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ISOMORPHISMS OF PRIME GOLDIE SEMI-PRINCIPAL LEFT IDEAL RINGS, II

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ABSTRACT. A prime Goldie ring K, in which each finitely generated left ideal is principal is the endomorphism ring $E(F, A)$ of a free module A, of finite rank, over an Ore domain F. We determine necessary and sufficient conditions to insure that whenever $K \cong E(F, A) \cong E(G, B)$ (with A and B free and finitely generated over domains F and G) then (F, A) is semi-linearly isomorphic to (G, B). We also show, by example, that it is possible for $K \cong E(F, A) \cong \mathrm{E}(G, B)$, with F and G, not isomorphic.

Introduction. A prime (left) Goldie ring K, in which each finitely generated left ideal is principal, is the endomorphism ring $E(F, A)$ of a free module A, of finite rank, over a (left) Ore domain F. (1, 3, 5, see also 2, Chapter 4). In a previous paper [8] we have found necessary and sufficient conditions to insure that whenever $K \cong E(F, A) \cong E(G, B)$, ($A$, B free and finitely generated over domains F and G) each isomorphism of $E(F, A)$ upon $E(G, B)$ is induced by a (semi-linear) module isomorphism of (F, A) upon (G, B). In this paper we show that isomorphism of the endomorphism rings implies (semi-linear) isomorphism of the modules, (not required to induce the ring isomorphism) precisely when eKe and fKf are isomorphic subrings of K for each pair of primitive idempotents e, f in K, or equivalently when $K \cong E(F, A)$ (A free of finite rank over an Ore domain F) where F has the property that $E(F, J) \cong F$ for each non-zero finitely generated left ideal J of F.

We also show (by example) that it is possible to have the endomorphism rings isomorphic when the underlying modules are not (semi-linearly) isomorphic, answering in the negative a conjecture of Jategaonkar. Finally we exhibit a procedure for constructing all the isomorphic images of a fixed $E(F, A)$ using the non-zero finitely generated left ideals of F.

1. **Definitions and Preliminaries.** We shall follow the definitions and notation of [8]. In particular, (F, A) indicates a unitary left module A over a ring F, and $E(F, A)$ its endomorphism ring, whose elements operate on the right of elements of A. We shall say $(F, A) \in \mathscr{F}$ when A is free and finitely generated,

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and F is an (not necessarily commutative) integral domain. Goldie ring means left Goldie ring, and Ore domain always means left Ore domain. We shall use the notation $(F, A) \cong (F, B)$ or just $A \cong B$ to mean the modules are isomorphic in the usual (linear) manner over F. The construction given by Herstein (2, Chapter 4, Lemma 4.11) for representing a prime Goldie semi-pli ring K, as an endomorphism ring, shows that if A is any minimal right annihilator of K, and $F = \{\sigma_a | a \in A\}$, where $\sigma_a(x) = ax$, for $x \in A$, then F is an Ore domain, and A is a free module of finite rank over F. Furthermore the mapping $k \to k^\phi$ for $k \in K$ defined by $xk^\phi = x \cdot k$ for all $x \in A$, is an isomorphism of K onto $E(F, A)$. But A is a minimal right annihilator precisely when $A = eK$, for e a primitive idempotent in K (8, Lemmas 2.2, 2.3). If $x, a \in A = eK$, then $x = ex$, $a = ea$, so that $\sigma_a(x) = (ea)(ex) = (eae)x$. It is easily verified that the mapping $\sigma_a \to eae$ is a ring isomorphism of F onto $eAe = eKe$, so that $K \cong E(eKe, eK)$ for each primitive idempotent e in K.

LEMMA 1.1. *Let (F, A) and (G, B) be modules, and assume (F, A) possesses a free cyclic summand Ae [where e is an idempotent in $E(F, A)$]. Let ϕ be an isomorphism of $E(F, A)$ upon $E(G, B)$. Then*

(1) *ϕ is induced by a (semi-linear) module isomorphism of (F, A)* upon *(G, B) if, and only if, Be^ϕ is free and cyclic over G.*

(2) *$F \cong G$, if, and only if, $E(G, Be^\phi) \cong G$.*

PROOF. (1) If ϕ is induced by a semi-linear module isomorphism, $A\sigma$ and $B\sigma^\phi$ are (semi-linearly) isomorphic for each $\sigma \in E(F, A)$. In particular Ae and Be^ϕ are semi-linearly isomorphic, so that Be^ϕ must be free and cyclic (over G). The converse is essentially known, and can be proven using Lemma 1.1 of [8].

(2) Since $F \cong E(F, Ae) \cong eE(F, A)e \cong e^\phi E(G, B)e^\phi \cong E(G, Be^\phi)$, the conclusion follows.

LEMMA 1.2. *Let (F, A) and (G, B) be free modules (of arbitrary rank) over Ore domains F and G. If $E(F, A) \cong E(G, B)$ then $r(A) = r(B)$.*

PROOF. The proof of this (7, Lemma 2.2) when F and G are principal left ideal domains goes over without change, if we interpret $r(H)$, for H a submodule of A, to be the cardinal number of a maximal linearly independent subset of H.

LEMMA 1.3. *Let $E(F, A)$ be a Goldie semi-pli ring where $(F, A) \in \mathscr{F}$. If e is a primitive idempotent in $E(F, A)$, there exists a finitely generated left ideal J of F such that $E(F, Ae) \cong E(F, J)$.*

PROOF. By (8, Lemma 2.1) each finitely generated left ideal of F is projective, so that each finitely generated submodule S of A is isomorphic to a direct sum of finitely generated left ideals of F. But Ae is finitely generated and indecomposable so that $(F, Ae) \cong (F, J)$ for a finitely generated $J \subseteq F$. This module isomorphism induces a ring isomorphism $E(F, Ae) \cong E(F, J)$.

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2. **The main results.**

THEOREM 2.1. *Let K be a prime Goldie semi-pli ring. Then the following statements are equivalent*:

(1) *If $K \cong E(F, A)$, $K \cong E(G, B)$ with $(F, A), (G, B) \in \mathcal{F}$, then there exists a semi-linear isomorphism of (F, A) onto (G, B).*

(2) *$eKe \cong fKf$ (as rings) for each pair of primitive idempotents, e, f in K.*

(3) *If $K \cong E(F, A)$ with $(F, A) \in \mathcal{F}$, then $E(F, J) \cong F$ for each non-zero finitely generated left ideal J of F.*

(4) *There exists $(F, A) \in \mathcal{F}$, such that $K \cong E(F, A)$ and $E(F, J) \cong F$ for each non-zero finitely generated left ideal J of F.*

PROOF. (1) $\Rightarrow$ (2). If e, f are primitive idempotents in K, then $K \cong E(eKe, eK)$ and $K \cong E(fKf, fK)$ with the modules free of finite rank over Ore domains, so that $(eKe, eK) \cong (fKf, fK)$ as modules. In particular eKe is ring isomorphic to fKf.

(2) $\Rightarrow$ (3). Assume $K \cong E(F, A)$ with $(F, A) \in \mathcal{F}$. Let J be a non-zero finitely generated left ideal of F. By (8, Lemma 2.1) there exists an F-module Q such that $(F, J \oplus Q) \cong (F, A)$. Let $g \in E(F, J \oplus Q)$ be defined by $xg = x$, if $x \in J$, and $Qg = 0$. Since F is an Ore domain, J is indecomposable and g is primitive. The module isomorphism $(F, J \oplus G) \to (F, A)$ induces a ring isomorphism $E(F, J \oplus Q) \to E(F, A)$ under which g maps onto a primitive idempotent f. Then $E(F, J) \cong gE(F, J \oplus Q)g \cong fE(F, A)f$. Now if e is a primitive idempotent for which Ae is free and cyclic, we have $F \cong E(F, Ae) \cong eE(F, A)e$. Since (2) implies that $eEe \cong fEf$, we have $E(F, J) \cong F$.

(3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (1). Assume $K \cong E(F_0, A_0) \cong E(F, A) \cong E(G, B)$ with (F_0, A_0), (F, A), $(G, B) \in \mathcal{F}$, and such that $E(F_0, J) \cong F_0$ for all non-zero finitely generated left ideals J of F_0. Let ϕ be an isomorphism of $E(F, A)$ onto $E(F_0, A_0)$ and let Ae be free and cyclic [e an idempotent in $E(F, A)$]. By Lemma 1.3, $E(F_0, A_0e^{\phi}) \cong E(F_0, J)$ for a finitely generated left ideal J of F_0. By hypothesis $E(F_0, J) \cong F_0$, so $E(F_0, A_0e^{\phi}) \cong F_0$. By Lemma 1.1, this implies $F \cong F_0$. Similarly $G \cong F_0$, so $F \cong G$. An appeal to Lemma 1.2 completes the proof.

COROLLARY 2.2. *If $(F, A) \in \mathcal{F}$, and $E(F, A)$ is a Goldie semi-pli ring, then $E(F, J)$ is an Ore domain, for each non-zero finitely generated left ideal J of F.*

PROOF. In the proof of the preceding theorem, it was shown that $E(F, J) \cong fE(F, A)f$ for a primitive idempotent f in $E(F, A)$. By the discussion in section 1, fEf is an Ore domain, whenever f is a primitive idempotent in E.

COROLLARY 2.3. *Let K be a prime Goldie semi-pli ring. Assume $K \cong E(F, A)$, with $(F, A) \in \mathcal{F}$. Then, there exists a non-zero finitely generated left ideal J of F such that $E(F, J) \not\cong F$ if, and only if, there exists $(G, B) \in \mathcal{F}$ such that $E(F, A) \cong E(G, B)$, but (F, A) and (G, B) are not semi-linearly isomorphic.*

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We show next that, in fact, the situation of Corollary 2.3 does occur. The example, originally due to Swan [6] involves a construction of O'Meara (4, p. 139-141). Let $\tau = \sqrt{2 + \sqrt{2}}$, $\tau' = \sqrt{2 - \sqrt{2}}$. If Z, Q are the integers and rationals respectively, let $R = Z[\tau]$, $K = Q(\tau)$, and $D = K \cdot 1 + Ki + Kj + Kk$ the quaternion division algebra over the real subfield K. Let $\alpha = \tau^{-1}\sqrt{2^{-1}}(\sqrt{2} + i + j)$, $\beta = \tau'^{-1}(1 + j)$, $\gamma = \alpha\beta$, and $F = R \cdot 1 + R\alpha + R_\beta + R_\gamma$. Similarly let G be the R-module spanned by 1, $\sqrt{2^{-1}}(1 + i)$, $\sqrt{2^{-1}}(1 + j)$, and $(1 + i + j + k)/2$. Both F and G are closed under multiplication, and are maximal R-orders in D. F is not a pli domain, but F_2 (and hence F_n, $n \geqq 2$), (3, p. 52, Corollary 2.12) is a pli ring. We show first that F and G are not isomorphic rings. Let F^* be the unit group of F, G^* the unit group G, and R^* the unit group of R. It is easy to verify that $F^* \cap R = R^*$, $G^* \cap R = R^*$, and that R is the centre of both F and G. Since any ring isomorphism of F onto G must map F^* onto G^*, and the centre of F onto the centre of G, it must map R^* onto itself. Such a ring isomorphism therefore induces an isomorphism of the groups F^*/R^* and G^*/R^*. But these groups have different orders, and are not isomorphic (6, p. 58).

Now with F, G has non-isomorphic maximal orders in the common quotient division ring D, the arguments of O'Meara produce a finite-dimensional left vector space V over D with bases $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ and which contains the free F-module $A = Fy_1 \oplus Fy_2 \oplus \ldots \oplus Fy_n$, and the free G-module $B = Gx_1 \oplus Gx_2 \oplus \ldots \oplus Gx_n$. $E(F, A)$ and $E(G, B)$ may be identified as a common subring of the ring of linear transformations $E(D, V)$. Hence a ring isomorphism of G_n onto F_n can be obtained by mapping the matrix representation of each of these common linear transformations in the $\{x_i\}$ basis onto its matrix representation in the $\{y_i\}$ basis. Since the class number of F is two, n may be chosen to be any positive even integer. We then have $F_n \cong G_n$, but $F \not\cong G$, where F, G are domains, and F_n, G_n are prime pli rings (for $n = 2, 4, 6, \ldots$). This answers in the negative a conjecture of Jategaonkar (3, p. 45, p. 137).

It should be pointed out that if ρ is the inverse of the matrix which represents the mapping $x_i \to y_i$ $(i = 1, 2, \ldots, n)$ with respect to the $\{x_i\}$ basis, and ϕ is the isomorphism of G_n onto F_n, then for each $\sigma \in G_n$,

$$\sigma^\phi = \rho^{-1}\sigma\rho, \quad \rho \in D_n$$

so that ϕ is induced by an automorphism of the module (D, V). This is, of course, a special case of the fact that each isomorphism of $E(G, B)$ onto $E(F, A)$ is induced by a (semi-linear) module isomorphism of appropriate extensions of the underlying modules (8, Theorem 3.2), even when $F \cong G$.

Continuing the notation of the example, we know by Corollary 2.3 that there exists a non-zero left ideal J of F, such that $E(F, J) \not\cong F$. We can identify such

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a J, as follows: Let $e \in E(G, B)$ be an idempotent such that Be is free and cyclic (a projection on the submodule spanned by a fixed basis element of B). If ϕ is an isomorphism of $E(G, B)$ onto $E(F, A)$, then by Lemmas 1.1 and 1.3, if $J \cong Ae^{\phi}$, then $E(F, J) \ncong F$.

We can apply this "construction" to the general situation. Let (F, A), $(G, B) \in \mathscr{F}$ and assume $E(F, A)$ is a Goldie semi-pli ring, and ϕ is an isomorphism of $E(G, B)$ onto $E(F, A)$. Then if e is defined as before $G \cong E(F, Ae^{\phi}) \cong E(F, J)$ where $J \cong Ae^{\phi}$. Clearly (F, A) is semi-linearly isomorphic to (G, B) if, and only if, $E(F, J) \cong F$, and ϕ is induced by a (semi-linear) module isomorphism if, and only if, J is a principal left ideal.

We shall now show how to identify all isomorphic images (but not necessarily all isomorphisms) of a Goldie semi-pli ring $E(F, A)$ where $(F, A) \in \mathscr{F}$. Let J be a non-zero finitely generated left ideal of F, and let $J_i \cong J$ for $i = 1, 2, \ldots, n$ (where $r(A) = n$). Then $E(F, J_1 \oplus J_2 \oplus \ldots \oplus J_n) \cong E(F, A)$ and also using the standard matrix representation $E(F, J_1 \oplus J_2 \oplus \ldots \oplus J_n) \cong [E(F, J)]_n$. If $G = E(F, J)$ (an Ore domain by Corollary 2.2) and $B = G^{(n)}$, we have

$$\text{(s)} \quad E(G, B) \cong G_n \cong [E(F, J)]_n \cong E(F, J_1 \oplus J_2 \oplus \ldots \oplus J_n) \cong E(F, A)$$

and every isomorphic image of $E(F, A)$ can be obtained in this way. For if ϕ is an isomorphism of some $E(G, B)$ onto our $E(F, A)$ with $(G, B) \in \mathscr{F}$, we can by Lemma 1.1, find a finitely generated left ideal J of F, such that $G \cong E(F, J)$. Using this J in the sequence (s) produces an isomorphism of $E(G, B)$ onto $E(F, A)$. Using Lemma 1.1, it follows that the isomorphism produced by the sequence (s) is induced by a (semi-linear) module isomorphism of (G, B) onto (F, A) if, and only if, J is a principal left ideal, and (F, A) is (semi-linearly) isomorphic to (G, B) if, and only if $E(F, J) \cong F$.

We have proved the following:

THEOREM 2.4. *Let K be a Goldie semi-pli ring, such that $K \cong E(F, A)$ with $(F, A) \in \mathscr{F}$. Then there exists $(G, B) \in \mathscr{F}$ such that*

(a) *There is a ring isomorphism ϕ of $E(G, B)$ onto $E(F, A)$,*

(b) *There is a (semi-linear) module isomorphism of (G, B) onto (F, A),*

(c) *ϕ is not induced by any (semi-linear) module isomorphism of (G, B) onto (F, A),*

if, and only if, there exists a finitely generated non-principal left ideal J of F, such that $E(F, J) \cong F$.

REFERENCES

1. A. W. Goldie, *Non-commutative principal ideal rings*, Arch. Math. **13** (1962), pp. 214-221.
2. I. N. Herstein, *Topics in ring theory*, Univ. of Chicago Press, Chicago Ill. (1969).
3. A. V. Jategoankar, *Left principal ideal rings*, Lecture Notes in Math. (123), Springer-Verlag, Berlin, 1970.

https://doi.org/10.4153/CMB-1988-053-3 Published online by Cambridge University Press

4. O. T. O'Meara, *A general isomorphism theory for linear groups*, J. Algebra **44** (1977), pp. 93-142.

5. J. C. Robson, *Rings in which finitely generated right ideals are principal*, Proc. London Math. Soc. **17** (1967), pp. 617-628.

6. R. G. Swan, *Projective modules over group rings and maximal orders*, Ann. of Math. **76** (1962), pp. 55-61.

7. K. G. Wolfson, *Isomorphisms of the endomorphism ring of a free module over a principal left ideal domain*, Mich. Math. J. **9** (1962), pp. 69-75.

8. ———, *Isomorphisms of prime Goldie semi-principal left ideal rings*, Proc. Amer. Math. Soc. (to appear).

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