SOME TORSION-FREE GROUP RINGS WITH NILPOTENT PRIME RADICALS

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1. Introduction

Let R be a ring with identity. We will use $J(R)$ and $P(R)$ to denote the Jacobson and prime radicals of R, respectively. If G is a group, the group ring of G over R will be denoted by RG.

In this paper, we will prove the following results:

THEOREM 1. *If R is a left Goldie ring (with identity) and G the infinite cyclic group, then $P(RG)$ is nilpotent and*

$$P(RG) = J(RG) = P(R)G = NG,$$

where $N = J(R[X]) \cap R$.

THEOREM 2. *If R is left Noetherian and G is torsion-free abelian, then $P(RG)$ is nilpotent and $P(RG) = P(R)G$.*

The first result is an analogue of a theorem of Amitsur on polynomial rings [1, page 358, Theorem 1]:

If R is a ring, then $J(R[X]) = N[X]$, where $N = J(R[X]) \cap R$ is nil. Furthermore, N contains the locally nilpotent radical $\sigma(R)$ of R, that is, the largest ideal of R whose finitely generated subrings are nilpotent.

In fact, we will make use of this result to prove Theorem 1.

2. Proof of the theorems

In this section, unless otherwise stated, R will always denote a left Goldie ring and G the infinite cyclic group.

LEMMA 1. $P(R) = \sigma(R) = N$.

PROOF. By the result of Amitsur quoted above, $\sigma(R) \subseteq N$ and N is nil, hence N is nilpotent, since R is left Goldie. Trivially, every finitely generated subring of N is nilpotent, thus $\sigma(R) = N$ by the maximality of $\sigma(R)$ with respect to the property that finitely generated subrings be nilpotent. Similarly, $P(R)$ is nil, so it is nilpotent and one infers in like manner that $P(R) = \sigma(R) = N$. This proves Lemma 1.

LEMMA 2. $J(RG) = NG$.

PROOF: Since N is nilpotent, so is NG. Therefore $NG \subseteq J(RG)$.

To show the reverse inclusion, let $a \in J(RG)$, $a \neq 0$. Let g be a generator of G. Then

$$a = g^n f(g)$$

for some integer n and for some $f(g) \in R[g]$, the polynomial ring of g considered as a variable over R, and the constant term of $f(g)$ is not zero. Then

$$gf(g) \in J(RG).$$

We want to show that $gf(g) \in J(R[g])$. To this end, it suffices to show that $gf(g)R[g]$ is a right quasi regular right ideal of $R[g]$. Let $h(g) \in R[g]$, then

$$gf(g)h(g) \in J(RG).$$

Hence there exists $g^r k(g) \in RG$ such that

$$gf(g)h(g) + g^r k(g) + gf(g)g^r k(g) = 0$$

where r is an integer and $k(g) \in R[g]$ has non-zero constant term. We claim that $r \geqq 0$. For if $r < 0$, then

$$g^{1-r} f(g)h(g) + k(g) + gf(g)k(g) = 0.$$

It would follow that the constant term of $k(g)$ is zero, a contradiction. Thus

$$g^r k(g) \in R[g].$$

This shows that $gf(g)R[g]$ is right quasi regular, so it is contained in $J(R[g]) = N[g]$. It follows that the coefficients of $gf(g)$ and hence of $f(g)$ are in N. This proves the reverse inclusion.

We remark that $J(RG) \subseteq NG$ is always valid for any ring with identity.

LEMMA 3. $P(RG) = P(R)G = NG$.

PROOF: Since $P(R)$ is nilpotent, so is $P(R)G$. Thus $P(R)G \subseteq P(RG) \subseteq J(RG) = NG = P(R)G$ by Lemma 2 and Lemma 1. Hence they are all equal.

This also completes the proof of Theorem 1.

We now prove Theorem 2. Let R be left Noetherian (with identity) and G be torsion-free abelian.

We first note that $P(R)$ is nilpotent, since R is left Noetherian. Hence $P(R)G$ is nilpotent, thus

$$P(R)G \subseteq P(RG).$$

We now assume that G is free abelian of finite rank n. If $n = 1$, then we have nothing to prove by Theorem 1. Let $n > 1$. Assume that the assertion is valid for all free abelian groups of ranks less than n. Let $G = HK$, where H is free abelian of rank $n - 1$ and K is infinite cyclic. Then RH is Noetherian, hence

$$P[(RH)K] = P(RH)K.$$

Also, by induction hypothesis,

$$P(RH) = P(R)H.$$

However, since

$$RG = R(HK) = (RH)K,$$

we have

$$\begin{aligned} P(RG) &= P[(RH)K] \\ &= [P(RH)]K = [P(R)H]K = P(R)(HK) = P(R)G. \end{aligned}$$

This proves the assertion for the case G is of finite rank.

Now let G be arbitrary. We are left to prove that $P(RG) \subseteq P(R)G$. Let $x \in P(RG)$. Write

$$x = x_1 g_1 + \cdots + x_n g_n$$

where $x_i \neq 0$ for all i and $g_i \neq g_j$ for $i \neq j$. Let G_0 be the subgroup of G generated by $g_1, \cdots, g_n$. Then G_0 is free abelian of finite rank and $x \in RG_0$. By the previous paragraph,

$$P(RG_0) = P(R)G_0.$$

However, $x \in P(RG)$ implies that x is strongly nilpotent in RG, in particular, x is strongly nilpotent in RG_0 ([3], page 55). Hence $x \in P(RG_0)$. Thus $x \in P(R)G_0 \subseteq P(R)G$. This completes the proof of Theorem 2.

3. Some related questions

It is not known to the author whether Theorem 2 is true for left Goldie rings. The answer would be affirmative if we know that R is left Goldie and G is infinite cyclic imply that RG is left Goldie.

In the sequel, unless otherwise stated, R will denote an arbitrary ring with identity, G the infinite cyclic group, $M = J(RG) \cap R$ and $N = J(R[X]) \cap R$.

As we noted before, $J(RG) \subseteq NG$. Since N is nil, so $N \subseteq J(R)$, it follows that $NG \subseteq J(R)G$. Trivially, $M \subseteq J(RG)$, thus $MG \subseteq J(RG)$. In summary, we have

PROPOSITION 4. $MG \subseteq J(RG) \subseteq NG \subseteq J(R)G$.

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Corollary 5. *If $J(R) = (0)$ and G is torsion-free abelian, then $J(RG) = (0)$.*

Proof: We may assume that G is finitely generated. Then G is of finite rank, say n. If $n = 1$, then the claim is obvious by Proposition 4. We may now complete the proof by induction on n.

We would like to know when will

$$MG = J(RG) = NG.$$

We note that if $J(RG) = NG$, then $N \subseteq J(RG)$. Thus

$$N = N \cap R \subseteq J(RG) \cap R = M.$$

Hence $M = N$ and so $MG = J(RG) = NG$. We further note that if $N = \sigma(R)$, then NG is nil, therefore $NG \subseteq J(RG)$, and so $MG = J(RG) = NG$. For instance if R is commutative or left Goldie, then $\sigma(R) = N$. Thus we have proved the following interesting

Proposition 6. *If R is commutative or left Goldie, then $MG = J(RG) = NG$.*

The author is unaware of any ring for which the three ideals above are not identical.

We now consider an example. Let p be a fixed prime and R the ring of all rational numbers whose denominators are not divisible by p. Then $J(R) = pR \neq (0)$, since pR is the set of all non-units of R and it forms the unique maximal ideal of R. However, $J(R[X]) = (0)$, so $N = (0)$. Thus, $(0) = NG \subsetneqq J(R)G$.

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