SETS WITH NO EMPTY CONVEX 7-GONS

by

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ABSTRACT. Erdős has defined g(n) as the smallest integer such that any set of g(n) points in the plane, no three collinear, contains the vertex set of a convex n-gon whose interior contains no point of this set. Arbitrarily large sets containing no empty convex 7-gon are constructed, showing that g(n) does not exist for n ≥ 7. Whether g(6) exists is unknown.

Esther Klein raised the following combinatorial geometry problem [5]. For n ≥ 3, let f(n) be the smallest integer such that for any set of f(n) points in the plane, no three collinear, contains the vertex set of a convex n-gon. Determine f(n). It is easy to show that f(3) = 3 and f(4) = 5. That f(5) = 9 was proved in [4]. Erdős and Szekeres [1], [2] determined that \(2^{n-2} + 1 ≤ f(n) ≤ (\frac{2n-4}{n-2}) + 1\).

Erdős has raised a similar question. For n ≥ 3, define g(n) to be the smallest integer such that any set of g(n) points in the plane, no three collinear, contains the vertex set of a convex n-gon whose interior contains no point of the set. We call a n-gon, with no points of the set in its interior, empty. Again, g(3) = 3 and g(4) = 5. Harborth [3] has proved that g(5) = 10. However, it is not known whether g(6) exists. The main result of this note is that g(7), and hence g(n) for all n ≥ 7, does not exist.

We construct, for any k, a set of \(2^k\) points with no empty convex 7-gon. Let \(a_1a_2 \cdots a_k\) be the binary expansion of the integer \(i\), \(0 ≤ i < 2^k\). Note that leading 0's are not omitted. Let \(c = 2^k + 1\), and define \(d(i) = \sum a_k c^{k-1}\), summing from \(j = 1\) to \(j = k\). Let \(p_i\) be the point \((i, d(i))\), and define \(S_k\) to be the set of points \(\{p_i \mid i = 0, 1, \ldots, 2^k - 1\}\). Observations:

(a) \(\{p_i \mid i < 2^k - 1\}\) = the left half of \(S_k = L\).
(b) \(\{p_i \mid i ≥ 2^k - 1\}\) = the right half of \(S_k = R\), which is a translate of \(L\).
(c) \(\{p_i \mid i \text{ is even}\}\) = the bottom half of \(S_k = B\).
(d) \(\{p_i \mid i \text{ is odd}\}\) = the top half of \(S_k = T\), which is a translate of \(B\).
(e) \(L, R, B,\) and \(T\) are all scaled translates of each other. For example, halving the first coordinate while multiplying the second coordinate by \(c\), takes \(B\) onto \(L\).
(f) The 180° rotation of the plane about \((2^k - 1)/2, \sum c^i/2\) takes \(T\) onto \(B\).

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(g) All points of $T$ are above any line joining two points of $B$. The value of $c$ was chosen large enough to make this true. Similarly, all points of $B$ are below any line joining two points of $T$.

(h) If $i$ and $j$ both have the same last $x$ digits in their binary expansions, and $h$ has a different sequence of $x$ rightmost digits, then whether $p_h$ is above or below the line joining $p_i$ and $p_j$ is determined by the sequences of the last $x$ digits.

Consider any empty convex $n$-gon $A$ in $S_k$. We may assume $A$ is contained entirely in neither $T$ nor $B$. Otherwise if $A$ is contained in $B$, apply the linear transformation that takes $B$ onto $L$. $A$ will be transformed into an empty convex $n$-gon in $L$. Similarly, if $A$ is contained in $T$, apply the linear transformation that takes $T$ onto $L$. Repeat this procedure until a transformed image of $A$ meets both $T$ and $B$.

Next, consider how many points of $A$ can be in $B$. Assume $p_i$ and $p_j$ are in $A \cap B$. By (g) above, no point $p_h$ of $B$ with $i < h < j$, can be above the line segment joining $p_i$ and $p_j$, since otherwise no point of $T$ could be in $A$. As well, I claim that $d(h) < d(i)$ and $d(h) < d(j)$. Since $p_h$ is below the line joining $p_i$ and $p_j$, clearly one of these statements is true. Assume $d(h) < d(i)$, but $d(h) > d(j)$. Let $x$ be the position of the right-most digit at which $h$ and $i$ differ in their binary expansions; let $y$ be the position of the right-most digit at which $h$ and $j$ differ. In both cases, the number with the larger functional value must have a 1 in the position, and the other number a 0. If $x < y$ then $p_i$ must be below the line joining $p_i$ and $p_h$, by observation (h). But then $p_h$ is above the line joining $p_i$ and $p_j$, a contradiction. Hence we can assume that $y < x$. In this case, consider $l = j - 2^{k-x}$. The right-most position in which the binary expansions of $l$ and $j$ differ is $x$, where $l$ has a 1 and $j$ has a 0. On the other hand, $l$ and $i$ must agree in the last $k-x$ positions. By observation (h), $p_i$ is below the line joining $p_i$ and $p_l$. But since $j - i > j - h \geq 2^{k-x} > 2^{k-x} = j - l$, $i < l < j$. Then $p_i$ must be both above and below the line joining $p_i$ and $p_h$, a contradiction. Similarly, $d(j) < d(h) < d(i)$ leads to a contradiction. Therefore $d(h) < d(i)$ and $d(h) < d(j)$.

If $A \cap B$ contained four points $i < h < l < j$, then $d(h) < d(l)$ and $d(l) < d(h)$. Hence $A \cap B$ cannot contain more than three points. By observation (f) above, $A \cap T$ cannot contain more than three points either. Hence $A$ has no more than 6 points.

Whether $g(6)$ exists is still unknown, although the author believes that $g(6)$ does exist.

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**REFERENCES**