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SUBDIRECT PRODUCTS OF E-INVERSIVE SEMIGROUPS

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Abstract

A semigroup S is called E-inversive if for every $a \in S$ there is an $x \in S$ such that $(ax)^2 = ax$. A construction of all E-inversive subdirect products of two E-inversive semigroups is given using the concept of subhomomorphism introduced by McAlister and Reilly for inverse semigroups. As an application, E-unitary covers for an E-inversive semigroup are found, in particular for those whose maximum group homomorphic image is a given group. For this purpose, the explicit form of the least group congruence on an arbitrary E-inversive semigroup is given. The special case of full subdirect products of a semilattice and a group (that is, containing all idempotents of the direct product) is investigated and, following an idea of Petrich, a construction of all these semigroups is provided. Finally, all periodic semigroups which are subdirect products of a semilattice or a band with a group are characterized.

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1. Introduction

A semigroup S is called *E-inversive* if for every $a \in S$ there exists $x \in S$ such that $ax \in E_S$, the set of all idempotents of S. This concept was introduced by G. Thierrin (1955) and was studied by Petrich (1967) and in a somewhat different form by Lallement and Petrich (1966). This class of semigroups recently reappeared in a paper by Hall and Munn (1985). The special case of E-inversive semigroups with commuting idempotents, called *E-dense*, was considered by Margolis and Pin (1987). We note that the definition is not one-sided (see Thierrin (1955)).

LEMMA 1. *Let S be a semigroup; if for all $a \in S$ there is an $x \in S$ such that $ax \in E_S$, then there exists $y \in S$ with ay, $ya \in E_S$.*

PROOF. Put $y = xax$; then $ay = axax = ax \in E_S$ and

$$(ya)^2 = (xaxa)(xaxa) = x(ax)^3a = (xax)a = ya,$$

hence $ya \in E_S$.

Consequently, one can say that S is E-inversive if and only if $I(a) = \{x \in S | ax, xa \in E_S\} \neq \varnothing$ for every $a \in S$.

EXAMPLES. (1) Regular semigroups ($a = axa$ implies that $ax \in E_S$).

(2) Periodic semigroups and in particular, finite semigroups ($a^n \in E_S$, $n > 1$, implies that $a \cdot a^{n-1} \in E_S$).

(3) Eventually regular semigroups (a^n regular for some $n > 1$ implies that $a^n x a^n = a^n$ and $a(a^{n-1}x) \in E_S$ for some $x \in S$).

(4) Bruck-semigroups over monoids are E-inversive, since for every $(m, a, n) \in S$: $(m, a, n)(n + 1, (a\alpha)^{-1}, m + 1) = (m + 1, e, m + 1) \in E_S$. (Note that S is regular if and only if the monoid T is regular.)

(5) Generalized Brandt-semigroups over a semigroup are E-inversive, since for every $(i, a, j) \in S$ there is $(k, b, l) \in S$ $(k \neq j)$ such that $(i, a, j)(k, b, l) = 0 \in E_S$, if $|I| > 1$.

(6) Rees-matrix semigroups over a group with zero and *arbitrary* sandwich matrix, $S = \mathscr{M}^0(I, G, M; P)$, are E-inversive, since for arbitrary $(i, a, \lambda) \in S$ we have: if there is some $k \in I$ with $p_{\lambda k} = 0$, then $(i, a, \lambda)(k, a, \lambda) = 0 \in E_S$; if $p_{\lambda_j} \neq 0$ for all $j \in I$ then $(i, a, \lambda)(i, p_{\lambda i}^{-1} a^{-1} p_{\lambda i}^{-1}, \lambda) \in E_S$. (Note that S is regular if and only if P is regular).

(7) Semigroups S which contain idempotents and are totally ordered with respect to the *natural partial order*

$$a \leq b \text{ if and only if } a = xb = by, \quad xa = a = ay \quad \text{for some } x, y \in S^1,$$

(see Mitsch (1986)) are E-inversive. In fact, if $a \in S$ and $a \geq e$ for some $e \in E_S$, then $e = xa = ay$ and $ay \in E_S$; if $a \leq e$, then $a = xe = ey$, $xa = a$ implies that $a^2 = (xe)(ey) = x(ey) = xa = a$ hence $aa = a \in E_S$.

(8) Let B be a band, T a regular semigroup of endomorphisms of B; on $S = B \times T$ define the multiplication $(e, \alpha)(f, \beta) = (e\alpha(f), \alpha \circ \beta)$. S is E-inversive since for any $(e, \alpha) \in S$ there exist $\beta \in T$ with $\alpha \circ \beta \circ \alpha = \alpha$ and $f = \beta(e) \in B$ such that $(e, \alpha)(f, \beta) \in E_S$. (Note that S is not regular in general; it suffices to consider for B the two element chain and for T the semigroup consisting of the two constant mappings of B.)

The following propositions give some elementary facts on E-inversive semigroups. The first of them is known (see Petrich (1973a), p. 18).

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Proposition 2. *If for every element a of a semigroup S there is exactly one $x \in S$ such that $ax \in E_S$, then X is a group.*

Proof. Suppose that there are two idempotents $e, f \in E_S$; then there exists $x \in S$ such that $efx \in E_S$. Since $e \cdot e = e \in E_S$ we conclude by hypothesis that $fx = e \in E_S$, and again since $f \cdot f = f \in E_S$, that $x = f$; hence $e = fx = f^2 = f$. Next, let $a \in S$; then by Lemma 1 there exists $y \in S$ such that $ya \in E_S = \{e\}$. Since $ya = e$ then $yae = e^2 = e \in E_S$ and by hypothesis $ae = a$. Since for every $a \in S$ there is an $x \in S$ with $ax \in E_S = \{e\}, S$ is a group.

Proposition 3. *Let S be an E-inversive semigroup without zero; then the following are equivalent:*

(i) *S is weakly cancellative (that is, $ax = bx$ and $xa = xb$ imply $a = b$);*
(ii) *S is trivially ordered with respect to its natural partial order;*
(iii) *S is completely simple.*

Proof. (i) implies (ii). Let $a \leq b$ for some $a, b \in S$; then $a = xb = by$, $xa = a = ay (x, y \in S^1)$. Thus $yxa = ya = yxb$ and $ayx = ax = byx$; by hypothesis we conclude $a = b$.

(ii) implies (iii). Let $a \in S$; then $ax \in E_S$ for some $x \in S$ and $axa = (ax)a = a(xa)$ with $(ax)axa = axa$ and $axa(xa) = axa$. Thus $axa \leq a$ in the natural partial order; by hypothesis, $axa = a$, that is, S is regular. Since by hypothesis every idempotent of S is primitive, we conclude that S is completely simple (see Petrich (1973a)).

(iii) implies (i). This follows by elementary calculation in the representation of S as a Rees matrix semigroup over a group.

Proposition 4. *Let S be an E-inversive semigroup. Then the following are equivalent:*

(i) *S is left cancellative;*
(ii) *S is a right group;*
(iii) *S is trivially ordered with respect to its natural order and $E_S a \subseteq aS$ for all $a \in S$.*

Proof. (i) implies (ii). Let $a \in S$ and $x \in I(a)$; then $xa \in E_S$ and $E_S \neq \emptyset$. We show that S is right simple; for every $b \in S$, $e \in E_S$ we have $eb = e(eb)$, thus by hypothesis, $b = eb$. Let $a \in S$, $a' \in I(a)$; then $aa' \in E_S$ and we have $b = a(a'b)$. Hence S is a right group.

(ii) implies (iii). This follows by calculation in the representation of S as the direct product of a group and a right zero semigroup.

(iii) implies (i). If $xa = xb$ then $x'xa = x'xb$ for some $x' \in I(x)$. So $x'xa = ay$ and $x'xb = bz$ for some $y, z \in S$, by hypothesis. Hence $c = x'xa = ay$ and $c = x'xb = bz$ so $c \leq a$, $c \leq b$ (see Mitsch (1986)). Thus $c = a = b$.

Lallement and Petrich (1966) considered the case with zero. They defined a semigroup S with zero to be *0-inversive* if for every $a \in S$, $a \neq 0$, there is some $x \in S$ such that $ax \in E_S \backslash \{0\}$, and *weakly 0-cancellative* if $ax = bx \neq 0$, $ya = yb \neq 0$ imply $a = b$.

PROPOSITION 5. *Let S be an 0-inversive semigroup. Then the following are equivalent:*

(i) *S is weakly 0-cancellative;*
(ii) *S is primitive regular;*
(iii) *$S \backslash \{0\}$ is trivially ordered with respect to its natural partial order.*

PROOF. (i) if and only if (ii). This equivalence was shown by Lallement and Petrich (1966).

(ii) implies (iii). This follows by calculations in the representation of S as 0-direct sum of completely 0-simple semigroups (that is, Rees matrix semigroups over groups with zero).

(iii) implies (ii). Let $a \in S$, $a \neq 0$; then $ax \in E_S \backslash \{0\}$ for some $x \in S$. Since $axa \leq a$ and $axa \neq 0$, we conclude $axa = a$, hence S is regular. Since every nonzero idempotent of S is primitive, we obtain that S is primitive regular.

2. Subdirect products

It is easy to see that every direct product of E-inversive semigroups is again E-inversive. So is every homomorphic image, but not every subsemigroup of an E-inversive semigroup (in the group $(\mathbb{Z}, +)$ the subsemigroup $(\mathbb{N}, +)$ is not E-inversive).

A semigroup isomorphic with a subsemigroup H of the direct product of two semigroups S, T is called *subdirect product of S and T* if the two projections $\pi_1 \colon H \to S$, $\pi_1(s, t) = s$, and $\pi_2 \colon H \to T$, $\pi_2(s, t) = t$ are surjective. It is well-known, that a semigroup H is a subdirect product of two semigroups S and T if and only if there are congruences ρ, τ on H such that $\rho \cap \tau = \varepsilon$; in this case $S \cong H/\rho$ and $T \cong H/\tau$. Since S and T are homomorphic images of H we have:

LEMMA 6. *If H is an E-inversive semigroup which is a subdirect product of two semigroups S and T, then S and T are E-inversive.*

Conversely, a subdirect product of two E-inversive semigroups is *not* E-inversive in general. Let $S = T = (\mathbb{Z}, +)$ and H be the subsemigroup of $S \times T$ generated by $\{(1,1),(-1,-3)\}$. Then H is a subdirect product of S and T which are E-inversive; but H is not, because $(-1,-1)$ is the only solution of $(1,1)+(x,y)=(0,0)$, but does not belong to H.

The problem of when a subdirect product of two inverse semigroups is again inverse, was considered by McAlister and Reilly (1977) and Petrich and Reilly (1983). In the first paper mentioned the concept of subhomomorphism of inverse semigroups was introduced in order to construct all subdirect products of two inverse semigroups which are inverse.

The appropriate generalization to the E-inversive case is the following.

DEFINITION. Let S, T be two E-inversive semigroups. A mapping $\alpha: S \to \mathscr{P}(T)$ (the power set of T) is called *surjective subhomomorphism of S onto T* if the following conditions are satisfied:

(i) $s\alpha \neq \emptyset$ for all $s \in S$;
(ii) $(s_1\alpha)(s_2\alpha) \subseteq (s_1s_2)\alpha$ for all $s_1, s_2 \in S$;
(iii) $\bigcup_{s\in S} s\alpha = T$;
(iv) for every $t \in s\alpha$ $(s \in S, t \in T)$ there are

$$s' \in I(s), \ t' \in I(t) \quad \text{such that} \quad t' \in s'\alpha.$$

The existence of surjective subhomomorphisms will be seen to be a necessary and sufficient condition for subdirect products of two E-inversive semigroups to be again E-inversive.

THEOREM 7. *Let S and T be E-inversive semigroups and let α be a surjective subhomomorphism of S onto T. Then $\pi(S,T,\alpha) = \{(s,t) \in S \times T \mid t \in s\alpha\}$ is an E-inversive semigroup which is a subdirect product of S and T. Conversely, every E-inversive semigroup which is a subdirect product of S and T can be obtained in this way.*

PROOF. *Sufficiency.* $\pi = \pi(S,T,\alpha)$ is a semigroup by condition (ii). It is a subdirect product of S and T by (i) and (iii). It is E-inversive, since $(s,t) \in \pi$ implies $t \in s\alpha$, so that by condition (iv) there are $s' \in I(s)$, $t' \in I(t)$ with $t' \in s'\alpha$, hence $(s',t') \in \pi$ and $(s,t)(s',t') = (ss',tt') \in E_\pi$.

Necessity. For any E-inversive semigroup H which is a subdirect product of S and T define $\alpha: S \to \mathscr{P}(T)$ by $s\alpha = \{t \in T \mid (s,t) \in H\}$ for every $s \in S$.

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Since H is a subdirect product of S and T, for every $s \in T$ there is some $t \in T$ with $(s,t) \in H$, hence $t \in s\alpha$ and (i) is satisfied. (ii) holds since H is a subsemigroup of $S \times T$, and (iii) holds because it is a subdirect product of S and T. Finally, let $t \in s\alpha (s \in S, t \in T)$. Then $(s,t) \in H$, so that there is $(x,y) \in H$ such that $(s,t)(x,y), (x,y)(s,t) \in E_H$ (by Lemma 1). Hence $xs, sx \in E_S$, $yt, ty \in E_T$ and $(x,y) \in H$, thus $x \in I(s)$, $y \in I(t)$ and $y \in x\alpha$, that is, (iv) holds. Evidently, $H = \pi(S,T,\alpha)$.

3. E-unitary covers

McAlister and Reilly (1977) used surjective subhomomorphisms of inverse semigroups for the construction of E-unitary covers of inverse semigroups. Recall that a semigroup S with idempotents is called *E-unitary* if $ea, e \in E_S$, $a \in S$ imply $a \in E_S$. Note that this condition is equivalent to: $ae, e \in E_S$, $a \in S$ imply $a \in E_S$ (see the proof of Lemma 2.1 in Howie and Lallement (1965)). In this section we shall consider E-unitary covers for E-inversive semigroups S. A semigroup P is called an *E-unitary cover of S* if P is E-unitary and if there is an idempotent separating homomorphism of P onto S.

In the following we shall give some sufficient conditions for the construction of E-unitary, E-inversive covers for an arbitrary E-inversive semigroup. The first is the existence of a group G and of a surjective subhomomorphism with the following property:

DEFINITION. Let S be an E-inversive semigroup and let G be a group; a subhomomorphism α of S into G is called *unitary* if $1 \in s\alpha$ for some $s \in S$ implies that $s \in E_S$.

THEOREM 8. *Let S be an E-inversive semigroup. If there exist a group G and a unitary surjective subhomomorphism α of S onto G, then $\pi(S,G,\alpha)$ is an E-inversive, E-unitary cover of S.*

PROOF. By Theorem 7, $P = \pi(S,G,\alpha)$ is E-inversive. It is also E-unitary, since $(s,g)(e,1), (e,1) \in E_P \subseteq \{(t,a) \in S \times G | t \in E_S,\ a = 1 \in G\}$ imply $(se,g) \in E_P$, hence $g = 1$ and $(s,1) \in P$. Thus $1 \in s\alpha$ by definition of P and $s \in E_S$ since α is unitary; hence $(s,g) \in E_P$. S is a homomorphic image of P under the projection $\pi_1(s,g) = s$ for all $(s,g) \in P$ which evidently separates idempotents of P.

McAlister and Reilly (1977) considered particular E-unitary covers of inverse semigroups, namely those whose maximum group homomorphic image is a given group. The general concept is the following.

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Definition. Let S be a semigroup and let G be a group; then a semigroup P is called an E-unitary cover of S *through* G if (i) P is an E-unitary cover of S and (ii) $P/\sigma \cong G$ where σ is the least group congruence on P (if σ exists).

For E-inversive semigroups S the existence of the least group congruence was noted by Hall and Munn (1985). We shall give an explicit form of it in the proposition below. It is strongly reminiscent of that of the least group congruence on a regular semigroup described by Feigenbaum (1979).

Proposition 9. *Let S be an E-inversive semigroup. Then the least group congruence on S is given by*

$$a\sigma b \text{ if and only if } xa = by \text{ for some } x, y \in U,$$

where U is the intersection of all subsemigroups T of S such that (1) $E_S \subseteq T$ *and* (2) $ata', a'ta \in T$ *for every* $t \in T$, $a \in S$ *and* $a' \in I(a)$.

Proof. We first show that for every subsemigroup T of S satisfying (1) and (2) of the proposition, the relation

$$a\sigma_T b \text{ if and only if } xa = by \quad \text{for some } x, y \in T$$

is a group congruence on S.

Since $(aa')a = a(a'a)$ with aa', $a'a \in E_S \subseteq T$ for $a' \in I(a)$ then σ is reflexive. σ is symmetric since $a\sigma_T b$ implies $xa = by(x, y \in T)$, thus

$$a(a'xy \cdot xa \cdot b'b) = (aa'xy \cdot by \cdot b')b \quad \text{for } a' \in I(a), b' \in I(b)$$

and $b\sigma_T a$. σ is transitive because $a\sigma_T b, b\sigma_T c$ imply

$$xa = by, \quad zb = cv(x, y, z, v \in T),$$

thus

$$(zx)a = z(xa) = z(by) = (zb)y = (cv)y = c(vy)$$

and $a\sigma_T c$. Then σ is a congruence since $a\sigma_T b$, $c \in S$ imply $xa = by(x, y \in T)$, thus

$$(bcc'b'x)ac = bc(c'b'byc) \quad \text{for } b' \in I(b),\ c' \in I(c),$$

hence $ac\sigma_T bc$, and

$$(cxaa'c')ca = cb(ya'c'ca) \quad \text{for } a' \in I(a),\ c' \in I(c),$$

hence $ca\sigma_T cb$. S/σ_T contains a unique idempotent, because $a\sigma_T, b\sigma_T \in E_{S/\sigma_T}$, imply $a^2\sigma_T a$, $b^2\sigma_T b$, thus $xa = a^2y$, $bt = sb^2$ $(x, y, st, \in T)$ and

$$a(a'x \cdot a\,ay \cdot a'b'bt \cdot b'b) = (aa'x \cdot xa \cdot a'b' \cdot sbb \cdot b')b \qquad \text{for } a' \in I(a), b' \in I(b),$$

hence $a\sigma_T = b\sigma_T$.

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Finally, suppose $(ax)\sigma_T$ and $(ay)\sigma_T$ are idempotents of S/σ_T. Since $|E_{S/\sigma_T}| = 1$ then $ax\sigma_T ay$ so $(yy'a'a)x\sigma_T y(y'a'ay)$. Hence for some $w, z \in T$, $(wyy'a'a)x = y(y'a'ayz)$. Thus $x\sigma_T y$ and $x\sigma_T = y\sigma_T$. Since S/σ_T is E-inversive then by Proposition 2, S/σ_T is a group.

Conversely, let τ be a group congruence on S and let $T = \{t \in S | t\tau$ is the identity of $S/\tau\}$. Clearly $E_S \subseteq T$. For each $a \in S$ and $a' \in I(a)$, $a'\tau$ is the group inverse of $a\tau \in S/\tau$. So for any $t \in T$, $(ata')\tau = (aa')\tau$ is the identity of S/τ. Thus $ata' \in T$, and likewise $a'ta \in T$. It is easy to see that $\sigma_T \leq \tau$. Let $a\tau b$; then $ab'\tau bb'$ for $b' \in I(b)$ and $ab' \in T$. Hence $(bb' \cdot ab'ba')a = b(b'ab'b \cdot a'a)$ and $a\sigma_T b$; so $\tau = \sigma_T$. Clearly U satisfies (1) and (2) and $\sigma = \sigma_U$.

If $\psi: S \to T$ is a homomorphism of semigroups define $\ker\psi = \{s \in S | s\psi \in E_T\}$. Also, if ρ is a congruence on S define $\ker\rho = \{s \in S | s\rho \in E_{S/\rho}\}$.

Corollary 10. *Let S be an E-unitary, E-inversive semigroup with least group congruence σ. Then* $\ker\sigma = U = E_S$.

Proof. Suppose $e, f \in E_S$ and $x \in I(ef)$. Then $e, e(fx) \in E_S$ and since S is E-unitary, $fx \in E_S$. Likewise $x \in E_S$. But then $(ef)x, x \in E_S$ so $ef \in E_S$. Thus E_S is a subsemigroup of S. If $a \in S$ and $a' \in I(a)$ then for each $n \in \mathbb{N}$, $n > 2$,

$$(aea')^n = a(ea'ae)^{n-1}a' = a(ea'ae)a' = (aea')^2.$$

Hence $(aea')^2$, $(aea')^3 \in E_S$, so $aea' \in E_S$ since S is E-unitary. Likewise $a'ea \in E_S$. Thus $E_S = U$.

Since $E_{S/\sigma} = \{e\sigma\}$ for any $e \in E_S$ then $a \in \ker\sigma$ if and only if $a\sigma = e\sigma$. In this case $xa = ey$ for some $x, y \in E_S$ by Proposition 9; so $x, xa \in E_S$ and therefore $a \in E_S$. Thus $\ker\sigma = E_S$.

First we give a sufficient condition for the existence of an E-unitary, E-inversive cover of an E-inversive semigroup through a group.

Theorem 11. *Let S be an E-inversive semigroup and let G be a group. If there is a surjective, unitary subhomomorphism α of S onto G, then $P = \pi(S, G, \alpha)$ is an E-unitary, E-inversive cover of S through G.*

Proof. By Theorem 8, P is an E-inversive, E-unitary cover of S. We have to show that $P/\sigma \cong G$, where σ is the least group congruence on P given in Proposition 9.

By Theorem 7, P is a subdirect product of S and G. Thus G is a homomorphic image of P under the projection $\pi_2(a, g) = g$ for all $(a, g) \in P$.

Since G is a group, $\sigma \leq \pi_2 \circ \pi_2^{-1}$ (the congruence induced by π_2 on P). Conversely, suppose $(a, g)(\pi_2 \circ \pi_2^{-1})(b, h)$; then $(a, g)\pi_2 = (b, h)\pi_2$ and $g = h$. Since P is E-inversive, for $(b, g) = (b, h) \in P$ there exists $(x, y) \in P$ such that $(b, g)(x, y) = (bx, gy) \in E_P = \{(e, 1) \in P | e \in E_S\}$. Hence $bx \in E_S$ and $gy = 1$, that is $y = g^{-1}$. Thus $(x, g^{-1}) \in P$ and $(bx, 1) \in E_P$. Consequently, $(xa, 1) = (x, g^{-1})(a, g) \in P$. Hence $1 \in (xa)\alpha$ by definition of P, and $xa \in E_S$ (since α is unitary); thus $(xa, 1) \in E_P$. Therefore,

$$(bx, 1)(a, g) = (bxa, g) = (b, g)(xa, 1) = (b, h)(xa, 1)$$

with $(bx, 1), (xa, 1) \in E_p \subseteq U$. Hence $(a, g)\sigma(b, h)$, thus $\pi_2 \circ \pi_2^{-1} \leq \sigma$ and equality follows. Consequently, $P/\sigma = P/\pi_2 \circ \pi_2^{-1} \cong G$.

Petrich and Reilly (1983) gave a construction of all unitary, surjective subhomomorphisms of an inverse semigroup onto a group. Their result can be generalized to E-inversive semigroups.

THEOREM 12. *Let S be an E-inversive semigroup and G be a group. Then α is a unitary surjective subhomomorphism of S onto G if and only if $\alpha = \phi^{-1} \circ \psi$ for some E-inverse semigroup R and some surjective homomorphisms $\phi: R \to S$ and $\psi: R \to G$ such that* $\ker \psi \subseteq \ker \phi$.

PROOF. Sufficiency follows by a straightforward generalization of the corresponding proof for inverse semigroups by Petrich and Reilly (1983).

Suppose α is a unitary surjective subhomomorphism from S onto G. By Theorem 11, $P = \pi(S, G, \alpha)$ is an E-unitary, E-inversive cover of S through G. Let $\pi_1: P \to S$ and $\pi_2: P \to G$ be the projection mappings. By the proof of Theorem 11, $\pi_2 \circ \pi_2^{-1}$ is the least group congruence on P, so by Corollary 10, $\ker \pi_2 = E_P$. Clearly $\ker \pi_1 \supseteq E_P$, so by the sufficiency of the condition of the theorem there is a unitary surjective subhomomorphism $\beta = \pi_1^{-1} \circ \pi_2$ from S onto G. It follows easily that

$$\pi(S, G, \beta) = \{(p\pi_1, p\pi_2) \mid p \in P\} = P.$$

So for $s \in S$,

$$s\beta = \{g \in G \mid (s, g) \in \pi(S, G, \beta)\} = \{g \in G \mid (s, g) \in P\} = s\alpha.$$

Thus $\alpha = \beta = \pi_1^{-1} \circ \pi_2$.

The converse of Theorem 11 (and Theorem 8) can now be proved.

THEOREM 13. *Let S be an E-inversive semigroup, G be a group and P be an E-inversive, E-unitary cover of S through G. Then there is a unitary surjective subhomomorphism α of S onto G such that $\pi(S, G, \alpha)$ is a homomorphic image* *of P, and $\pi(S,G,\alpha)$ is an E-unitary cover of S through G. Furthermore, if P is also a subdirect product of S and G then $P \cong \pi(S,G,\alpha)$.*

PROOF. There are surjective homomorphisms $\phi: P \to S$ and $\psi: P \to G$ such that $\psi \circ \psi^{-1}$ is the least group congruence on P. By Corollary 10, $E_p = \ker\psi \subseteq \ker\phi$. By Theorem 12 we may choose $\alpha = \phi^{-1} \circ \psi$; for by Theorem 11, $\pi(S,G,\alpha)$ is an E-unitary cover of S through G, and there is a homomorphism $\theta: P \to \pi(S,G,\alpha)$ given by $p\theta = (p\phi, p\psi)$. Also θ is surjective, since for $(s,g) \in \pi(S,G,\alpha)$ then $g \in s(\phi^{-1} \circ \psi)$, so there exists $p \in P$ such that $p\phi = s, p\psi = g$.

Now suppose that P is also a subdirect product of S and G. Then by Theorem 7, $P = \pi(S,G,\beta)$ for some surjective subhomomorphism β of S onto G. Let $\pi_1: P \to S$ and $\pi_2: P \to G$ be the projections. We may choose $\phi = \pi_1$. Since $\psi \circ \psi^{-1}$ is the least group congruence on P then $\psi \circ \psi^{-1} \leq \pi_2 \circ \pi_2^{-1}$. Hence if $a\theta = b\theta$ for some $a, b \in P$ then $a\phi = b\phi$, $a\psi = b\psi$ whence $a\pi_1 = b\pi_1$, $a\pi_2 = b\pi_2$. But P is a subdirect product of S and G, so $a = b$. Thus θ is an isomorphism.

4. Subdirect products of semilattices and groups

Regular semigroups S which are a subdirect product of a semilattice Y and a group G were characterized by Howie and Lallement (1965), see also Petrich (1973 b), as sturdy semilattices of groups, that is strong semilattice of groups with injective linking homomorphisms (or equivalently, as E-unitary Clifford semigroups). They have the additional property, that $(\alpha, 1) \in S$ for every $\alpha \in Y$. We shall characterize *all* subdirect products of Y and G with this property. Note that every subdirect product of Y and G is E-inversive.

DEFINITION. A subdirect product H of a semigroup S and a group G is called *full*, if $(e, 1) \in H$ for ever $e \in E_S$.

THEOREM 14. *For a semigroup S the following are equivalent:*

(i) *S is a full subdirect product of a semilattice and a group;*
(ii) *S is an E-inversive sturdy semilattice of cancellative monoids;*
(iii) *S is an E-inversive sturdy semilattice of weakly cancellative monoids;*
(iv) *S is an E-inversive sturdy semilattice of trivially ordered monoids;*
(v) *S is an E-inversive sturdy semilattice of monoids with a single idempotent.*

PROOF. (i) implies (ii). Let $(\alpha, g) \in S$; then for $g^{-1} \in G$ there is $\beta \in Y$ with $(\beta, g^{-1}) \in S$, hence $(\alpha, g)(\beta, g^{-1}) = (\alpha\beta, 1) \in E_S$. Thus S is E-inversive. For every $\alpha \in Y$ define $S_\alpha = (\{\alpha\} \times G) \cap S$. Since by hypothesis $(\alpha, 1) \in S$ for every $\alpha \in Y$, $S_\alpha \neq \varnothing$ for every $\alpha \in Y$. S_α is easily seen to be cancellative submonoid of S. For every $\beta \leq \alpha$ in Y and every $(\alpha, g) \in S$, also $(\beta, g) = (\alpha, g)(\beta, 1) \in S$ by hypothesis. Hence the mapping $\varphi_{\alpha,\beta}: S_\alpha \to S_\beta$, $(\alpha, g)\varphi_{\alpha,\beta} = (\beta, g)$ is defined for all $\beta \leq \alpha$. It is easily shown that all $\varphi_{\alpha,\beta}$ are injective homomorphisms satisfying the conditions which show that $S = \bigcup_{\alpha \in Y} S_\alpha$ is a sturdy semilattice of the semigroups $S_\alpha, \alpha \in Y$.

(ii) implies (iii). Every cancellative semigroup is weakly cancellative.

(iii) implies (iv). Let $\alpha \in Y$ and $a \leq b$ in S_α; then $a = xb = by$, $xa = a = ay (x, y \in S^1)$. Thus $xa = a = xb$ and $by = a = ay$, hence $(yx)a = (yx)b$ and $b(yx) = a(yx)$. By hypothesis it follows that $a = b$.

(iv) implies (v). Since $e \leq 1_\alpha$ for every $e \in E_{S_\alpha}$, 1_α the identity of S_α, we have by hypothesis $e = 1_\alpha$ and $E_{S_\alpha} = \{1_\alpha\}$.

(v) implies (i). Since S is a sturdy semilattice Y of semigroup $S_\alpha (\alpha \in Y)$, by Petrich (1973 b) S is a full subdirect product of Y and the semigroup S/θ, where θ is the congruence on S defined by:

$$a\theta b \text{ if and only if } a\varphi_{\alpha,\alpha\beta} = b\varphi_{\beta,\alpha\beta} \qquad (a \in S_\alpha, b \in S_\beta).$$

We show that S/θ admits a unique idempotent which is the identity of S/θ. Since every homomorphism maps the unique idempotent $1_\alpha \in S_\alpha$ onto the unique idempotent $1_{\alpha\beta} \in S_{\alpha\beta}$ and since $1_\alpha\varphi_{\alpha,\alpha\beta} = (1_\alpha\varphi_{\alpha,\alpha\beta})\varphi_{\alpha\beta,\alpha\beta}$ we have

$$1_\alpha \theta 1_\alpha \varphi_{\alpha,\alpha\beta} = 1_{\alpha\beta} = 1_\beta \varphi_{\beta,\alpha\beta} \theta 1_\beta.$$

Let I denote the θ-class containing all the identity elements 1_α $(\alpha \in Y)$. Clearly I is the identity of S/θ. Suppose $a\theta \in E_{S/\theta}$ and $a \in S_\alpha$ for some $\alpha \in Y$. S_α is a monoid so $a^2 \in S_\alpha$. We have $a^2\theta a$ so $a^2\varphi_{\alpha,\alpha} = a\varphi_{\alpha,\alpha}$ and hence $a^2 = a$ in S_α. Thus $a = 1_\alpha$ and therefore $a\theta = I$. Since S/θ is E-inversive it follows easily that S/θ is a group.

Petrich (1973 b) gave a construction of all regular semigroup which are a subdirect product of a band (semilattice Y) and a group G. Following his ideas we can generalize it to that of all semigroups which are full subdirect products of Y and G.

THEOREM 15. *Let Y be a semilattice and G be a group. Suppose that there is a mapping f of $(Y, \leq)$ into the lattice $(\mathscr{U}(G), \subseteq)$ of all submonoids of G ordered by inclusion, which is order-inverting and satisfies $\bigcup_{\alpha \in Y} \alpha f = G$. Then $S = \{(\alpha, a) \in Y \times G \mid a \in \alpha f\}$ is an E-inversive semigroup which is a* *full subdirect product of Y and G. Conversely, every such semigroup can be constructed in this way.*

PROOF. *Sufficiency.* S is a semigroup: $(\alpha, a), (\beta, b) \in S$ imply that $a \in \alpha f \subseteq (\alpha\beta)f$, $b \in \beta f \subseteq (\alpha\beta)f$, thus $ab \in (\alpha\beta)f$ and $(\alpha\beta, ab) \in S$. S is a subdirect product of Y and G, since $\alpha f \neq \varnothing$ for every $\alpha \in Y$ and since $G = \bigcup_{\alpha \in Y} \alpha f$. It is full, since every $\alpha f \in \mathscr{U}(G)$ is a submonoid of G with identity 1 (the identity of G), hence $1 \in \alpha f$ and $(\alpha, 1) \in S$ for every $\alpha \in Y$.

Necessity. Let S be a full subdirect product of Y and G; define $\alpha f = \{a \in G \mid (\alpha, a) \in S\}$ for every $\alpha \in Y$. It is easy to see that αf is a submonoid of G, so that the mapping $f\colon Y \to \mathscr{U}(G)$, $\alpha \to \alpha f$ is defined. If $\alpha \leq \beta$ in Y and $x \in \beta f$, then $\alpha = \alpha\beta$ and $(\beta, x) \in S$, thus $(\alpha, x) = (\alpha, 1)(\beta, x) \in S$ by hypothesis and $x \in \alpha f$, that is, f is order-inverting. Since S is a subdirect product of Y and G, also the second property is satisfied by f. Evidently, $S \cong \{(\alpha, a) \in Y \times G \mid a \in \alpha f\}$.

Finally we give a characterization of all periodic semigroups which can be represented as a subdirect product of a semilattice and a group. They are the same as the regular ones found by Howie and Lallement (1965); see also Petrich (1973 b) and the beginning of this section. Note that every periodic semigroup is E-inversive.

THEOREM 16. *Let S be a periodic (in particular, finite) semigroup. Then S is a subdirect product of a semilattice and a group if and only if S is an E-unitary Clifford-semigroup.*

PROOF. *Sufficiency.* By Howie and Lallement (1965), every E-unitary Clifford semigroup is a subdirect product of a semilattice and a group.

Necessity. Let S be a periodic semigroup which is a subdirect product of a semilattice Y and a group G. For every $\alpha \in Y$ define $S_\alpha = (\{\alpha\} \times G) \cap S$. $S_\alpha \neq \varnothing$ since S is a subdirect product of Y and G. We show that for every $\alpha \in Y$, S_α is a group. First, each S_α is a cancellative semigroup; it is periodic, since S is. Thus for every $a \in S_\alpha$, the semigroup $\langle a \rangle$ generated by a in S_α is finite, hence a group. So S_α is a cancellative union of groups, and is therefore a group. It follows that S is a full subdirect product. By the proof of Theorem 14, (i) implies (ii), then S is a sturdy semilattice Y of groups S_α $(\alpha \in Y)$. Thus S is an E-unitary Clifford semigroup.

This result can be extended to a characterization of all periodic semigroups which are subdirect products of a band B and a group G. Again, the regular case was considered by Howie and Lallement (1965), see also Petrich (1973b), who obtained the same class of semigroups as given in the following result.

THEOREM 17. *A periodic semigroup S is a subdirect product of a band B and a group G if and only if S is a band of groups and E_S is a unitary subset.*

PROOF. By Howie and Lallement (1965), the condition is sufficient. Necessity: By the same proof as of Theorem 16, S is the union of the groups $S_\alpha = (\{\alpha\} \times G) \cap S$, $\alpha \in B$. Also, it is easily shown that S is E-unitary. Since Green's relation $\mathscr{H}$ on S is given by $(\alpha, a)\mathscr{H}(\beta, b)$ if and only if $\alpha = \beta$, it follows immediately that $\mathscr{H}$ is a congruence. Hence S is a band of groups (see Petrich (1973b).

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