|F_{n+e(\nu)} \wedge T_{e(\nu)} F_n\|$$

for large enough n , and then

$$\pi(n + e(\nu))^{-1} \|(F_{n+e(\nu)} \wedge T_{e(\nu)} F_n) - F_{n+e(\nu)}\| < \varepsilon.$$

Together with the first assertion this proves the second assertion. □

Remarks. The theorem is not true if B is not a UMB-lattice. This may be seen with the help of example 4.2. If $n^{-1} \sum_{i=0}^{n-1} \varepsilon_i$ does not converge to 1 then $n^{-1} \|F_n - TF_n\|$ fails to converge to zero. To show that the first assertion may also fail, consider the identity operator T on $B = l_\infty(\mathbb{N})$ and $F_n = n 1_{[n, \infty[}$. Even if $B = L_2$ we also cannot replace the condition $\|T_u\| \leq 1$ in the theorem by the condition $\sup \{\|T_u\| : u \in \mathfrak{S}\} < \infty$. To see this consider again example 4.2, but replace the sup-norm by the L_2 -norm. Then we have $\|T\| = 2^{1/2}$ and $T^n = T$ for all n . We also cannot replace the condition $F_n \geq 0$ by the weaker condition $\sup n^{-1} \|F_n\| < \infty$ even if T is an isometry and $B = L_2$. This will be shown with the help of example 6.1.

We are now ready to prove the subadditive mean ergodic theorem for UMB-lattices.

THEOREM 4.4. *Let $\mathfrak{T} = \{T_u, u \in \mathfrak{S}\}$ be a d -parameter semigroup of positive contractions in a UMB-lattice B and $\{F_I : I \in \mathfrak{I}\}$ a non-negative subadditive process for \mathfrak{T} . For $f \in B$ the following assertions are equivalent:*

- (i) $f = \lim_{n \rightarrow \infty} \pi(n)^{-1} F_n$ in the norm topology.
- (ii) f is a weak cluster point of $\pi(n)^{-1} F_n$ when $n \rightarrow \infty$.
- (iii) f is fixed under \mathfrak{T} and it belongs for all $v \in \mathfrak{S}$ to the closed convex hull of $\{\pi(u)^{-1} F_u : u \geq v\}$.

Proof. The implication (i) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (iii). By theorem 4.3, if f is a weak cluster point of $\pi(n)^{-1} F_n$ where $n \rightarrow \infty$ then f is fixed under \mathfrak{T} . By the Mazur theorem f belongs for all $v \in \mathfrak{S}$ to the closed convex hull of $\{\pi(u)^{-1} F_u : u \geq v\}$.

(iii) \Rightarrow (i). Since $\lim_{n \rightarrow \infty} \pi(n)^{-1} \|F_n\| = \inf \{\pi(u)^{-1} \|F_u\| : u \in \mathfrak{S}\} = \gamma \geq 0$ we have $\|f\| \leq \gamma$.

By theorem 3.1 we have $\|(\pi(n)^{-1} F_n - f)^+\| \rightarrow 0$. Since $\pi(n)^{-1} F_n = ((\pi(n)^{-1} F_n) \wedge f) + (\pi(n)^{-1} F_n - f)^+$ it is enough to prove that $\|f - (\pi(n)^{-1} F_n) \wedge f\| \rightarrow 0$. If $\gamma = 0$ we have finished, if not we may again assume $\gamma = 1$. Given $\varepsilon > 0$ choose $\delta > 0$ according to the UMB-condition (4.1). Since

$$1 \leq \pi(n)^{-1} \|F_n\| \leq \|\pi(n)^{-1} F_n \wedge f\| + \|(\pi(n)^{-1} F_n - f)^+\|,$$

we shall have

$$1 - \delta \leq \|(\pi(n)^{-1} F_n) \wedge f\| \leq \|f\| \leq 1$$

for large enough $n \in \mathfrak{S}$. The UMB-condition now yields $\|\pi(n)^{-1}F_n \wedge f - f\| < \varepsilon$, which proves the theorem. \square

Our next aim is to derive some consequences of theorem 4.4. The example $B = L_1$ shows that $\pi(n)^{-1}F_n$ need not converge even in the additive case. We want to show that in our situation additive convergence implies subadditive convergence. To do this we need the following proposition which may also be of independent interest.

For $u = (u_1, \dots, u_d)$, $w = (w_1, \dots, w_d) \in \mathfrak{S}$ we employ the notation

$$u \circ w = (u_1 w_1, u_2 w_2, \dots, u_d w_d).$$

PROPOSITION 4.5. *Let B be a Banach-lattice with order continuous norm and let $\mathfrak{T} = \{T_u, u \in \mathfrak{S}\}$ be a bounded d -parameter semigroup of positive linear operators in B . Then for any $w > 0$ and $f \in B^+$ the following assertions are equivalent:*

- (i) $\pi(n)^{-1} \sum_{v < n} T_v f$ converges strongly when $n \rightarrow \infty$.
- (ii) $\pi(n)^{-1} \sum_{v < n} T_{w \circ v} f$ converges strongly when $n \rightarrow \infty$.

Proof. The implication (ii) \Rightarrow (i) follows easily from

$$\sum_{t < w} T_t \left(\sum_{v < n} T_{w \circ v} f \right) = \sum_{v < n \circ w} T_v f$$

and holds in any Banach space without any positivity or norm-conditions.

(i) \Rightarrow (ii). Since $f \geq 0$, $\sum_{v < n} T_{w \circ v} f \leq \sum_{v < n \circ w} T_v f$. Hence with $h_n = \pi(n)^{-1} \sum_{v < n} T_{w \circ v} f$ and $g_n = \pi(w)\pi(n \circ w)^{-1} \sum_{v < n \circ w} T_v f$ we find that $0 \leq h_n \leq g_n$. By assumption, g_n is converging strongly to some $g \in B^+$ when $n \rightarrow \infty$. As B has an order continuous norm, the set $\{h \in B^+ : h \leq g\}$ is weakly compact by theorem 5.10 in [14]. It follows that the sequence $h_n \wedge g$ has a weak cluster point and $(h_n - g)^+$ converges strongly to 0. Hence $(h_n \wedge g) + (h_n - g)^+$ has a weak cluster point when $n \rightarrow \infty$. From the general mean ergodic theorem [9] this is enough to imply (ii). \square

In general, the implication (i) \Rightarrow (ii) is not true even for positive contractions in a Banach lattice B without the condition that B has an order continuous norm. To see this one can take $B = l_\infty(\mathbb{N})$ and $Tf(k) = f(k + 1)$: it is not difficult to build a sequence $(\varepsilon_i)_{i \in \mathbb{N}}$ of zeros and ones such that the averages $n^{-1} \sum_{i=j}^{j+n-1} \varepsilon_i$ converge uniformly in j but for which the implication (i) \Rightarrow (ii) is false.

In this example the convergence holds only for certain elements $f \in B$. We do not know if for any positive contraction in an arbitrary Banach lattice the strong convergence of $n^{-1} \sum_{i=0}^{n-1} T^i f$ for all f implies the strong convergence of $n^{-1} \sum_{i=0}^{n-1} T^{2i} f$ for all f . At any rate such a conclusion is not true for arbitrary contractions in a Banach space. Consider the subspace B of $L_1(\mathbb{Z})$ (with the counting measure) consisting of all elements f with $\sum_{k \text{ odd}} f(k) = - \sum_{k \text{ even}} f(k)$. If T is again the operator induced by the translation we have $n^{-1} \sum_{k=0}^{n-1} T^k f \rightarrow 0$ for all f , but there are elements $f \in B$ for which $n^{-1} \sum_{k=0}^{n-1} T^{2k} f$ does not converge.

THEOREM 4.6. *Let $\mathfrak{X} = \{T_u : u \in \mathfrak{G}\}$ be a d -parameter semigroup of positive contractions acting on a UMB-lattice B . Let $\{F_t\}$ be a non-negative subadditive process for \mathfrak{X} . If $\pi(n)^{-1} \sum_{v < n} T_v F_e$ converges strongly in B when $n \rightarrow \infty$ then $\pi(n)^{-1} F_n$ also converges strongly. Furthermore, the following strong limits exist:*

$$(4.3) \quad \lim_{n \rightarrow \infty} \pi(n)^{-1} \sum_{v < n} T_{w \circ v} F_w =: \hat{F}_w \quad (w \in \mathfrak{G}).$$

$$(4.4) \quad \lim_{w \rightarrow \infty} \pi(w)^{-1} \hat{F}_w = \lim_{n \rightarrow \infty} \pi(n)^{-1} F_n.$$

Proof. From the definition of a non-negative subadditive process we obtain $0 \leq F_n \leq \sum_{v < n} T_v F_e$. If $\pi(n)^{-1} \sum_{v < n} T_v F_e$ converges strongly the argument used in the proof of proposition 4.5 shows that $\pi(n)^{-1} F_n$ has a weak cluster point. Thus, by theorem 4.4, $\pi(n)^{-1} F_n$ converges strongly to some $f \in B$. By the preceding proposition $\pi(n)^{-1} \sum_{v < n} T_{w \circ v} F_e$ converges strongly for all $w \in \mathfrak{G}$. Since $0 \leq \sum_{v < n} T_{w \circ v} F_w \leq \sum_{v < n} T_{w \circ v} (\sum_{t < w} T_t F_e)$ we can again use the same argument to infer that the sequence $\pi(n)^{-1} \sum_{v < n} T_{w \circ v} F_w$ has a weak cluster point. Hence the limits \hat{F}_w exist for all w . Let $\varepsilon > 0$ be given. If $\|f\| = 0$ we have, for large enough w , $\pi(w)^{-1} \|F_w\| < \varepsilon$ and therefore $\|\hat{F}_w\| < \varepsilon \pi(w)$. In this case both limits in (4.4) are zero. In the case $\|f\| > 0$ we may assume $\|f\| = 1$. Find $\delta > 0$ from the UMB-condition (4.1). If w is so large that $\pi(w)^{-1} \|F_w\| < 1 + \delta$ we have $\pi(w)^{-1} \|\hat{F}_w\| < 1 + \delta$. Now

$$\begin{aligned} f &= \lim_{n \rightarrow \infty} \pi(n)^{-1} F_n = \lim_{n \rightarrow \infty} \pi(n \circ w)^{-1} F_{n \circ w} \leq \lim_{n \rightarrow \infty} \pi(w)^{-1} \pi(n)^{-1} \sum_{v < n} T_{v \circ w} F_w \\ &= \pi(w)^{-1} \hat{F}_w \end{aligned}$$

implies that $\|f - \pi(w)^{-1} \hat{F}_w\| < \varepsilon$. □

Remarks. (1) The examples given after theorem 4.3 show that in the preceding theorem neither the assumption UMB nor the assumption $\|T_u\| \leq 1$ can be removed.

(2) In theorem 4.3 the assumption of convergence $\pi(n)^{-1} \sum_{v < n} T_v F_e$ is satisfied if there is a \mathfrak{X} -invariant $p \in B^+$ with $\|(F_e - kp)^+\| \rightarrow 0$ ($k \rightarrow \infty$). In this case for all $0 \leq g \leq kp$ the averages $\pi(n)^{-1} \sum_{v < n} T_v g$ converge strongly since they belong to the weakly compact order interval $[0, kp]$ and F_e can be approximated by such g s. If B is a function space it therefore suffices that there exists a strictly positive \mathfrak{X} -invariant $p \in B$.

5. Subadditive processes in L_p

We now drop the condition that the considered subadditive process $\{F_t\}$ be non-negative, but we assume $B = L_p$ ($1 \leq p < \infty$). We shall prove theorems on norm-convergence for $p = 1$ and on weak convergence for $1 < p < \infty$ and $d = 1$. We first recall the following additive mean ergodic theorem for $p = 1$.

THEOREM 5.1. *Let $\mathfrak{X} = \{T_u, u \in \mathfrak{G}\}$ be a semigroup of (not necessarily positive) contractions in $L_1 = L_1(\mu)$. Assume that there exists a strictly positive $h \in L_1$ such that*

$|T_u f| \leq h$ whenever $u \in \mathfrak{S}$ and $|f| \leq h$. Then for any $f \in L_1$ there exists a \mathfrak{I} -invariant $\bar{f} \in L_1$ such that $\pi(n)^{-1} \sum_{u < n} T_u f$ converges strongly to \bar{f} .

This follows from the general mean ergodic theorem [9], if one observes that f can be approximated by $f_k = ((-kh) \vee f) \wedge (kh)$, because all averages $\pi(n)^{-1} \sum_{u < n} T_u f_k$ lie in the weakly compact order interval $[-kh, kh]$. Alternatively, theorem 5.1 can be deduced from theorem 9 of Dunford & Schwartz [8, p. 679]: passing to the new reference measure $h \cdot \mu$, i.e. using the isomorphism of $L_1(h \cdot \mu)$ and $L_1(\mu)$, we may assume that the L_1 -norm and the L_∞ -norm of the operators T_u is ≤ 1 , and therefore also the L_2 -norm. One can then approximate f with L_2 -elements.

In the subadditive case it is natural to assume that the operators T_u are positive. For non-negative F_I we have then already obtained a positive result combining theorem 5.1 with theorem 4.6. For general F_I the assumption on h must be strengthened as may be seen by considering the operators $T_u f = 0$ ($u \neq 0$). The following theorem has been proved by Kingman ($d = 1$) [13] and Smythe [15] in the case $T_u f = f \cdot \tau_u$, where the τ_u are measure preserving transformations in a probability space. Our proof is different.

THEOREM 5.2. *Let $\mathfrak{I} = \{T_u, u \in \mathfrak{S}\}$ be a d -parameter semigroup of positive contractions in L_1 . Assume that there exists a strictly positive \mathfrak{I} -invariant $h \in L_1$. If $\{F_I\}$ is subadditive and $\inf \left\{ \pi(n)^{-1} \int F_n d\mu : n \in \mathfrak{S} \right\} > -\infty$, then $\pi(n)^{-1} F_n$ converges in L_1 -norm to a \mathfrak{I} -invariant limit \bar{f} .*

Proof. It is convenient to prove the result for the superadditive process $\{-F_I\}$ instead; we therefore assume that $\{F_I\}$ is superadditive and $\sup \left\{ \pi(n)^{-1} \int F_n d\mu : n \in \mathfrak{S} \right\} < \infty$. Subtracting the additive process $\sum_{u \in I} T_u F_e$, for which theorem 5.1 applies, we may further assume $F_I \geq 0$. The existence of a strictly positive T_u -invariant h implies that the operators T_u are conservative and therefore Markovian, i.e. $\int T_u g d\mu = \int g d\mu$ holds for all $g \geq 0$. Therefore the d -parameter family of real numbers $\gamma_n = \int F_n d\mu$ ($n \in \mathfrak{S}$) is superadditive, so that $\pi(n)^{-1} \gamma_n$ converges to

$$\gamma = \sup \{ \pi(n)^{-1} \gamma_n : n \in \mathfrak{S} \} \quad \text{as } n \rightarrow \infty.$$

For $k \in \mathbb{N}$, $\nu = 1, \dots, d$ and $n = (n_1, \dots, n_d) \in \mathfrak{S}$ let $h_\nu(n, k) = [n_\nu/k]$ be the largest integer $\leq n_\nu/k$. Put $N(n, k) = k^d h_1(n, k) \cdots h_d(n, k)$. Then $\pi(n)^{-1} N(n, k) \rightarrow 1$ ($n \rightarrow \infty$) for fixed k . If $I_{k,n}$ denotes the interval

$$[0, (k \cdot h_1(n, k), k \cdot h_2(n, k), \dots)]$$

then the superadditivity and non-negativity of $\{F_I\}$ yield

$$F_{I_{k,n}} \geq \sum_{i_1=0}^{h_1(n,k)-1} \cdots \sum_{i_d=0}^{h_d(n,k)-1} T_{(ki_1, \dots, ki_d)} F_e \cdot k.$$

Denote the right-hand side by $\sum_{k,n}$. For any $\varepsilon > 0$ we may find k with $k^{-d} \gamma_{ke} > \gamma - \varepsilon$.

By theorem 5.1, $N(n, k)^{-1} \sum_{k,n}$ converges in L_1 -norm to some $g_k \in L_1$ invariant under T_{ke} .

As all T_u are Markovian we have

$$\int \sum_{k,n} d\mu = h_1(n, k) \cdots h_d(n, k) \int F_{ke} d\mu \geq (\gamma - \varepsilon)N(n, k).$$

It follows that $\int g_k d\mu \geq (\gamma - \varepsilon)$. Using $\pi(n)^{-1}N(n, k) \rightarrow 1$ we obtain $\pi(n)^{-1} \sum_{k,n} \rightarrow g_k$.

Therefore $F_n \geq \sum_{k,n}$ implies that

$$\int (\pi(n)^{-1}F_n - g_k)^- d\mu \leq \int (\pi(n)^{-1} \sum_{k,n} - g_k)^- d\mu < \varepsilon$$

holds for large enough n . On the other hand,

$$\int (\pi(n)^{-1}F_n - g_k) d\mu \leq \gamma - \int k^{-d}F_{ke} < \gamma - (\gamma - \varepsilon) = \varepsilon.$$

Together this yields $\|\pi(n)^{-1}F_n - g_k\|_1 < 3\varepsilon$.

If k, l are two integers with $k\gamma_{ke} > \gamma - \varepsilon$ and $l\gamma_{le} > \gamma - \varepsilon$ we obtain that $\|g_k - g_l\|_1 < 6\varepsilon$. Thus the sequence g_k is a Cauchy-sequence and converges in L_1 to some \bar{f} with

$$\int \bar{f} d\mu = \gamma. \text{ Clearly } \|\pi(n)^{-1}F_n - \bar{f}\|_1 \rightarrow 0.$$

It remains to prove the \mathfrak{T} -invariance of \bar{f} . Instead of working with F_{ke} above we may do the same with $F_u, u = (u_1, u_2, \dots, u_d) \in \mathfrak{S}$, when $u_\nu \geq 1 (\nu = 1, \dots, d)$. We then obtain functions g_u invariant under T_u . Let $w \in \mathfrak{S}$ be fixed. For large enough u we have $\|g_u - \bar{f}\|_1 < \varepsilon$ and $\|g_{u+w} - \bar{f}\|_1 < \varepsilon$. Hence

$$\begin{aligned} \|T_w \bar{f} - \bar{f}\|_1 &\leq \|T_w \bar{f} - T_w g_u\|_1 + \|T_w g_u - T_{u+w} g_{u+w}\|_1 + \|g_{u+w} - \bar{f}\|_1 \\ &\leq \varepsilon + 2\varepsilon + \varepsilon. \end{aligned}$$

Thus \bar{f} is \mathfrak{T} -invariant. □

Remarks on almost-sure convergence. We say that $n \in \mathfrak{S}$ tends to infinity in a restricted way, and write $n(\rightarrow)\infty$, if there exist $0 < \alpha < \beta < \infty$ such that $\alpha \leq n_i/n_j \leq \beta$ holds for all n under consideration and all $1 \leq i, j \leq d$. The main result on a.e.-convergence for d -parameter semigroups of contractions in L_1 in the additive case is:

THEOREM 5.3. *Let $\mathfrak{T} = \{T_u, u \in \mathfrak{S}\}$ be a d -parameter semigroup of contractions in $L_1(\mu)$ (μ σ -finite) such that $\|T_u f\|_\infty \leq \|f\|_\infty$ holds for $u \in \mathfrak{S}, f \in L_\infty \cap L_1$. Then the averages $\pi(n)^{-1} \sum_{u < n} T_u f$ converge a.e. for all $f \in L_1$ as $n(\rightarrow)\infty$.*

This is a slight variant of a theorem of Brunel [6]. Brunel has studied only the case $n_1 = n_2 = \dots = n_d$, but his maximal inequality is sufficient to yield theorem 5.3 without change of the argument. If the assumption on the L_∞ -norm is replaced by the assumption of existence of a strictly positive $h \in L_1$ with $|f| \leq h \Rightarrow |T_u f| \leq h$, the assertion of theorem 5.3 remains valid because we can pass to $L_1(h \cdot \mu)$. In the superadditive case we only have:

PROPOSITION 5.4. *If $\mathfrak{T} = \{T_u, u \in \mathfrak{S}\}$ is a d -parameter semigroup of positive contractions in $L_1(\mu)$, $\{F_I\}$ superadditive, $\sup \left\{ \pi(n)^{-1} \int F_I d\mu : n \in \mathfrak{S} \right\} < \infty$, and there exists a strictly positive \mathfrak{T} -invariant $h \in L_1$, then*

$$(5.1) \quad \liminf_{n \rightarrow \infty} \pi(n)^{-1} F_n = \bar{f} \quad \text{a.e.}$$

where \bar{f} is the L_1 -limit.

Proof. It follows from L_1 -convergence that $\liminf \leq \bar{f}$. To prove \geq we may again assume $F_I \geq 0$. For all k , $N(k, n)^{-1} \sum_{k,n}$ tends to g_k a.e. by theorem 5.3 as $n \rightarrow \infty$.

Because $F_n \geq \sum_{k,n}$ and $\pi(n)^{-1} N(k, n) \rightarrow 1$ we obtain

$$\liminf_{n \rightarrow \infty} \pi(n)^{-1} F_n \geq g_k$$

for all k . Now $\bar{f} = \lim_k g_k$ yields the proposition. □

In the 1-parameter case one can use theorem 2.1 of Akcoglu & Sucheston [4] on the existence of exact dominants to show that the limit in (5.1) actually exists a.e. Extending Kingman's theorem ($d = 1$) to $d \geq 1$ Akcoglu & Krengel have recently shown [2] that $\pi(n)^{-1} F_n$ converges a.e. as $n \rightarrow \infty$ if $\{F_I\}$ is superadditive for operators T_u given by measure preserving transformations τ_u and $T_u f = f \circ \tau_u$, provided – of course – that $\sup \left\{ \pi(n)^{-1} \int F_n d\mu \right\} < \infty$. The case of Markovian operators T_u with $\|T_u f\|_\infty \leq \|f\|_\infty$ ($f \in L_1 \cap L_\infty, u \in \mathfrak{S}$) seems to require completely new methods.

In the 1-parameter case the following observation may be an interesting complement to the results of Akcoglu & Sucheston [4]. Let T be a positive contraction in L_1 and let $\{F_I\}$ be subadditive and non-negative for $\mathfrak{T} = \{T^k, k \geq 0\}$. In this case

$$\sum_{n=1}^{\infty} (F_{n+1} - F_n)^+ \leq \sum_{n=1}^{\infty} T^n F_e < \infty$$

on the dissipative part D , and this proves the a.e.-convergence of the sequence F_n on D . Together with the results in [4] this implies that $F_n / \sum_{i=0}^{n-1} T^i p$ converges a.e. on $\left\{ \sum_{i=0}^{\infty} T^i p > 0 \right\}$ for $p \in L_1^+$, i.e. the full statement of the Chacon–Ornstein theorem holds for such processes without the restrictions that T be Markovian or $\{F_I\}$ additive on D .

Fava [10] has proved unrestricted convergence in theorem 5.3 under stronger integrability conditions, which are needed even in the case of measure preserving transformations. Smythe [15] has discussed the difficulties connected with unrestricted a.e.-convergence for subadditive processes.

Weak convergence in L_p ($1 < p < \infty$). We now consider positive contractions in L_p and study questions of a.e.-convergence and weak convergence. We require the following lemma due to Akcoglu & Sucheston [3, proposition 3.4 and corollary 3.5.]:

LEMMA 5.5. Let $(\Omega, \mathcal{F}, \mu)$ be a σ -finite measure space, $1 < p < \infty$, and let T be a positive contraction in L_p . Then Ω decomposes uniquely (mod μ) into two sets E and E^c having the following properties:

- (i) E is the support of a T -invariant function $h \in L_p^+$ and the support of any such function is contained in E .
- (ii) E and E^c are invariant sets, i.e. the subspaces $L_p(E)$ and $L_p(E^c)$ are both invariant under T .

THEOREM 5.6. Let $(\Omega, \mathcal{F}, \mu)$ be σ -finite, $1 < p < \infty$, and let $\{F_I\}$ be a superadditive process for the positive contraction T in L_p satisfying $\sup \{n^{-1}\|F_n\|_p : n \in \mathbb{N}\} < \infty$. Then on the part E defined in lemma 5.5, $n^{-1}F_n$ converges a.e. If T is even a positive isometry then $n^{-1}F_n$ converges weakly in L_p .

Proof. Because of Akcoglu’s ergodic theorem [1] and the mean ergodic theorem we may subtract the additive process $\sum_{u \in I} T^u F_1$ and assume $F_I \geq 0$. It is clear from (ii) in the lemma that the processes $\{1_E F_I\}$ and $\{1_{E^c} F_I\}$ are then both non-negative and superadditive with $\sup \{n^{-1}\|1_E F_n\|_p\} < \infty$ and $\sup \{n^{-1}\|1_{E^c} F_n\|_p\} < \infty$. It therefore suffices to discuss the cases $\Omega = E$ and $\Omega = E^c$ separately. In the first case there exists a strictly positive T -invariant $h \in L_p$. Let m denote the measure $h^p \mu$, i.e. $m(A) = \int h^p d\mu$. It is a finite measure equivalent to μ . Given $f \in L_\infty(m)$ define $Sf = h^{-1}T(f \cdot h)$. The operator S is linear and positive and we have $S1 = 1$. The function h satisfies $\int h \cdot T^*h^{p-1} d\mu = \int Th \cdot h^{p-1} d\mu = \|h\|_p^p$. Since h^{p-1} is the unique function g in $L_q = L_{p/p-1}$ such that $\int hg d\mu = \|h\|_p^p$, we see that $T^*h^{p-1} = h^{p-1}$. Hence

$$\int Sf dm = \int h^{p-1}T(fh) d\mu = \int fh^p d\mu = \int f dm$$

and S can be extended as a Markovian operator in $L_1(m)$. Moreover, $S^n f = h^{-1}T^n(fh)$, and $\{h^{-1}F_I\}$ is non-negative and superadditive for $\{S^n, n \in \mathbb{N}\}$. Because of

$$\|h^{-1}F_n\|_{L_1(m)} \leq \|h\|_{L_p(\mu)}^{1/p-1} \|F_I\|_{L_p(\mu)},$$

the new process satisfies the boundedness condition $\sup \{n^{-1}\|h^{-1}F_n\|_{L_1(m)}\} < \infty$. Akcoglu & Sucheston’s extension of Kingman’s theorem [4] now asserts the a.e.-convergence of $n^{-1}h^{-1}F_n$ and the first statement is proved.

The weak convergence statement in the case $\Omega = E$ follows from the a.e.-convergence and $\sup \{n^{-1}\|F_n\|_p\} < \infty$.

It remains to discuss the case $\Omega = E^c$. By assumption all the sequences $((n+j)^{-1}F_{n+j})_{n \geq 1} (j \geq 0)$ are norm-bounded in L_p . Let g_0 be a weak cluster point of $(n^{-1}F_n)$. By the diagonal procedure there exists a strictly increasing sequence of integers (n') such that, for any $j \geq 0$, the weak limit g_j of $(n'+j)^{-1}F_{n'+j}$ in L_p exists. We have $\|g_j\|_p \leq \sup \{ \|n^{-1}F_n\|_p \}$ and $g_j \geq 0$. From $F_{k+1} \geq F_k + T^k F_1$ and $\lim_k k^{-1}\|T^k F_1\|_p = 0$ we obtain $g_{j+1} \geq g_j (j \geq 0)$. Therefore the strong limit g of (g_j) exists in L_p . Since the strong continuity of T implies the weak continuity, we obtain

$g_{j+1} \geq Tg_j$ from $F_{k+1} \geq F_1 + TF_k$ and $\lim k^{-1}\|F_1\| = 0$. Taking the limit we obtain $g \geq Tg$. As we have assumed that T is an isometry in the second assertion, $g = Tg$. Since the support of any T -invariant $g \geq 0$ was contained in E and now $\Omega = E^c$ we must have $g = 0$, and also $g_0 = 0$ since $g_0 \leq g$. Therefore the sequence $n^{-1}F_n$ has only 0 as a weak cluster point in L_p . This proves the weak convergence to 0. \square

Remarks. Using the operator $Tf = 0$ ($f \in L_p$) it is easy to see that the second assertion does not hold for all contractions. The norm-convergence may fail in both cases considered above as will be shown in the next section. The question of a.e.-convergence in the case $\Omega = E^c$ remains unsolved. In the additive case Akcoglu [1] has proved a *strong* maximal inequality: if T is a positive contraction in L_2 there is a constant $c_p < \infty$ such that $\|\sup n^{-1}F_n\|_p \leq c_p\|F_1\|_p$. In the superadditive non-negative case one might hope for $\|\sup n^{-1}F_n\|_p \leq c_p\delta_p$ with $\delta_p = \sup \|n^{-1}F_n\|_p$, but this fails even in the case where the operator is given by a measure preserving transformation. It is even possible that $\delta_2 < \infty$ and $\mu(\{\sup n^{-1}F_n > 1\}) = \infty$.

6. Main counterexamples

In this section we collect the main examples promised in the introduction and in the previous sections.

Example 6.1. We first show that there exists a measure preserving transformation τ in a probability space $(\Omega, \mathcal{F}, \mu)$ and a non-negative superadditive process $\{F_I\}$ for $\mathcal{I} = \{\tau^n, n \geq 0\}$ such that $\sup \left\{ n^{-1} \int F_n d\mu : n \geq 1 \right\}$ and $\sup \{n^{-1}\|F_n\|_2 : n \geq 1\}$ are finite but $n^{-1}F_n$ does not converge in L_2 -norm.

We shall inductively define a sequence of integers $n_1 < n_2 < \dots$. The space Ω shall be the union of the spaces $\Omega_k = \{(k, i) : 1 \leq i \leq 2^k n_k\}$. Put $\tau(k, i) = (k, i - 1)$ for $i > 1$ and $\tau(k, 1) = (k, 2^k n_k)$, i.e. τ permutes Ω_k cyclically. The measure will be given by $\mu(\{(k, i)\}) = 2^{-2^k n_k^{-1}}$. Let f_k be the function with $f_k((k, 1)) = 2^k n_k, f_k((j, i)) = 0$ otherwise. Once we know n_k expressions like $n^{-1} \sum_{i=0}^{n-1} f_k \circ \tau^i$ make sense even though the definition of Ω is not yet complete. This average will be zero on the undefined part of Ω . Start with $n_1 = 1$. When n_k is defined find $n_{k+1} > n_k$ so that

$$\left\| n^{-1} \sum_{i=0}^{n-1} f_j \tau^i - 1_{\Omega_j} \right\|_2 < 2^{-k}$$

for all $n \geq n_{k+1}, j \leq k$. Put

$$F_n = \sum_{i=0}^{n-1} (f_1 + \dots + f_k) \circ \tau^i$$

when $n_k \leq n < n_{k+1}$. Using $F_I \circ \tau^{-i} = F_{I+k}$ this defines all F_I and the process $\{F_I\}$ is superadditive. $\int f_k d\mu = 2^{-k}$ implies the finiteness of $\sup \left\{ n^{-1} \int F_n d\mu \right\}$. To show that the sequence $n^{-1}\|F_n\|_2$ is bounded first note that, for $n \geq n_k$ and $j \leq k - 1$, we have

$$\left\| n^{-1} \sum_{i=0}^{n-1} f_j \circ \tau^i \right\|_2 \leq \|1_{\Omega_j}\|_2 + 2^{-k} \leq 2^{-j/2} + 2^{-k}$$

which implies

$$(6.1) \quad \left\| n^{-1} \sum_{j=1}^{k-1} \sum_{i=0}^{n-1} f_j \circ \tau^i \right\|_2 \leq \sum_{j=1}^{\infty} 2^{-j/2} + 1 < \infty.$$

To estimate the contribution of f_k we use that, for $n_k \leq n \leq 2^k n_k$,

$$(6.2) \quad n^{-1} \left\| \sum_{i=0}^{n-1} f_k \circ \tau^i \right\|_2 = n^{-1} (n \cdot 2^{2k} n_k^2 2^{-2k} n_k^{-1})^{1/2} = (n^{-1} n_k)^{\frac{1}{2}} \leq 1.$$

For general n with $n_k \leq n < n_{k+1}$ we can put the sum $\sum_{i=0}^{n-1} f_k \circ \tau^i$ together from pieces with at least n_k and at most $2^k n_k$ terms. Therefore

$$\left\| n^{-1} \sum_{i=0}^{n-1} f_k \circ \tau^i \right\|_2 \leq 1.$$

Together with the estimate (6.1) this proves the boundedness of the sequence $n^{-1} \|F_n\|_2$. It follows from

$$\begin{aligned} n_k^{-1} \|F_{n_k} - F_{n_{k-1}}\|_2 &= n_k^{-1} \left\| (f_1 + \dots + f_{k-1}) \circ \tau^{n_k-1} + \sum_{i=0}^{n_k-1} f_k \circ \tau^i \right\|_2 \\ &\geq n_k^{-1} \left\| \sum_{i=0}^{n_k-1} f_k \circ \tau^i \right\|_2 = 1 \end{aligned}$$

that the sequence $n^{-1} F_n$ does not converge in L_2 .

The process $\{-F_l\}$ is subadditive. Therefore this example also shows that the first assertion in theorem 4.3 does not remain true if the non-negativity condition for the subadditive process is replaced by the boundedness condition

$$\sup \{ \pi(n)^{-1} \|F_n\| : n \in \mathfrak{S} \} < \infty.$$

It is easy to give also an example to this extent for the second assertion in theorem 4.3. Just define F_n as above for $n \neq n_k$ and for $n = n_k$ put

$$F_n = \sum_{i=0}^{n-1} (f_1 + \dots + f_{k-1}) \circ \tau^i + \sum_{i=0}^{\lceil n/2 \rceil - 1} f_k \circ \tau^{2i}.$$

In the above example τ is not ergodic. It is not difficult to give a similar example with an ergodic τ in $[0, 1]$ with Lebesgue-measure. The necessary sets Ω_k are then replaced by $2^k n_k$ disjoint images of suitable sets E_k obtained with the help of Rohlin's lemma.

Example 6.2. In the terminology of lemma 5.5, example 6.1 shows that for non-negative superadditive F_n the sequence $n^{-1} F_n$ need not converge in L_2 in the case $\Omega = E$. To find an example with $\Omega = E^c$ take $\Omega = \mathbb{R}^1$, $\mu =$ Lebesgue-measure, and let τ be given by $\tau\omega = \omega + 1$. Start with some $f_1 \geq 0$ with $\|f_1\|_1 \leq 2^{-1}$ and $\|f_1\|_2 = 1$.

There exists an $n_1 \in \mathbb{N}$ such that $n \geq n_1$ implies $\left\| n^{-1} \sum_{k=0}^{n-1} f_1 \circ \tau^k \right\|_2 < 2^{-1}$. Find $f_2 \geq 0$

with $\|f_2\|_1 \leq 2^{-2}$, $\left\| (n_1 + 1)^{-1} \sum_{k=0}^{n_1} f_2 \circ \tau^k \right\|_2 = 1$ and $\left\| n^{-1} \sum_{k=0}^{n-1} f_2 \circ \tau^k \right\|_2 \leq 1$ for $n \geq n_1 + 1$.

When $n_1, \dots, n_k, f_1, \dots, f_{k+1}$ have been constructed, find n_{k+1} such that $n \geq n_{k+1}$

implies $\left\| n^{-1} \sum_{i=0}^{n-1} (f_1 + \dots + f_{k+1}) \circ \tau^i \right\|_2 < 2^{-(k+1)}$. Then find $f_{k+2} \geq 0$ with

$$\|f_{k+1}\|_1 < 2^{-(k+1)}, \left\| (n_{k+1} + 1) \sum_{i=0}^{n_{k+1}} f_{k+2} \circ \tau^i \right\|_2 = 1$$

and

$$\left\| n^{-1} \sum_{i=0}^{n-1} f_{k+2} \circ \tau^i \right\|_2 \leq 1 \quad (n \geq n_{k+1} + 1).$$

Put $F_n = \sum_{i=0}^{n-1} (f_1 + \dots + f_{k+1}) \circ \tau^i$ for $n_k < n \leq n_{k+1}$. It is easy to check the desired properties of the process.

Example 6.3. If τ is a uniquely ergodic transformation (see [7]) in a compact metric space Ω and $f: \Omega \rightarrow \mathbb{R}$ continuous, then $n^{-1} \sum_{i=0}^{n-1} f \circ \tau^i$ converges uniformly. The following example shows that this classical result does not extend to non-negative subadditive processes, i.e. we construct a non-negative subadditive process $\{F_I\}$, for which the F_I are continuous, but $n^{-1}F_n$ fails to converge uniformly.

Let τ be any uniquely ergodic transformation in a compact metric space Ω such that the unique invariant probability measure μ is non-atomic. We construct a sequence of continuous non-negative functions f_0, f_1, \dots and a sequence $0 = n_0 < n_1 < \dots$ of integers. Let $\sum_{n,k} = \sum_{i=0}^{n-1} (f_0 - f_1 - f_2 - \dots - f_k) \circ \tau^i$. We shall put $F_n = \sum_{n,k}$ for $n_k \leq n < n_{k+1}$.

We start with $f_0 \equiv 1, n_1 = 1$ and any continuous f_1 with $0 \leq f_1 \leq 2^{-1}$ and $\int f_1 d\mu < 2^{-2}$. After the k th step we shall know n_k and f_k and we shall have $0 \leq f_k \leq 2^{-1} (1 \leq k)$ and $\int f_k d\mu < 2^{-(k+1)}$. We find $n_{k+1} > n_k$ so large that $n \geq n_{k+1}$ implies

$$\left| n^{-1} \sum_{n,k} - \int (f_0 - f_1 - \dots - f_k) d\mu \right| < 2^{-(k+2)}$$

uniformly on Ω . In particular, for $n \geq n_{k+1}$, we then have that $n^{-1} \sum_{n,k} > 2^{-1}$ on Ω . With the help of Rohlin's lemma we now find a measurable set E_k with $\mu(E_k) > 0$ such that the sets $\tau^{-i}E_k (i = 0, 1, 2, \dots, n_{k+1}2^{(k+2)})$ are disjoint.

Let $U_{k+1} = \bigcup_{i=0}^{n_{k+1}^{-1}-1} \tau^{-i}E_k$. f_{k+1} shall be any continuous function with $0 \leq f_{k+1} \leq \frac{1}{2}$ which approximates $2^{-1}1_{U_{k+1}}$ in L_1 -norm so closely that $\int f_{k+1} d\mu < 2^{-(k+2)}$ and $n_{k+1}^{-1} \sum_{i=0}^{n_{k+1}-1} f_{k+1} \circ \tau^i$ is $\geq 4^{-1}$ on a set of positive measure. This implies

$$|F_{n_{k+1}-1} - F_{n_{k+1}}| \geq 4^{-1}n_{k+1} - (k + 1).$$

We may assume $4^{-1}n_{k+1} - (k + 1) \geq 8^{-1}n_{k+1}$. Thus $n^{-1}F_n$ cannot converge uniformly.

This research was conducted during a visit of the first named author to the University of Göttingen.

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