Can. J. Math., Vol. XXV, No. 6, 1973, pp. 1274–1284

THE SC_nP-INTEGRAL AND THE P^{n+1}-INTEGRAL

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Introduction. In [**2**], we have briefly described, as examples of the general theory developed there, a scale of symmetric Cesaro-Perron integrals, namely SC_nP-integral for $n = 1, 2, 3, \ldots$. The purpose of this paper is to consider the integrals in a greater detail.

As a preliminary, we prove some lemmas, which are also interesting for their own sake, concerning the de la Vallée Poussin derivatives in Section 1, and we also state two deep theorems concerning the n-convex functions in Section 2. Our main effort is to establish Theorem 3 in Section 3, which is essential to the theory of the SC_nP-integral. In Section 4, the definition of the SC_nP-integral is given, while its usual properties are only briefly indicated since they follow from the general theory in [**2**]. The last section is devoted to the connection between the SC_nP-integral and the symmetric P^{n+1}-integral of James [**9**].

1. The symmetric de la Vallée Poussin derivatives. Let F be a function defined on a bounded closed interval $[a, b]$, and let x be a point in the open interval $]a, b[$. If there are constants $\beta_0, \beta_2, \beta_4, \ldots, \beta_{2r}$ ($r \geqq 0$), depending on x but not on h such that

$$(1) \qquad \tfrac{1}{2}\{F(x+h) + F(x-h)\} - \sum_{k=0}^{r} \beta_{2k} \frac{h^{2k}}{(2k)!} = o(h^{2r})$$

as $h \to 0$, then β_{2r} is called the symmetric de la Vallée Poussin (s.d.l.V.P.) derivative of order $2r$ of F at x, and we write $\beta_{2r} = D_{2r}F(x)$. It is clear that if $D_{2r}F(x)$ exists, so does $D_{2k}F(x)$ for $k = 0, 1, 2, \ldots, r-1$, and $D_{2k}F(x) = \beta_{2k}$.

If $D_{2k}F(x)$ exists for $0 \leqq k \leqq m-1$, ($m \geqq 1$), define $\theta_{2m}(x, h) = \theta_{2m}(F; x, h)$ by

$$(2) \qquad \frac{h^{2m}}{(2m)!}\theta_{2m}(x, h) = \tfrac{1}{2}\{F(x+h) + F(x-h)\} - \sum_{k=0}^{m-1} \frac{h^{2k}}{(2k)!} D_{2k}F(x),$$

and let

$$\overline{D}_{2m}F(x) = \limsup_{h\to 0} \theta_{2m}(x, h),$$

$$\underline{D}_{2m}F(x) = \liminf_{h\to 0} \theta_{2m}(x, h).$$

Received July 24, 1972 and in revised form, August 3, 1972.

Then a finite common value for $^-D_{2m}F(x)$ and $_-D_{2m}F(x)$ implies that $D_{2m}F(x)$ exists and equals this common value.

In a similar way, the odd-ordered s.d.l.V.P. derivative is defined by replacing (1) by

$$(1') \qquad \tfrac{1}{2}\{F(x+h) - F(x-h)\} - \sum_{k=1}^{r} \beta_{2k+1} \frac{h^{2k+1}}{(2k+1)!} = o(h^{2r+1}).$$

Similar changes can be made in (2), (3).

The following lemma is an extension and generalization of Lemma 4, (i) in [**13**]. For a partial converse in the non-symmetric case, see Lemma 10 in [**12**].

LEMMA 1. *Let H be a function and $H'(x) = G(x)$ in a neighborhood of x_0. If for some n, $D_nG(x_0)$ exists, then $D_{n+1}H(x_0)$ exists and is equal to $D_nG(x_0)$.*

Proof. The proof is by induction on n. To see that it is true for $n = 1$, consider, for sufficiently small $h > 0$,

$$\theta_2(H; x_0, h) = \frac{2!}{h^2}\{\tfrac{1}{2}[H(x_0+h) + H(x_0-h)] - H(x_0)\}.$$

Letting $f(h) = 1/2\ \{H(x_0+h) + H(x_0-h)\} - H(x_0)$, and $g(h) = h^2/2!$, one has $f(h) \to 0$ as $h \to 0$ since H is clearly continuous in a neighborhood of x_0, and also $g(h) \to 0$ as $h \to 0$, and $g'(h) = h \neq 0$. Furthermore

$$\frac{f'(h)}{g'(h)} = \frac{H'(x_0+h) - H'(x-h)}{2h} = \frac{G(x_0+h) - G(x_0-h)}{2h},$$

which approaches to $D_1G(x_0)$ as $h \to 0$ if $D_1G(x_0)$ exists. Hence by l'Hôpital's rule, $D_2H(x_0) = \lim_{h\to 0}\theta_2(H;\ x_0,\ h) = D_1G(x_0)$ if $D_1G(x_0)$ exists, completing the proof for $n = 1$.

Now, suppose that the conclusion of the lemma is true for $n < r$, where $r \geqq 2$. We prove that it is also true for $n = r$ as follows. For r even, $r = 2m$, say, suppose that $D_{2m}G(x_0)$ exists. Then $D_{2k}G(x_0)$ exists for $0 \leqq k \leqq m-1$, and hence by induction hypotheses, $D_{2k+1}H(x_0)$ exists and equals $D_{2k}G(x_0)$ for $0 \leqq k \leqq m-1$. Consider

$$\theta_{2m+1}(H; x_0, h) = \frac{(2m+1)!}{h^{2m+1}} \times \left\{\tfrac{1}{2}[H(x_0+h) - H(x_0-h)] - \sum_{k=0}^{m-1} \frac{h^{2k+1}}{(2k+1)!} D_{2k+1}H(x_0)\right\}.$$

Applying l'Hôpital's rule, one gets $\lim_{h\to 0}\theta_{2m+1}(H;\ x_0,\ h) = D_{2m}G(x_0)$, completing the proof for even r. A similar argument will give the case for r odd.

Following James [**9**], we say that a function F is n-smooth at x if $D_{n-2}F(x)$ exists and $\lim_{h\to 0}h\theta_n(F; x, h) = 0$. By an argument similar to that in the proof of Lemma 1, one has

LEMMA 2. *Let H be a function and $H'(x) = G(x)$ in a neighborhood of x_0. Then H is $(n+1)$-smooth at x_0 if G is n-smooth at x_0.*

LEMMA 3. *Let H be a function and $H'(x) = G(x)$ in a neighborhood of x_0. Then for $n \geqq 1$,*

$$(4) \qquad {}^-D_nG(x_0) \geqq {}^-D_{n+1}H(x_0) \geqq {}_-D_{n+1}H(x_0) \geqq {}_-D_nG(x_0)$$

whenever $\theta_n(G; x_0, h)$ makes sense.

Proof. By Lemma 1, if $\theta_n(G; x_0, h)$ makes sense, so does $\theta_{n+1}(H; x_0, h)$. The inequalities (4) then follow from the inequalities [**8**, p. 359]

$$\limsup_{h\to 0} \frac{f'(h)}{g'(h)} \geqq \limsup_{h\to 0} \frac{f(h)}{g(h)} \geqq \liminf_{h\to 0} \frac{f(h)}{g(h)} \geqq \liminf_{h\to 0} \frac{f'(h)}{g'(h)}$$

for suitable choices of f and g.

2. Some properties of n-convex functions. For the definition of n-convex function, we refer to [**1**] and [**9**]. To state two deep results concerning n-convex functions, we recall some concepts first.

A function F defined on $[a, b]$ is said to satisfy the condition (C_{2r}) in $[a, b]$ if

(a) F is continuous in $[a, b]$;

(b) $D_{2k}F$ exists, is finite and has no simple discontinuities in $]a, b[$ for $0 \leqq k \leqq r - 1$;

(c) F is $2r$-smooth at all points in $]a, b[$ except perhaps for points of a countable set.

Similarly, the condition (C_{2r+1}) is defined, so that the condition (C_n) makes sense for all integer $n \geqq 2$.

If it is true that

$$F(x+h) - F(x) = \sum_{k=1}^{r} \alpha_k \frac{h^k}{k!} + o(h^r) \quad \text{as } h \to 0,$$

then α_k $(1 < k < r)$ is called the Peano derivative of order k of F at x, written $\alpha_k = F_{(k)}(x)$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants depending on x only, not on h. It is clear that if $F_{(k)}(x)$ exists, so does $D_kF(x)$ and the two are equal. But the converse is not true in general.

If F possesses Peano derivatives $F_{(k)}(x)$, $1 \leqq k \leqq r - 1$, write

$$\frac{h^r}{r!}\gamma_r(F; x, h) = F(x+h) - F(x) - \sum_{k=1}^{r-1} F_{(k)}(x).$$

Then define

$${}^-F_{(r),+}(x) = \limsup_{h\to 0+} \gamma(F; x, h).$$

${}_-F_{(r),+}$, ${}_-F_{(r),-}$, ${}_-F_{(r),-}$ are similarly defined, and then $F_{(r),+}$, $F_{(r),-}$ are defined in the usual way. It is easy to show that $F_{(r)}(x)$ exists if and only if $F_{(r),+}$, $F_{(r),-}$ exist and are equal and in this case, $F_{(r)}(x) = F_{(r),+}(x) = F_{(r),-}(x)$.

A linear set is called a scattered set if it contains no subset that is dense-in-itself. For properties of scattered sets, we refer to [**11**].

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THEOREM 1. *Let F satisfy the condition (C_n) in $[a, b]$, and*
(i) $^{-}D_nF(x) \geqq 0$ *almost everywhere in* $]a, b[$;
(ii) $^{-}D_nF(x) > -\infty$ *for* $x \in]a, b[\sim S$, *S a scattered set;*
(iii) $\limsup_{h\to 0} h\theta_n(F; x, h) \geqq 0 \geqq \liminf_{h\to 0} h\theta_n(F; x, h)$ *for* $x \in S$.
Then F is n-convex in $[a, b]$.

Note that for $n = 2m$, even, this is just [**1**, Theorem 16], of which the similar argument gives the case $n = 2m + 1$ (odd), too.

THEOREM 2 [**1**, Theorem 7]. *Let F be n-convex in $[a, b]$. Then*
(i) $F^{(r)}$ *exists and is continuous in* $[a, b]$ *for* $1 \leqq r \leqq n - 2$, *where* $F^{(r)}(x)$ *denotes the ordinary rth derivative of F at x;*
(ii) *both* $F_{(n-1),-}$, $F_{(n-1),+}$ *are monotone increasing in* $[a, b]$;
(iii) $F_{(n-1),+} = (F^{(n-2)})_+{}'$ *and* $F_{(n-1),-} = (F^{(n-2)})_-{}'$;
(iv) $F^{(n-1)}(x)$ *exists at all except a countable set of points.*

3. The SC_r-derivative and the SC_r-continuity. We assume the theory of C_nP-integral in [**4**]. For $r \geqq 1$, and for a $C_{r-1}P$-integrable function F on $[a, b]$, let

$$\Delta_r(F; x, h) = \frac{r+1}{2h}\{C_r(F; x, x+h) - C_r(F; x, x-h)\},$$

$$SC_rD_*F(x) = \liminf_{h\to 0} \Delta_r(F; x, h),$$

where $x \in]a, b[$ and $C_r(F; x, x + h)$ is as defined in [**4**]. The notations SC_rD^* and SC_rD then have the obvious meanings. We call $SC_rDF(x)$, if exists, the symmetric Cesáro derivative of order r of F at x, or simply SC_r-derivative of F at x. If $\lim_{h\to 0+} h\Delta_r(F; x, h) = 0$, F is said to be SC_r-continuous at x. It is clear that F is SC_r-continuous at x whenever it is C_r-continuous at x, and $SC_rDF(x)$ exists and equals $C_rDF(x)$ whenever $C_rDF(x)$ exists. But neither of the converses is true. It is also easy to check that the SC_r-derivates and derivatives are measurable.

LEMMA 4. *For $r \geqq 0$, let F be C_r-continuous in $[a, b]$. Then F has no simple discontinuities in $[a, b]$. In particular, every C_rP-integral of a function has no simple discontinuities.*

Proof. For $r = 0$, the result is immediate since C_0-continuity is just the ordinary continuity. For $r \geqq 1$, suppose that $x_0 \in]a, b]$, and $\lim_{x\to x_0-} F(x) = B$. Then for $\epsilon > 0$, there exists $\delta > 0$ such that

$$B - \epsilon < F(x) < B + \epsilon \quad \text{for} \quad x_0 - \delta < x < x_0,$$

or

$$B - \epsilon < F(x) < B + \epsilon \quad \text{for} \quad x_0 - h \leqq x < x_0,$$

https://doi.org/10.4153/CJM-1973-134-7 Published online by Cambridge University Press

where h is such that $0 < h < \delta$. Hence.

$$(B - \epsilon)(x - x_0 + h)^{r-1} \leqq (x - x_0 + h)^{r-1}F(x) \leqq (B + \epsilon)(x - x_0 + h)^{r-1}$$

for $x_0 - h \leqq x < x_0$, which implies that

$$B - \epsilon \leqq \frac{r}{h^r}(C_{r-1}P) - \int_{x_0-h}^{x_0} (x - x_0 + h)^{r-1}F(x)dx \leqq B + \epsilon$$

for $0 < h < \delta$, so that $\lim_{h\to 0+} C_r(F; x_0, x_0 - h) = B$. But

$$F(x_0) = \lim_{h\to 0} C_r(F; x_0, x_0 - h) = \lim_{h\to 0+} C_r(F; x_0, x_0 - h).$$

Hence $F(x_0) = B$.

Similarly, if $x_0 \in [a, b[$, and $\lim_{x\to x_0+} F(x) = B'$, then $F(x_0) = B'$. Hence F has no simple discontinuities in $[a, b]$.

The last statement of the lemma is now immediate since it is well-known that each C_rP-integral is C_r-continuous.

LEMMA 5. *For $n \geqq 0$, let F be C_nP-integrable on $[a, b]$, and for $x \in [a, b]$, let*

$$G_n(x) = (C_nP) - \int_a^x F(t)dt,$$

$$G_k(x) = (C_kP) - \int_a^x G_{k+1}(t)dt, \qquad 0 \leqq k \leqq n - 1,$$

$$G(x) = G_0(x).$$

Then

(i) *G is continuous in $[a, b]$;*

(ii) *if F is SC_{n+1}-continuous at x, then D_nG exists and $D_{n-2k}G(x) = G_{n-2k}(x)$ for $0 \leqq k \leqq [n/2]$, and G is $(n + 2)$-smooth at x, and $\theta_{n+2}(G; x, x + h) = \Delta_{n+1}(F; x, h)$;*

(iii) *if F is C_{n+1}-continuous at x, then $G_{(n+1)}(x)$ exists and $G_{(k)}(x) = G_k(x)$ for $0 \leqq k \leqq n + 1$, where $G_{n+1} = F$.*

Proof. (i) is immediate since G is just a C_0P-integral. For (ii) and (iii), note that by integration by parts,

$$(5) \quad C_{n+1}(F; x, x + h) = \frac{(n + 1)!}{h^{n+1}}\left\{G(x + h) - G(x) - \sum_{k=1}^{n} \frac{h^k}{k!} G_k(x)\right\},$$

and

$$C_{n+1}(F; x, x - h) = \frac{(n + 1)!}{(-h)^{n+1}}\left\{G(x - h) - G(x) - \sum_{k=1}^{n} \frac{(-h)^k}{k!} G_k(x)\right\}$$

for $h \neq 0$ with $x + h \in [a, b]$. Hence for n even, say $n = 2m$,

$$(5e) \quad C_{n+1}(F; x, x+h) - C_{n+1}(F; x, x-h) = \frac{(2m+1)!}{h^{2m+1}} \times \left\{ G(x+h) + G(x-h) - 2 \sum_{k=1}^{m} \frac{h^{2k}}{(2k)!} G_{2k}(x) \right\};$$

and for n odd, say $n = 2m + 1$,

$$(5o) \quad C_{n+1}(F; x, x+h) - C_{n+1}(F; x, x-h) = \frac{(2m+2)!}{h^{2m+2}} \times \left\{ G(x+h) - G(x-h) - 2 \sum_{k=0}^{m} \frac{h^{2k+1}}{(2k+1)!} G_{2k+1}(x) \right\}.$$

For both cases, if F is SC_{n+1}-continuous at x, then $D_nG(x)$ exists and $D_{n-2k}G(x) = G_{n-2k}(x)$ for $0 \leqq k \leqq [n/2]$, and G is $(n + 2)$-smooth at x, where $[n/2]$ = the greatest integer less than $n/2 + 1$. Furthermore, $\theta_{n+2}(G; x, h) = \Delta_{n+1}(F; x, h)$, proving (ii). (iii) follows from the equality (5).

Remark. If $D_{n-2k}G(x) = G_{n-2k}(x)$ for $0 \leqq k \leqq [n/2]$, and G is $(n + 2)$-smooth at x, then F is SC_{n+1}-continuous at x. This is clear since replacing $G_{n-2k}(x)$ by $D_{n-2k}G(x)$ in (5e) and (5o) one has that

$$C_{n+1}(F; x, x+h) - C_{n+1}(F; x, x-h) = \frac{2}{n+2} h\theta_{n+2}(G; x, h).$$

LEMMA 6. *For $n \geqq 0$, let F be C_nP-integrable on $[a, b]$, and SC_{n+1}-continuous in $]a, b[$, and G be defined as in Lemma 5. If*

(a) $SC_{n+1}D^*F(x) \geqq 0$ *almost everywhere in* $[a, b]$, *and*

(b) $SC_{n+1}D^*F(x) > -\infty$ *for* $x \in]a, b[\sim S$, *S a scattered set,*

then G is $(n + 2)$-convex in $[a, b]$.

Proof. This is immediate since by Lemma 5, (ii), and Lemma 4, G satisfies all the conditions in Theorem 1 with $n + 2$ replacing n.

THEOREM 3. *For $n \geqq 0$, let F be C_nP-integrable on $[a, b]$ and SC_{n+1}-continuous in $]a, b[$. If*

(a) $SC_{n+1}D^*F(x) \geqq 0$ *almost everywhere in* $[a, b]$,

(b) $SC_{n+1}D^*F(x) > -\infty$ *for* $x \in]a, b[\sim S$, *S scattered, and*

(c) *F is C_{n+1}-continuous in a set $B \subset [a, b]$, then F is monotone increasing in B.*

Proof. Let G be defined as in Lemma 5. Then by Lemma 6, G is $(n + 2)$-convex in $[a, b]$, so that by Theorem 2, (iv), $G^{(n+1)}$ and hence $G_{(n+1)}$ exists at all except a countable set of points. By Theorem 2, (ii), $G_{(n+1)}$ is monotone increasing where it exists. Thus the condition (c) and Lemma 5, (iii) imply that F is monotone increasing in B.

THEOREM 4. *For $n \geqq 0$, let F be C_nP-integrable on $[a, b]$, and $x_0 \in]a, b[$. If F is SC_{n+1}-continuous at x_0, then F is SC_{n+2}-continuous at x_0, and*

$$(6) \quad SC_{n+1}D^*F(x_0) \geqq SC_{n+2}D^*F(x_0) \geqq SC_{n+2}D_*F(x_0) \geqq SC_{n+1}D_*F(x_0).$$

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Proof. Note first that F is $C_{n+1}P$-integrable on $[a, b]$ by the consistency of the CP-scale. For $x \in [a, b]$, let

$$G_n(x) = (C_nP) - \int_a^x F(t)dt,$$

$$G_k(x) = (C_kP) - \int_a^x G_{k+1}(t)dt \qquad \text{for } 0 \leqq k \leqq n-1,$$

$$H_{n+1}(x) = (C_{n+1}P) - \int_a^x F(t)dt,$$

$$H_k(x) = (C_kP) - \int_a^x H_{k+1}(t)dt \qquad \text{for } 0 \leqq k \leqq n.$$

Then $H_{k+1} = G_k$ for $0 \leqq k \leqq n$ and

$$H_0(x) = (L) - \int_a^x G_0(t)dt.$$

By Lemma 5, (ii), G_0 is $(n+2)$-smooth at x_0, so that H_0 is $(n+3)$-smooth at x_0 by Lemma 2. Hence by the remark following Lemma 5, F is SC_{n+2}-continuous at x_0. The inequalities (6) follow from Lemma 5 and Lemma 3, completing the proof.

THEOREM 5. *Let $\{M_k\}$ be a sequence of SC_n-continuous functions in $]a, b[$, and each M_k is C_n-continuous in a set $B \subset [a, b]$ with $a, b \in B$ and the measure of B being $b - a$. Suppose that $M_k(x) \to M(x)$ as $k \to +\infty$ uniformly in B. Then M is SC_n-continuous in $]a, b[$ and C_n-continuous in B.*

Proof. Given $\epsilon > 0$, choose k such that for all $x \in B$, $|M(x) - M_k(x)| < \frac{1}{3}\epsilon$. For each $c \in B$, choose $\delta > 0$ such that $|C_n(M_k; c, c+h) - M_k(c)| < \frac{1}{3}\epsilon$ whenever $|h| < \delta$ with $x + h \in [a, b]$. Then

$$|C_n(M; c, c+h) - C_n(M_k; c, c+h)| < \tfrac{1}{3}\epsilon,$$

so that $|C_n(M; c, c+h) - M(c)| < \epsilon$ whenever $|h| < \delta$ with $x + h \in [a, b]$, proving that M is C_n-continuous at c.

That M is SC_n-continuous at each point $c \in]a, b[$ is proved in a similar way, only replacing $M_k(c)$, $M(c)$ in the above argument by $C_n(M_k; c-h, c)$ and $C_n(M; c-h, c)$, h now being restricted to $c \pm h \in [a, b]$.

4. The SC_nP-Integral. We have defined in [**2**] a system $SC_nP = (S\mathscr{M}^n, SC_nD, \mathscr{B}, \mathscr{N}, {}^-I_n)$, SC_nD being the SC_nD_* here. By Theorem 3 and Theorem 5, it is easy to check that SC_nP is in fact a derivate system as defined in [**2**], and hence one obtains a SC_nP-integral and its usual properties follow from the general theory in [**2**]. For completeness, we give the direct definition of the SC_nP-integral here, $n = 1, 2, 3, \ldots$.

Suppose that f is a function defined and finite almost everywhere in $[a, b]$, and B a subset of $[a, b]$ of measure $b - a$, and $a, b \in B$. A $C_{n-1}P$-integrable function M will be called an SC_nP-major function of f on $[a, b]$ with base B if

(a) M is SC_n-continuous in $]a, b[$ and C_n-continuous in B;

(b) $SC_nD_*M(x) \geqq f(x)$ almost everywhere in $]a, b[$;

(c) $SC_nD_*M(x) > -\infty$ except perhaps in a scattered set;

(d) $M(a) = 0$.

An SC_nP-minor function is similarly defined. If f has SC_nP-major and -minor functions and if

$$\inf M(b) = \sup m(b) \neq \pm \infty,$$

then f is said to be SC_nP-integrable on $[a, b]$ with base B, and the common value, denoted by

$$(SC_nP) - \int_{[a,b]}^{B} f(t)dt,$$

is called the SC_nP-integral of f on $[a, b]$ with base B. As remarked in [**2**], we can often without ambiguity leave the base unspecified.

Except for those properties obtainable from the general theory in [**2**], it is easy to see that the SC_nP-integral is more general than the C_nP-integral [**4**] since $SC_nD_*M(x) \geqq C_nD_*M(x)$. Furthermore, we have the consistency theorem for the scale:

THEOREM 6. *If f is SC_nP-integrable on $[a, b]$ with base B, then f is $SC_{n+1}P$-integrable on $[a, b]$ with base B and the two integrals are equal.*

Proof. This is immediate from Theorem 4 and the general comparison theorem in [**2**].

Remarks. (i) Note that the SC_1P-integral is equivalent to Burkill's SCP-integral [**5**] as we have remarked in [**2**].

(ii) Burkill in [**5**] listed an integration by parts formula for his SCP-integral and stated that the proof followed from that given for the CP-integral in [**3**]. This is not true since the proof in [**3**] used essentially the following inequality

$$CD_*(MG)(x) \geqq M(x)G'(x) + [CD_*M(x)]G(x),$$

but we do not have a similar inequality for the SC_1D-derivate. For example, let

$$\begin{aligned} M(x) &= x^{-1/2}, && \text{for } x > 0, \\ &= (-x)^{-1/2}, && \text{for } x < 0, \\ &= k, && \text{for } x = 0, \text{ where } k \text{ is any constant,} \end{aligned}$$

and let $G(x) = -x$. Then

$$SC_1D(MG)(0) = -\infty \ngeqq -k = M(0)\,G'(0) + [SC_1DM(0)]G(0).$$

Thus, whether the formula for SCP-integral in [**5**] is true remains an open question.

If such an integration by parts formula exists for the SC_1P-integral, then one can use this to define the SC_2P-integral instead of using C_1P-integral. Then a more general scale would be obtained by induction.

5. The SC_nP-integral and the P^{n+1}-integral. As we mentioned in the introduction, in this section we are going to investigate the relation of the P^{n+1}-integral and the SC_nP-integral.

By P^{n+1}-integral, we mean the modified symmetric one as in [**10**]. For convenience, we give the definition of its major functions here.

Let f be a function defined almost everywhere in $[a, b]$, and let a_i, $i = 1, 2, 3, \ldots, n + 1$, be fixed points such that $a = a_1 < a_2 < \ldots < a_{n+1} = b$. A function Q is called a J_{n+1}-major function of f over (a_i) if

(a) Q satisfies the condition (C_{n+1}) in $[a, b]$ (cf. Section 2);

(b) $\underline{D}_{n+1}Q(x) \geqq f(x)$ almost everywhere in $[a, b]$;

(c) $\underline{D}_{n+1}Q(x) > -\infty$, $x \in \,]a, b[\sim S$, S a scattered set;

(d) $Q(a_i) = 0$ for $i = 1, 2, 3, \ldots, n + 1$.

THEOREM 7. *Let f be SC_nP-integrable on $[a, b]$ with base B. Then f is P^{n+1}-integrable over $(a_i; c)$, where $a = a_1 < a_2 < \ldots < a_n < a_{n+1} = b$, and $c \in [a, b]$. Moreover, letting*

$$F_n(x) = (SC_nP) - \int_a^x f(t)dt, \qquad x \in B,$$

$$F_k(x) = (C_kP) - \int_a^x F_{k+1}(t)dt, \qquad x \in [a, b], \qquad 0 \leqq k \leqq n - 1,$$

$$F = F_0,$$

one has for $a_s \leqq c < a_{s+1}$,

$$(-1)^s \int_{(a_i)}^c f(t)d_{n+1}t = F(c) - \sum_{i=1}^{n+1} \lambda(c; a_i)F(a_i), \tag{7}$$

where $\lambda(c; a_i) = \prod_{j\neq i}(c - a_j)/(a_i - a_j)$ is a polynomial in c of degree at most n.

Proof. Let M be an SC_nP-major function and let

$$G(x) = (C_0P) - \int_a^x (C_1P) - \int_a^{t_1} (C_2P) - \int_a^{t_2} \ldots (C_{n-1}P) - \int_a^{t_{n-1}} M(t_n)dt_ndt_{n-1}\ldots dt_2dt_1.$$

Then by Lemma 4 and Lemma 5, G satisfies conditions (a), (b), (c) in the above definition. Hence setting

$$Q(x) = G(x) - \sum_{i=1}^{n+1} \lambda(x; a_i)G(a_i),$$

we see that Q is a J_{n+1}-major function of f over (a_i). Similarly, an SC_nP-minor function m yields a J_{n+1}-minor function

$$q(x) = g(x) - \sum_{i=1}^{n+1} \lambda(x; a_i) g(a_i),$$

where g is defined similar to G.

For $\epsilon > 0$, if we choose M, m such that

$$M(b) - m(b) < \epsilon \Big/ \left[1 + \sum_{i=1}^{n+1} \lambda(c; a_i)\right](b - a)^n,$$

then the corresponding Q, q have

$$|Q(c) - q(c)| \leqq |G(c) - g(c)| + \sum |\lambda(c; a_i)| |G(a_i) - g(a_i)| \leqq \epsilon.$$

Hence, the P^{n+1}-integrability of f follows.

The equality (7) follows as above by using the property that F_n can be uniformly approximated in B by a sequence of SC_n-major or minor functions.

COROLLARY 1. *$F_{(n)}(x)$ exists for each x in B and $D_{n-1}F(x)$ exists for each $x \in]a, b[$. Furthermore, $F_{(n)} = F_n$ on B, and $D_kF = F_k$ on $]a, b[$ for $k = 0, 1, 2, \ldots, n - 1$, where F, F_k are those in Theorem 7.*

Proof. As F_n can be proved to be C_n-continuous in B and SC_n-continuous in $]a, b[$ (see property (F) in [**2**]), the required results follow from Lemma 5.

COROLLARY 2. *There exists a function which is P^{n+1}-integrable on $[a, b]$ but not SC_n-Pintegrable on $[a, b]$.*

Proof. This is similar to that of Cross in [**7**] for $n = 1$. In fact,

$$\begin{aligned} \text{if } n \text{ is odd, let } F(x) &= x \cos 1/x, && \text{for } x \neq 0,\\ &\quad 0, && \text{for } x = 0,\\ \text{if } n \text{ is even, let } F(x) &= x \sin 1/x, && \text{for } x \neq 0,\\ &\quad 0, && \text{for } x = 0. \end{aligned}$$

In either case, let

$$\begin{aligned} f(x) &= F^{(n+1)}(x), && \text{for } x \neq 0,\\ &= 0, && \text{for } x = 0. \end{aligned}$$

Then $D_{n+1}F(x) = f(x)$ for all x, including $x = 0$, and as shown in [**9**], f is P^{n+1}-integrable over any interval containing 0. However, f is not SC_nP-integrable over $[0, b]$ for any $b > 0$. For otherwise, it would follow from Corollary 1 that $F_{(n)}(0)$ exists. But not even $F_{(1)}(0)$ exists.

COROLLARY 3. *Let f be periodic of period $2b$, $b > 0$. For $n \geqq 1$, let $m = [n - 1/2]$. Then if f is SC_nP-integrable on $[-2(m + 1)b, 2(n - m)b]$ with a base B, one has*

$$\frac{1}{(2b)^n}\binom{n+1}{m+1}\int_{(a_i)}^{0} f(t) d_{n+1}t = (SC_nP) - \int_{[-b,b]} f(t) dt,$$

where $(a_i) = (-2(m + 1)b, -2mb, -2(m - 1)b, \ldots, -2b, 2b, 4b, \ldots, 2(n - m)b)$.

https://doi.org/10.4153/CJM-1973-134-7 Published online by Cambridge University Press

The proof, exactly similar to that of Cross in [**6**] for the unsymmetric case, is omitted.

Remarks. (i) Although P^{n+1}-integral is more general than the SC_nP-integral, the SC_nP-integral has the nice "additive" property that a function SC_nP-integral over two abutting intervals is SC_nP-integrable over their union, (see property (C) in [**2**]), while P^{n+1}-integral has no such property. For example, let F be as defined in the proof of Corollary 2. Consider the function f defined by

$$\begin{aligned} f(x) &= F^{(n+1)}(x), \text{ for } x \in\,]0, i/\pi], \\ &= 0, \qquad\quad \text{ for } x \in [-i/\pi, 0], \end{aligned}$$

where $i = 2$ if n is odd and 1 if n is even. Then f is P^{n+1}-integrable over each of the intervals $[-i/\pi, 0]$ and $[0, i/\pi]$, but not over $[-i/\pi, i/\pi]$. Note that this has been pointed out by Skvorcov in [**14**] for the case $n = 1$, i.e. for the P^2-integral.

(ii) As mentioned in [**2**], the SC_1P-integral solves the coefficient problem of convergent trigonometric series. Whether the SC_nP-integral (for $n \geqq 2$) solves the coefficient problem of $(C, n-1)$ summable trigonometric series, considered by James in [**10**], is under consideration.

References

1. P. S. Bullen, *A criterion for n-convexity*, Pacific J. Math. *36* (1971), 81–98.

2. P. S. Bullen and C. M. Lee, *On the integrals of Perron type* (to appear in Trans. Amer. Math. Soc.).

3. J. C. Burkill, *The Cesáro-Perron integral*, Proc. London Math. Soc. *34* (1932), 314–322.

4. ——— *The Cesáro-Perron scale of integration*, Proc. London Math. Soc. *39* (1935), 541–552.

5. ——— *Integrals and trigonometric series*, Proc. London Math. Soc. *1* (1951), 46–57.

6. G. E. Cross, *The relation between two definite integrals*, Proc. Amer. Math. Soc. *11* (1960), 578–579.

7. ——— *The relation between two symmetric integrals*, Proc. Amer. Math. Soc. *14* (1963), 185–190.

8. E. W. Hobson, *The theory of functions of a real variable*, 3rd ed. (Cambridge Univ. Press, Cambridge, 1927).

9. R. D. James, *Generalized nth primitives*, Trans Amer. Math. Soc. *76* (1954), 149–176.

10. ——— *Summable trigonometric series*, Pacific J. Math. *6* (1956), 99–110.

11. K. Kuratowski, *Topology* (Academic Press, New York, 1966).

12. J. Marcinkiewicz and A. Zygmund, *On the differentiability of functions and summability of trigonometric series*, Fund. Math. *27* (1937), 38–69.

13. W. L. C. Sargent, *On the generalized derivatives and Cesáro-Denjoy integrals*, Proc. London Math. Soc. *52* (1951), 365–376.

14. V. A. Skvorcov, *Concerning definitions of P^2- and SCP-integrals*, Vestnik Moskov. Univ. Ser. I Mat. Meh. *21* (1966), 12–19.

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https://doi.org/10.4153/CJM-1973-134-7 Published online by Cambridge University Press