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HARDY SPACES ON METRIC MEASURE SPACES WITH GENERALIZED SUB-GAUSSIAN HEAT KERNEL ESTIMATES

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Abstract

Hardy space theory has been studied on manifolds or metric measure spaces equipped with either Gaussian or sub-Gaussian heat kernel behaviour. However, there are natural examples where one finds a mix of both behaviours (locally Gaussian and at infinity sub-Gaussian), in which case the previous theory does not apply. Still we define molecular and square function Hardy spaces using appropriate scaling, and we show that they agree with Lebesgue spaces in some range. Besides, counterexamples are given in this setting that the H^p space corresponding to Gaussian estimates may not coincide with L^p. As a motivation for this theory, we show that the Riesz transform maps our Hardy space H^1 into L^1.

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1. Introduction

The study of Hardy spaces originated in the 1910s and at the very beginning was confined to Fourier series and complex analysis in one variable. Since the 1960s, it has been transferred to real analysis in several variables or more generally to analysis on metric measure spaces. There are many different equivalent definitions of Hardy spaces, which involve suitable maximal functions, the atomic decomposition, the molecular decomposition, singular integrals, square functions and so on. See, for instance, the classical references [16, 18, 23, 37]. More recently, a lot of work has been devoted to the theory of Hardy spaces associated with operators; see, for example, [4, 5, 30, 40] and the references therein.

In [5], Auscher *et al.* studied Hardy spaces with respect to the Hodge Laplacian on Riemannian manifolds with the doubling volume property by using Davies–Gaffney-type estimates. They defined Hardy spaces of differential forms of all degrees via

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molecules and square functions, on which the Riesz transform is H^p bounded for $1 \le p \le \infty$. Comparing with the Lebesgue spaces, we have $H^p \subset L^p$ for $1 \le p \le 2$ and $L^p \subset H^p$ for $p > 2$. Moreover, under the assumption of a Gaussian heat kernel upper bound, H^p coincides with L^p for $1 < p < \infty$.

In [30], Hofmann *et al.* further developed the theory of H^1 and BMO spaces adapted to a metric measure space (M, d, μ) with the volume doubling property endowed with a nonnegative self-adjoint operator L, which generates an analytic semigroup $\{e^{-tL}\}_{t>0}$ satisfying the so-called Davies–Gaffney estimate: there exist $C, c > 0$ such that for any open sets $U_1, U_2 \subset M$ and, for every $f_i \in L^2(M)$ with $\operatorname{supp} f_i \subset U_i$, $i = 1, 2$,

$$|\langle e^{-tL} f_1, f_2\rangle| \le C \exp\Big(-\frac{\operatorname{dist}^2(U_1, U_2)}{ct}\Big)\|f_1\|_2\|f_2\|_2, \quad \forall t > 0, \tag{1.1}$$

where $\operatorname{dist}(U_1, U_2) := \inf_{x\in U_1, y\in U_2} d(x, y)$. The authors extended results of [5] by obtaining an atomic decomposition of the H^1 space.

More generally, instead of (1.1), if M satisfies the Davies–Gaffney estimate of order m with $m \ge 2$: for all $x, y \in M$ and for all $t > 0$,

$$\|\mathbb{1}_{B(x,t^{1/m})} e^{-tL} \mathbb{1}_{B(y,t^{1/m})}\|_{2\to 2} \le C \exp\Big(-c\Big(\frac{d(x, y)}{t}\Big)^{m/(m-1)}\Big), \tag{1.2}$$

where the symbol $\mathbb{1}_E$ stands for the characteristic function of a Borel set $E \subset M$. Kunstmann and Uhl [33, 40] defined Hardy spaces via square functions and via molecules adapted to (1.2), where the two H^1 spaces are also equivalent. Here and in the sequel, $B(x, r)$ denotes the ball of centre $x \in M$ and radius $r > 0$ and $V(x, r) = \mu(B(x, r))$. In addition, if the $L^{p_0} - L^{p_0'}$ off-diagonal estimates of order m hold: for all $x, y \in M$ and for all $t > 0$,

$$\|\mathbb{1}_{B(x,t^{1/m})} e^{-tL} \mathbb{1}_{B(y,t^{1/m})}\|_{p_0\to p_0'} \le \frac{C}{V^{1/p_0-1/p_0'}(x, t^{1/m})} \exp\Big(-c\Big(\frac{d(x, y)}{t}\Big)^{m/(m-1)}\Big), \tag{1.3}$$

with p_0' the conjugate of p_0, then the Hardy space H^p defined via square functions coincides with L^p for $p \in (p_0, 2)$.

However, there are natural examples where one finds a mix of both behaviours (1.1) and (1.2), in which case the previous Hardy space theory does not apply. For example, on fractal manifolds, the heat kernel behaviour is locally Gaussian and at infinity sub-Gaussian (see Section 2.1 for more details). We aim to develop a proper Hardy space theory for this setting. An important motivation for our Hardy space theory is to study the Riesz transform on fractal manifolds, where the weak-type $(1, 1)$ boundedness has recently been proved in a joint work by the author with Coulhon *et al.* [14].

In this paper, we work in the metric measure space endowed with a measure which satisfies the doubling volume property and a nonnegative self-adjoint operator which satisfies the L^2 off-diagonal estimate with different local and global decays (see (DG_ρ) below). The specific description will be found below in Section 1.1. We define two classes of Hardy spaces in this setting, via molecules and via conical square functions;

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see Section 1.2. Both definitions have the scaling adapted to the off-diagonal decay (DG_ρ).

In Section 3, we identify the two different H^1 spaces. The molecular H^1 spaces are always convenient spaces to deal with Riesz transforms and other sublinear operators, while the H^p, $p \geq 1$, spaces defined via conical square functions possess certain good properties like real and complex interpolation. The identification of both spaces gives us a powerful tool to study the Riesz transform, Littlewood–Paley functions, boundary value problems for elliptic operators and so on.

In Section 4, we compare the Hardy spaces defined via conical square functions with the Lebesgue spaces. Assuming further an $L^{p_0} - L^{p_0'}$ off-diagonal estimate for some $1 \leq p_0 < 2$ with different local and global decays for the heat semigroup, we show the equivalence of our H^p spaces and the Lebesgue spaces L^p for $p_0 < p < p_0'$. We also justify that the scaling for the Hardy spaces is the right one, by disproving this equivalence of H^p and L^p for p close to 2 on some fractal Riemannian manifolds. As far as we know, no previous results are known in this direction.

In Section 5, we shall apply our theory to prove that the Riesz transform is $H^1 - L^1$ bounded on fractal manifolds. The proof is inspired by [14] (see [24, 25] for the original proof and the related Hardy spaces in the discrete setting), where the integrated estimate for the gradient of the heat kernel plays a crucial role.

In the following, we will introduce our setting, the definitions and the main results more specifically.

Notation. Throughout this paper, we denote $u \simeq v$ if $v \lesssim u$ and $u \lesssim v$, where $u \lesssim v$ means that there exists a constant C (independent of the important parameters) such that $u \leq Cv$.

For a ball $B \subset M$ with radius $r > 0$ and given $\alpha > 0$, we write αB as the ball with the same centre and the radius αr. We denote $C_1(B) = 4B$ and $C_j(B) = 2^{j+1}B \backslash 2^j B$ for $j \geq 2$.

1.1. The setting. We shall assume that M is a metric measure space satisfying the doubling volume property: for any $x \in M$ and $r > 0$,

$$V(x, 2r) \lesssim V(x, r), \tag{D}$$

and the L^2 Davies–Gaffney estimate with different local and global decays for the analytic semigroup $\{e^{-tL}\}_{t>0}$ generated by the nonnegative self-adjoint operator L, that is, for all $x, y \in M$,

$$\|\mathbb{1}_{B(x,t)} e^{-\rho(t)L} \mathbb{1}_{B(y,t)}\|_{2\to 2} \lesssim \begin{cases} \exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\beta_1/(\beta_1-1)}\right) & \text{if } 0 < t < 1, \\ \exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\beta_2/(\beta_2-1)}\right) & \text{if } t \geq 1, \end{cases} \tag{DG_ρ}$$

where $1 < \beta_1 \leq \beta_2$ and

$$\rho(t) = \begin{cases} t^{\beta_1} & \text{if } 0 < t < 1, \\ t^{\beta_2} & \text{if } t \geq 1. \end{cases} \tag{1.4}$$

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Recall a simple consequence of (D): there exists $\nu > 0$ such that

$$\frac{V(x,r)}{V(x,s)} \lesssim \left(\frac{r}{s}\right)^{\nu}, \quad \forall x \in M,\ r \geq s > 0. \tag{1.5}$$

It follows that

$$V(x,r) \lesssim \left(1 + \frac{d(x,y)}{r}\right)^{\nu} V(y,r), \quad \forall x \in M,\ r \geq s > 0.$$

Therefore,

$$\int_{d(x,y)<r} \frac{1}{V(x,r)} d\mu(x) \simeq 1, \quad \forall y \in M,\ r > 0. \tag{1.6}$$

If M is noncompact, we also have a reverse inequality of (1.5) (see, for instance, [27, page 412]). That is, there exists $\nu' > 0$ such that

$$\frac{V(x,r)}{V(x,s)} \gtrsim \left(\frac{r}{s}\right)^{\nu'}, \quad \forall x \in M,\ r \geq s > 0. \tag{1.7}$$

Also notice that in (1.4), if necessary we may smooth $\rho(t)$ as

$$\rho(t) = \begin{cases} t^{\beta_1} & \text{if } 0 < t \leq 1/2, \\ \text{smooth part} & \text{if } 1/2 < t < 2, \\ t^{\beta_2} & \text{if } t \geq 2, \end{cases}$$

with $\rho'(t) \simeq 1$ for $1/2 < t < 2$, which we still denote by $\rho(t)$. Since $\rho'(t)/\rho(t) = \beta_1/t$ for $0 < t \leq 1/2$ and $\rho'(t)/\rho(t) = \beta_2/t$ for $t \geq 2$, we have in a uniform way

$$\frac{\rho'(t)}{\rho(t)} \simeq \frac{1}{t}. \tag{1.8}$$

We say that M satisfies an $L^{p_0} - L^{p_0'}$ off-diagonal estimate for some $1 < p_0 < 2$ if

$$\|\mathbb{1}_{B(x,t)} e^{-\rho(t)L} \mathbb{1}_{B(y,t)}\|_{p_0 \to p_0'} \lesssim \begin{cases} \dfrac{1}{V^{1/p_0 - 1/p_0'}(x,t)} \exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\beta_1/(\beta_1-1)}\right) & \text{if } 0 < t < 1, \\ \dfrac{1}{V^{1/p_0 - 1/p_0'}(x,t)} \exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\beta_2/(\beta_2-1)}\right) & \text{if } t \geq 1 \end{cases} \tag{$DG_\rho^{p_0}$}$$

and a generalized pointwise sub-Gaussian heat kernel estimate if for all $x, y \in M$,

$$p_{\rho(t)}(x,y) \lesssim \begin{cases} \dfrac{1}{V(x,t)} \exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\beta_1/(\beta_1-1)}\right) & \text{if } 0 < t < 1, \\ \dfrac{1}{V(x,t)} \exp\left(-c\left(\dfrac{d(x,y)}{t}\right)^{\beta_2/(\beta_2-1)}\right) & \text{if } t \geq 1. \end{cases} \tag{UE_ρ}$$

Examples of fractal manifolds satisfy (UE_ρ) with $\beta_1 = 2$ and $\beta_2 > 2$; see Section 2 below for more information.

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1.2. Definitions. Recall that M satisfies (D) and (DG_ρ). We shall define the H^1 space via molecules and H^p spaces via square functions for $p \geq 1$.

DEFINITION 1.1. Let $\varepsilon > 0$ and K be an integer such that $K > \nu/2\beta_1$, where ν is in (1.5). A function $a \in L^2(M)$ is called a $(1, 2, \varepsilon)$-molecule associated to L if there exist a function $b \in \mathcal{D}(L)$ and a ball B with radius r_B such that:

(1) $a = L^K b$;

(2) we have for every $k = 0, 1, \ldots, K$ and $i = 0, 1, 2, \ldots,$

$$\|(\rho(r_B)L)^k b\|_{L^2(C_i(B))} \leq \rho^K(r_B)2^{-i\varepsilon}V(2^iB)^{-1/2}. \tag{1.9}$$

DEFINITION 1.2. We say that $f = \sum_{n=0}^\infty \lambda_n a_n$ is a molecular $(1, 2, \varepsilon)$-representation of f if $(\lambda_n)_{n\in\mathbb{N}} \in l^1$, each a_n is a molecule as above and the sum converges in the L^2 sense. We denote the collection of all the functions with a molecular representation by $\mathbb{H}^1_{L,\rho,\mathrm{mol}}$, where the norm of $f \in \mathbb{H}^1_{L,\rho,\mathrm{mol}}$ is given by

$$\|f\|_{\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M)} = \inf\Big\{\sum_{n=0}^\infty |\lambda_n| : f = \sum_{n=0}^\infty \lambda_n a_n \text{ is a molecular } (1, 2, \varepsilon)\text{-representation}\Big\}.$$

The Hardy space $H^1_{L,\rho,\mathrm{mol}}(M)$ is defined as the completion of $\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M)$ with respect to this norm.

Consider the following conical square function:

$$S^\rho_h f(x) = \bigg(\iint_{\Gamma(x)} |\rho(t)Le^{-\rho(t)L}f(y)|^2 \frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\bigg)^{1/2}, \tag{1.10}$$

where the cone $\Gamma(x) = \{(y, t) \in M \times (0, \infty) : d(y, x) < t\}$.

We define first the $L^2(M)$ adapted Hardy space $H^2(M)$ as the closure of the range of L in $L^2(M)$ norm, that is, $H^2(M) := \overline{R(L)}$.

DEFINITION 1.3. The Hardy space $H^p_{L,S^\rho_h}(M)$, $p \geq 1$, is defined as the completion of the set $\{f \in H^2(M) : \|S^\rho_h f\|_{L^p} < \infty\}$ with respect to the norm $\|S^\rho_h f\|_{L^p}$. The $H^p_{L,S^\rho_h}(M)$ norm is defined by $\|f\|_{H^p_{L,S^\rho_h}(M)} := \|S^\rho_h f\|_{L^p(M)}$.

For $p = 2$, the operator S^ρ_h is bounded on $L^2(M)$. Indeed, for every $f \in L^2(M)$,

$$\begin{aligned}
\|S^\rho_h f\|^2_{L^2(M)} &= \int_M \iint_{\Gamma(x)} |\rho(t)Le^{-\rho(t)L}f(y)|^2 \frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\,d\mu(x)\\
&\simeq \iint_{M\times(0,\infty)} |\rho(t)Le^{-\rho(t)L}f(y)|^2\,d\mu(y)\frac{dt}{t}\\
&\simeq \iint_{M\times(0,\infty)} |\rho(t)Le^{-\rho(t)L}f(y)|^2\,d\mu(y)\frac{\rho'(t)\,dt}{\rho(t)}\\
&= \int_0^\infty \langle(\rho(t)L)^2e^{-2\rho(t)L}f, f\rangle\frac{\rho'(t)\,dt}{\rho(t)} \simeq \|f\|^2_{L^2(M)}.
\end{aligned} \tag{1.11}$$

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Note that the second step follows from the Fubini theorem and (1.6) in Section 1.1. The third step is obtained by using the fact (1.8): $\rho'(t)/\rho(t) \simeq 1/t$. The last one is a consequence of spectral theory.

REMARK 1.4. The above definitions are similar to those in [30] (see also [5] for 1-forms on Riemannian manifolds) and [33, 40]. The difference is that we replace t^2 or t^m by $\rho(t)$ in (1.9) and (1.10).

In the case when $\rho(t) = t^2$, we denote S_h^ρ by S_h, that is,

$$S_h f(x) := \left(\iint_{\Gamma(x)} |t^2 L e^{-t^2 L} f(y)|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2}$$

and denote H^p_{L,S_h^ρ} by H^p_{L,S_h}.

1.3. Main results. We first obtain the equivalence between H^1 spaces defined via molecules and via square functions.

THEOREM 1.5. *Let M be a metric measure space satisfying the doubling volume property (D) and the L^2 off-diagonal heat kernel estimate (DG_ρ). Then $H^1_{L,\rho,\mathrm{mol}}(M) = H^1_{L,S_h^\rho}(M)$, which we denote by $H^1_{L,\rho}(M)$. Moreover,*

$$\|f\|_{H^1_{L,\rho,\mathrm{mol}}(M)} \simeq \|f\|_{H^1_{L,S_h^\rho}(M)}.$$

Now we compare $H^p_{L,S_h^\rho}(M)$ and L^p for $1 < p < \infty$.

Recall that on a Riemannian manifold satisfying the doubling volume property (D) and the Gaussian upper bound for the heat kernel of the operator, we have $H^p_{L,S_h}(M) = L^p(M)$, $1 < p < \infty$; see, for example, [5, Theorem 8.5] for Hardy spaces of 0-forms on Riemannian manifolds. However, in general, the equivalence is not known. It is also proved in [33, 40] that if the $L^{p_0} - L^{p_0'}$ off-diagonal estimates of order m (1.3) hold, then the Hardy space $\rho(t) = t^m$ (see Remark 1.4) coincides with L^p for $p \in (p_0, 2)$.

Our result in this direction is the following theorem.

THEOREM 1.6. *Let M be a noncompact metric measure space as above. Let $1 \le p_0 < 2$ and ρ be as above. Suppose that M satisfies (D) and ($DG_\rho^{p_0}$). Then $H^p_{L,S_h^\rho}(M) = L^p(M)$ for $p_0 < p < p_0'$.*

If one assumes the pointwise heat kernel estimate, then Theorems 1.6 and 1.7 yield the following corollary.

COROLLARY 1.7. *Let M be a noncompact metric measure space satisfying the doubling volume property (D) and the pointwise heat kernel estimate (UE_ρ). Then $H^1_{L,\rho,\mathrm{mol}}(M) = H^1_{L,S_h^\rho}(M)$ and $H^p_{L,S_h^\rho}(M) = L^p(M)$ for $1 < p < \infty$.*

In the following theorem, we show that for $1 < p < 2$, the equivalence may not hold between L^p and H^p defined via a conical square function S_h with scaling t^2.

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The counterexamples we find are certain Riemannian manifolds satisfying (D) and two-sided sub-Gaussian heat kernel estimates: (UE_ρ) and its reverse, with $\beta_1 = 2$ and $\beta_2 = m > 2$. Notice that in this case, L is the nonnegative Laplace–Beltrami operator, which we denote by Δ. For simplicity, we denote (UE_ρ) by ($UE_{2,m}$) and the two-sided estimate by ($HK_{2,m}$). Also, we denote by $H^1_{\Delta,m,\mathrm{mol}}$ the H^1 space defined via molecules $H^1_{L,\rho,\mathrm{mol}}$ and by H^p_{Δ,S_h^m} the H^p space defined via square functions H^p_{L,S_h^ρ}.

Theorem 1.8. *Let M be a Riemannian manifold with polynomial volume growth*

$$V(x,r) \simeq r^d, \quad r \geq 1, \tag{1.12}$$

as well as a two-sided sub-Gaussian heat kernel estimate ($HK_{2,m}$) with $2 < m < d/2$, that is, ($UE_{2,m}$) and the matching lower estimate. Then

$$L^p(M) \subset H^p_{\Delta,S_h}(M)$$

does not hold for $p \in (d/(d-m), 2)$.

As an application of this Hardy space theory, we have the following result.

Theorem 1.9. *Let M be a manifold satisfying the doubling volume property (D) and the heat kernel estimate ($UE_{2,m}$), $m > 2$, that is, the upper bound of ($HK_{2,m}$). Then the Riesz transform $\nabla\Delta^{-1/2}$ is $H^1_{\Delta,m} - L^1$ bounded.*

Remark 1.10. Recall that under the same assumptions, it is proved in [14] that the Riesz transform is of weak-type $(1,1)$ and thus L^p bounded for $1 < p < 2$.

2. Preliminaries

2.1. More about sub-Gaussian off-diagonal and pointwise heat kernel estimates. Let us first give some examples that satisfy ($DG^{p_0}_\rho$) with $\beta_1 \neq \beta_2$. More examples of this case are metric measure Dirichlet spaces, for which we refer to [8, 29, 38, 39] for details.

Example 2.1. Fractal manifolds.

Fractal manifolds are built from graphs with a self-similar structure at infinity by replacing the edges of the graph with tubes of length 1 and then gluing the tubes together smoothly at the vertices. For instance, see [10] for the construction of Vicsek graphs. For any $D, m \in \mathbb{R}$ such that $D > 1$ and $2 < m \leq D + 1$, there exist complete connected Riemannian manifolds satisfying $V(x,r) \simeq r^D$ for $r \geq 1$ and (UE_ρ) with $\beta_1 = 2$ and $\beta_2 = m > 2$ in (1.4) (see [7] and [14]).

Example 2.2. Cable systems (quantum graphs) (see [41] and [9, Section 2]).

Given a weighted graph (G, E, ν), we define the cable system G_C by replacing each edge of G by a copy of $(0,1)$ joined together at the vertices. The measure μ on G_C is given by $d\mu(t) = \nu_{xy}\, dt$ for t in the cable connecting x and y, and μ assigns no mass to any vertex. The distance between two points x and y is given as follows: if x and y are on the same cable, the length is just the usual Euclidean distance $|x - y|$. If they are on different cables, then the distance is $\min\{|x - z_x| + d(z_x, z_y) + |z_y - y|\}$ (d is the usual graph distance), where the minimum is taken over all vertices z_x and z_y such that x is on a cable with one end at z_x and y is on a cable with one end at z_y. One takes as the core C the functions in $C(G_C)$ which have compact support and are C^1 on each cable, and sets

$$\mathcal{E}(f, f) := \int_{G_C} |f'(t)|^2 \, d\mu(t).$$

Let L be the associated nonnegative self-adjoint operator associated with $\mathcal{E}$ and $\{e^{-tL}\}_{t>0}$ be the generated semigroup. Then the associated kernel satisfies (UE_ρ). For example, the cable graph associated with the Sierpinski gasket graph (in $\mathbb{Z}^2$) satisfies ($UE_{2,\log 5/\log 2}$).

The following are some useful lemmas for the off-diagonal estimates. We first observe that (UE_ρ) $\Rightarrow$ ($DG_\rho^{p_0}$) $\Rightarrow$ (DG_ρ) for $1 \le p_0 \le 2$. Indeed, we have the following result.

Lemma 2.3 [13]. *Let (M, d, μ) be a metric measure space satisfying the doubling volume property. Let L be a nonnegative self-adjoint operator on $L^2(M, \mu)$. Assume that ($DG_\rho^{p_0}$) holds. Then, for all $p_0 \le u \le v \le p_0'$,*

$$\|\mathbb{1}_{B(x,t)} e^{-\rho(t)L} \mathbb{1}_{B(y,t)}\|_{u\to v} \lesssim \begin{cases} \dfrac{1}{V^{1/u-1/v}(x,t)} \exp\Big(-c\Big(\dfrac{d(x,y)}{t}\Big)^{\beta_1/(\beta_1-1)}\Big), & 0 < t < 1, \\ \dfrac{1}{V^{1/u-1/v}(x,t)} \exp\Big(-c\Big(\dfrac{d(x,y)}{t}\Big)^{\beta_2/(\beta_2-1)}\Big), & t \ge 1. \end{cases}$$

Remark 2.4. The estimate ($DG_\rho^{p_0}$) is equivalent to the $L^{p_0} - L^2$ off-diagonal estimate

$$\|\mathbb{1}_{B(x,t)} e^{-\rho(t)L} \mathbb{1}_{B(y,t)}\|_{p_0\to 2} \lesssim \begin{cases} \dfrac{1}{V^{1/p_0-1/2}(x,t)} \exp\Big(-c\Big(\dfrac{d(x,y)}{t}\Big)^{\beta_1/(\beta_1-1)}\Big), & 0 < t < 1, \\ \dfrac{1}{V^{1/p_0-1/2}(x,t)} \exp\Big(-c\Big(\dfrac{d(x,y)}{t}\Big)^{\beta_2/(\beta_2-1)}\Big), & t \ge 1. \end{cases}$$

We refer to [13, 20] for the proof.

In fact, we also have the following result.

Lemma 2.5 [13, 40]. *Let (M, d, μ) satisfy (D). Let L be a nonnegative self-adjoint operator on $L^2(M, \mu)$. Assume that ($DG_\rho^{p_0}$) holds. Then, for all $p_0 \le u \le v \le p_0'$ and $k \in \mathbb{N}$, we have the following results.*

(1) *For any ball $B \subset M$ with radius $r > 0$ and any $i \ge 2$,*

$$\|\mathbb{1}_B (tL)^k e^{-tL} \mathbb{1}_{C_i(B)}\|_{u\to v}, \|\mathbb{1}_{C_i(B)} (tL)^k e^{-tL} \mathbb{1}_B\|_{u\to v}$$
$$\lesssim \begin{cases} \dfrac{2^{iv}}{\mu^{1/u-1/v}(B)} e^{-c(2^{i\beta_1} r^{\beta_1}/t)^{1/(\beta_1-1)}} & \text{if } 0 < t < 1, \\ \dfrac{2^{iv}}{\mu^{1/u-1/v}(B)} e^{-c(2^{i\beta_2} r^{\beta_2}/t)^{1/(\beta_2-1)}} & \text{if } t \ge 1. \end{cases} \tag{2.1}$$

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(2) *For all $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1/u - 1/v$,*

$$\|V^{\alpha}(\cdot, t)(\rho(t)L)^k e^{-\rho(t)L} V^{\beta}(\cdot, t)\|_{u \to v} \leq C.$$

2.2. Tent spaces. We recall definitions and properties related to tent spaces on metric measure spaces with the doubling volume property, following [16, 35].

Let M be a metric measure space satisfying (*D*). For any $x \in M$ and, for any closed subset $F \subset M$, a saw-tooth region is defined as $\mathcal{R}(F) := \bigcup_{x \in F} \Gamma(x)$. If O is an open subset of M, then the 'tent' over O, denoted by $\widehat{O}$, is defined as

$$\widehat{O} := [\mathcal{R}(O^c)]^c = \{(x, t) \in M \times (0, \infty) : d(x, O^c) \geq t\}.$$

For a measurable function F on $M \times (0, \infty)$, consider

$$\mathcal{A}F(x) = \bigg(\iint_{\Gamma(x)} |F(y, t)|^2 \frac{d\mu(y)}{V(x, t)} \frac{dt}{t}\bigg)^{1/2}.$$

Given $0 < p < \infty$, we say that a measurable function $F \in T_2^p(M \times (0, \infty))$ if

$$\|F\|_{T_2^p(M)} := \|\mathcal{A}F\|_{L^p(M)} < \infty.$$

For simplicity, we denote $T_2^p(M \times (0, \infty))$ by $T_2^p(M)$ from now on.

Therefore, for $f \in H^p_{L,S_h}(M)$ and $0 < p < \infty$, write $F(y, t) = \rho(t)Le^{-\rho(t)L} f(y)$; then

$$\|f\|_{H^p_{L,S_h^\rho}(M)} = \|F\|_{T_2^p(M)}.$$

Consider another functional

$$CF(x) = \sup_{x \in B} \bigg(\iint_{\widehat{B}} |F(y, t)|^2 \frac{d\mu(y)\, dt}{t}\bigg)^{1/2};$$

we say that a measurable function $F \in T_2^{\infty}(M)$ if $CF \in L^{\infty}(M)$.

Proposition 2.6. *Suppose that $1 < p < \infty$ and let p' be the conjugate of p. Then the pairing $\langle F, G\rangle \longrightarrow \int_{M \times (0,\infty)} F(x, t)G(x, t)(d\mu(x)\, dt/t)$ realizes $T_2^{p'}(M)$ as the dual of $T_2^p(M)$.*

Denote by $[\,,\,]_\theta$ the complex method of interpolation described in [11]. Then we have the following result of interpolation of tent spaces, where the proof can be found in [1].

Proposition 2.7. *Suppose that $1 \leq p_0 < p < p_1 \leq \infty$ with $1/p = (1 - \theta)/p_0 + \theta/p_1$ and $0 < \theta < 1$. Then*

$$[T_2^{p_0}(M), T_2^{p_1}(M)]_\theta = T_2^p(M).$$

Next we review the atomic theory for tent spaces, which was originally developed in [16] and extended to the setting of spaces of homogeneous type in [35].

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DEFINITION 2.8. A measurable function A on $M \times (0,\infty)$ is said to be a T_2^1-atom if there exists a ball $B \in M$ such that A is supported in $\widehat{B}$ and

$$\int_{M\times(0,\infty)} |A(x,t)|^2 \, d\mu(x)\frac{dt}{t} \leq \mu^{-1}(B).$$

PROPOSITION 2.9 [30, 35]. *For every element $F \in T_2^1(M)$, there exist a sequence of numbers $\{\lambda_j\}_{j=0}^\infty \in l^1$ and a sequence of T_2^1-atoms $\{A_j\}_{j=0}^\infty$ such that*

$$F = \sum_{j=0}^{\infty} \lambda_j A_j \text{ in } T_2^1(M) \quad \text{and} \quad \text{a.e. in } M \times (0,\infty). \tag{2.2}$$

Moreover, $\sum_{j=0}^\infty \lambda_j \approx \|F\|_{T_2^1(M)}$, where the implicit constants depend only on the homogeneous space properties of M.

Finally, if $F \in T_2^1(M) \cap T_2^2(M)$, then the decomposition (2.2) also converges in $T_2^2(M)$.

3. The molecular decomposition

In this section, we shall prove Theorem 1.5. That is, under the assumptions of (D) and (DG_ρ), the two H^1 spaces $H^1_{L,\rho,\mathrm{mol}}(M)$ and $H^1_{L,S_h^\rho}(M)$ are equivalent. We denote

$$H^1_{L,\rho}(M) := H^1_{L,S_h^\rho}(M) = H^1_{L,\rho,\mathrm{mol}}(M).$$

Since $H^1_{L,\rho,\mathrm{mol}}(M)$ and $H^1_{L,S_h^\rho}(M)$ are completions of $\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M)$ and $H^1_{L,S_h^\rho}(M) \cap H^2(M)$, it is enough to show that $\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M) = H^1_{L,S_h^\rho}(M) \cap H^2(M)$ with equivalent norms. In the following, we will prove the two-sided inclusions separately. Before proceeding to the proof, we first note the lemma below to prove the $H^1_{L,\rho,\mathrm{mol}}(M) - L^1(M)$ boundedness of an operator, which is an analogue of [30, Lemma 4.3].

LEMMA 3.1. *Assume that T is a linear operator or a nonnegative sublinear operator satisfying the weak-type $(2,2)$ bound*

$$\mu(\{x \in M : |Tf(x)| > \eta\}) \lesssim \eta^{-2}\|f\|_2^2, \quad \forall \eta > 0$$

and that, for every $(1,2,\varepsilon)$-molecule a,

$$\|Ta\|_{L^1} \leq C,$$

with constant C independent of a. Then T is bounded from $\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M)$ to $L^1(M)$, with

$$\|Tf\|_{L^1} \lesssim \|f\|_{\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M)}.$$

Consequently, by density, T extends to be a bounded operator from $H^1_{L,\rho,\mathrm{mol}}(M)$ to $L^1(M)$.

For the proof, we refer to [30], which is also applicable here.

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3.1. The inclusion $\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M) \subseteq H^1_{L,S_h^\rho}(M) \cap H^2(M)$. We have the following theorem.

THEOREM 3.2. *Let M be a metric measure space satisfying the doubling volume property (D) and the heat kernel estimate (DG_ρ). Then $\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M) \subseteq H^1_{L,S_h^\rho}(M) \cap H^2(M)$ and*

$$\|f\|_{H^1_{L,S_h^\rho}(M)} \leq C\|f\|_{\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M)}.$$

PROOF. First observe that $\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M) \subseteq H^2(M)$. Indeed, by Definition 1.1, any $(1,2,\varepsilon)$-molecule belongs to $R(L)$. Thus, any finite linear combination of molecules belongs to $R(L)$. Since $f \in \mathbb{H}^1_{L,\rho,\mathrm{mol}}(M)$ is the $L^2(M)$ limit of a finite linear combination of molecules, we get $f \in \overline{R(L)} = H^2(M)$.

It remains to show that $\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M) \subseteq H^1_{L,S_h^\rho}(M)$, that is, S_h^ρ is bounded from $\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M)$ to $L^1(M)$. Note that S_h^ρ is L^2 bounded by spectral theory (see (1.11)); it follows from Lemma 3.1 that it suffices to prove that, for any $(1,2,\varepsilon)$-molecule a, there exists a constant C such that $\|S_h^\rho a\|_{L^1(M)} \leq C$. In other words, one needs to prove that $\|A\|_{T_2^1(M)} \leq C$, where

$$A(y,t) = \rho(t)Le^{-\rho(t)L}a(y).$$

Assume that a is a $(1,2,\varepsilon)$-molecule related to a function b and a ball B with radius r, that is, $a = L^K b$ and, for every $k = 0,1,\ldots,K$ and $i = 0,1,2,\ldots$,

$$\|(\rho(r)L)^k b\|_{L^2(C_i(B))} \leq \rho(r)2^{-i\varepsilon}\mu(2^iB)^{-1/2}.$$

Similarly as in [5], we divide A into four parts:

$$\begin{aligned} A &= \mathbb{1}_{2B\times(0,2r)}A + \sum_{i\geq 1}\mathbb{1}_{C_i(B)\times(0,r)}A + \sum_{i\geq 1}\mathbb{1}_{C_i(B)\times(r,2^{i+1}r)}A + \sum_{i\geq 1}\mathbb{1}_{2^iB\times(2^ir,2^{i+1}r)}A \\ &=: A_0 + A_1 + A_2 + A_3. \end{aligned}$$

Here $\mathbb{1}$ denotes the characteristic function and $C_i(B) = 2^{i+1}B\backslash 2^iB$, $i \geq 1$. It suffices to show that for every $j = 0,1,2,3$, we have $\|A_j\|_{T_2^1} \leq C$.

Firstly consider A_0. Observe that

$$\mathcal{A}(A_0)(x) = \left(\iint_{\Gamma(x)} |\mathbb{1}_{2B\times(0,2r)}(y,t)A(y,t)|^2 \frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\right)^{1/2}$$

is supported on $4B$. Indeed, denote by x_B the centre of B; then $d(x,x_B) \leq d(x,y) + d(y,x_B) < 4r$. Also,

$$\begin{aligned} \|A_0\|^2_{T_2^2(M)} = \|\mathcal{A}(A_0)\|_2^2 &\leq \int_M \iint_{\Gamma(x)} |\rho(t)Le^{-\rho(t)L}a(y)|^2 \frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\,d\mu(x) \\ &\lesssim \|a\|^2_{L^2(M)} \lesssim \mu^{-1}(B). \end{aligned}$$

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Here the second and the third inequalities follow from (1.11) and the definition of molecules, respectively. Now, applying the Cauchy–Schwarz inequality,

$$\|A_0\|_{T_2^1(M)} \le \|A\|_{T_2^2(M)} \mu(4B)^{1/2} \le C.$$

Secondly consider A_1. For each $i \ge 1$, we have $\operatorname{supp} \mathcal{A}(\mathbb{1}_{C_i(B)\times(0,r)}A) \subset 2^{i+2}B$. In fact, $d(x, x_B) \le d(x, y) + d(y, x_B) \le t + 2^{i+1}r < 2^{i+2}r$. Then

$$\begin{aligned}
\|\mathbb{1}_{C_i(B)\times(0,r)}A\|_{T_2^2} &= \|\mathcal{A}(\mathbb{1}_{C_i(B)\times(0,r)}A)\|_2 \\
&\le \bigg(\int_{2^{i+2}B} \iint_{\Gamma(x)} |\mathbb{1}_{C_i(B)\times(0,r_B)}(y,t)\rho(t)Le^{-\rho(t)L}a(y)|^2 \frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\,d\mu(x)\bigg)^{1/2} \\
&\le \bigg(\int_0^r \int_{C_i(B)} |\rho(t)Le^{-\rho(t)L}a(y)|^2\,d\mu(y)\frac{dt}{t}\bigg)^{1/2} \\
&\le \sum_{l=0}^{\infty}\bigg(\int_0^r \int_{C_i(B)} |\rho(t)Le^{-\rho(t)L}\mathbb{1}_{C_l(B)}a(y)|^2\,d\mu(y)\frac{dt}{t}\bigg)^{1/2} \\
&=: \sum_{l=0}^{\infty} I_l.
\end{aligned}$$

We estimate I_l with $|i-l| > 3$ and $|i-l| \le 3$, respectively. Firstly assume that $|i-l| \le 3$. Using (1.11) again,

$$I_l^2 \le \int_0^{\infty}\int_M |\rho(t)Le^{-\rho(t)L}\mathbb{1}_{C_l(B)}a(y)|^2\,d\mu(y)\frac{dt}{t} \lesssim \|a\|^2_{L^2(C_l(B))} \lesssim 2^{-2i\varepsilon}\mu^{-1}(2^iB).$$

Assume now that $|i-l| > 3$. Note that $\operatorname{dist}(C_l(B), C_i(B)) \ge c2^{\max\{l,i\}}r_B \ge c2^ir_B$. Then it follows from Lemma 2.5 that

$$\begin{aligned}
I_l^2 &\le \int_0^r \exp\bigg(-c\Big(\frac{\rho(2^ir)}{\rho(t)}\Big)^{\beta_2/(\beta_2-1)}\bigg)\|a\|^2_{L^2(C_l(B))}\,d\mu(y)\frac{dt}{t} \\
&\lesssim 2^{-2l\varepsilon}\mu^{-1}(2^lB)\int_0^r \Big(\frac{\rho(t)}{\rho(2^ir)}\Big)^c\frac{dt}{t} \lesssim 2^{-ci}2^{-2l\varepsilon}\mu^{-1}(2^iB). \qquad (3.1)
\end{aligned}$$

The last inequality comes from (1.5).

It follows from the above that

$$\|\mathbb{1}_{C_i(B)\times(0,r)}A\|_{T_2^2} \lesssim \sum_{l:|l-i|\le 3} 2^{-i\varepsilon}\mu^{-1/2}(2^iB) + \sum_{l:|l-i|>3} 2^{-ic}2^{-l\varepsilon}\mu^{-1/2}(2^iB) \lesssim 2^{-ic}\mu^{-1/2}(2^iB),$$

where c depends on ε, M. Therefore,

$$\|A_1\|_{T_2^1} \le \sum_{i\ge1}\|\mathbb{1}_{C_i(B)\times(0,r)}A\|_{T_2^2}\mu^{1/2}(2^{i+2}B) \lesssim \sum_{i\ge1}2^{-ic} \le C.$$

We estimate A_2 in a similar way as before except that we replace a by L^Kb. Note that for each $i \ge 1$, we have $\operatorname{supp}\mathcal{A}(\mathbb{1}_{C_i(B)\times(r,2^{i+1}r)}A) \subset 2^{i+2}B$. Indeed,

$$d(x, x_B) \le d(x, y) + d(y, x_B) \le t + 2^{i+1}r \le 2^{i+2}r.$$

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Then

$$\begin{aligned}
\|\mathbb{1}_{C_i(B)\times(r,2^{i+1}r)}A\|_{T_2^2} &= \|\mathcal{A}(\mathbb{1}_{C_i(B)\times(r,2^{i+1}r)}A)\|_2 \\
&\leq \Big(\int_{2^{i+2}B}\iint_{\Gamma(x)} |\mathbb{1}_{C_i(B)\times(r,2^ir)}(y,t)A(y,t)|^2 \frac{d\mu(y)\,dt}{V(x,t)t}\,d\mu(x)\Big)^{1/2} \\
&\leq \Big(\int_r^{2^{i+1}r}\int_{C_i(B)} |(\rho(t)L)^{K+1}e^{-\rho(t)L}b(y)|^2\,d\mu(y)\frac{dt}{t\rho^{2K}(t)}\Big)^{1/2} \\
&\leq \Big(\sum_{l=0}^{\infty}\int_r^{2^{i+1}r}\int_{C_i(B)} |(\rho(t)L)^{K+1}e^{-\rho(t)L}\mathbb{1}_{C_l(B)}b(y)|^2\,d\mu(y)\frac{dt}{t\rho^{2K}(t)}\Big)^{1/2} \\
&=: \sum_{l=0}^{\infty} J_l.
\end{aligned}$$

When $|i-l|\leq 3$, by the spectral theorem, we get $J_l^2 \leq C2^{-2i\varepsilon}V^{-1}(2^iB)$. And, when $|i-l|>3$, we have $\operatorname{dist}(C_l(B),C_i(B)) \geq c2^{\max\{l,i\}}r \geq c2^ir$. Then we estimate J_l in the same way as for (3.1),

$$\begin{aligned}
J_l^2 &\leq \int_r^{2^{i+1}r} \exp\Big(-c\Big(\frac{\rho(2^ir)}{\rho(t)}\Big)^{\beta_2/(\beta_2-1)}\Big)\|b\|_{L^2(C_l(B))}^2\,d\mu(y)\frac{dt}{t\rho^{2K}(t)} \\
&\leq \rho^{2K}(r)2^{-2l\varepsilon}\mu^{-1}(2^{l+1}B)\int_r^{2^{i+1}r}\Big(\frac{\rho(t)}{\rho(2^ir)}\Big)^c\frac{dt}{t\rho^{2K}(t)} \\
&\lesssim 2^{-ic}2^{-l(2\varepsilon+\nu)}\mu^{-1}(2^iB).
\end{aligned}$$

Here c in the second and the third lines are different. We can carefully choose c in the second line to make sure that c in the third line is positive.

Hence,

$$\|\mathbb{1}_{C_i(B)\times(r,2^ir)}A\|_{T_2^2}^2 \lesssim 2^{-ic}\mu^{-1}(2^iB)$$

and

$$\|A_2\|_{T_2^1} \leq \sum_{i\geq 1}\|\mathbb{1}_{C_i(B)\times(r,2^ir)}A\|_{T_2^2}\mu^{1/2}(2^{i+2}B) \lesssim \sum_{i\geq 1}2^{-ic/2} \leq C.$$

It remains to estimate the last term A_3. For each $i\geq 1$, we still have

$$\operatorname{supp}\mathcal{A}(\mathbb{1}_{2^iB\times(2^ir,2^{i+1}r)}A) \subset 2^{i+2}B.$$

Then we obtain as before that

$$\begin{aligned}
\|\mathbb{1}_{2^iB\times(2^ir,2^{i+1}r)}A\|_{T_2^2} &= \|\mathcal{A}(\mathbb{1}_{2^iB\times(2^ir,2^{i+1}r)}A)\|_2 \\
&\leq \Big(\int_{2^{i+2}B}\iint_{\Gamma(x)} |\mathbb{1}_{2^iB\times(2^ir_B,2^{i+1}r)}(y,t)A(y,t)|^2\frac{d\mu(y)\,dt}{V(x,t)t}\,d\mu(x)\Big)^{1/2} \\
&\leq \Big(\int_{2^ir}^{2^{i+1}r}\int_{2^iB} |(\rho(t)L)^{K+1}e^{-\rho(t)L}b(y)|^2\frac{d\mu(y)\,dt}{t\rho^{2K}(t)}\Big)^{1/2}
\end{aligned}$$

$$\leq \sum_{l=0}^{\infty}\bigg(\int_{2^i r}^{2^{i+1}r}\int_{2^i B}|(\rho(t)L)^2 e^{-\rho(t)L}\mathbb{1}_{C_l(B)}b(y)|^2\frac{d\mu(y)\,dt}{t\rho^{2K}(t)}\bigg)^{1/2}$$
$$=: \sum_{l=0}^{\infty}K_l.$$

In fact, due to the doubling volume property, (1.11) as well as the definition of molecules,

$$K_l^2 \leq \int_{2^i r}^{2^{i+1}r}\|\mathbb{1}_{C_l(B)}b\|_{L^2}^2\frac{dt}{t\rho^{2K}(t)} \lesssim \rho^{2K}(r)2^{-2l\varepsilon}\mu^{-1}(2^l B)\int_{2^i r}^{2^{i+1}r}\frac{dt}{t\rho^{2K}(t)}$$
$$\lesssim 2^{-2l\varepsilon}2^{-ic}\mu^{-1}(2^i B).$$

Hence,

$$\|A_3\|_{T_2^1} \leq \sum_{i\geq 1}\|\mathbb{1}_{2^i B\times(2^i r,2^{i+1}r)}A\|_{T_2^2}\mu^{1/2}(2^{i+2}B) \lesssim \sum_{i\geq 1}2^{-2i} \leq C.$$

This finishes the proof. □

3.2. The inclusion $H^1_{L,S_h^\rho}(M)\cap H^2(M)\subseteq \mathbb{H}^1_{L,\rho,\mathrm{mol}}(M)$. We closely follow the proof of Theorem 4.13 in [30] and get the following result.

THEOREM 3.3. *Let M be a metric measure space satisfying (D) and (DG_ρ). If $f\in H^1_{L,S_h^\rho}(M)\cap H^2(M)$, then there exist a sequence of numbers $\{\lambda_j\}_{j=0}^\infty\subset l^1$ and a sequence of $(1,2,\varepsilon)$-molecules $\{a_j\}_{j=0}^\infty$ such that f can be represented in the form $f=\sum_{j=0}^\infty\lambda_j a_j$, with the sum converging in $L^2(M)$, and*

$$\|f\|_{\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M)} \leq C\sum_{j=0}^{\infty}\lambda_j \leq C\|f\|_{H^1_{L,S_h^\rho}(M)},$$

where C is independent of f. In particular, $H^1_{L,S_h^\rho}(M)\cap H^2(M)\subseteq\mathbb{H}^1_{L,\rho,\mathrm{mol}}(M)$.

PROOF. For $f\in H^1_{L,S_h^\rho}(M)\cap H^2(M)$, denote $F(x,t)=\rho(t)Le^{-\rho(t)L}f(x)$. Then, by the definition of $H^1_{L,S_h^\rho}(M)$, we have $F\in T_2^1(M)\cap T_2^2(M)$.

From Theorem 2.9, we decompose F as $F=\sum_{j=0}^\infty\lambda_j A_j$, where $\{\lambda_j\}_{j=0}^\infty\in l^1$, $\{A_j\}_{j=0}^\infty$ is a sequence of T_2^1-atoms supported in a sequence of sets $\{\widehat{B}_j\}_{j=0}^\infty$ and the sum converges in both $T_2^1(M)$ and $T_2^2(M)$. Also,

$$\sum_{j=0}^{\infty}\lambda_j \lesssim \|F\|_{T_2^1(X)} = \|f\|_{H^1_{L,S_h^\rho}(M)}.$$

For $f\in H^2(M)$, by functional calculus, we have the following 'Calderón reproducing formula':

$$f = C\int_0^\infty(\rho(t)L)^{K+1}e^{-2\rho(t)L}f\frac{\rho'(t)\,dt}{\rho(t)} = C\int_0^\infty(\rho(t)L)^K e^{-\rho(t)L}F(\cdot,t)\frac{\rho'(t)\,dt}{\rho(t)} =: C\pi_{h,L}(F).$$

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Denote $a_j = C\pi_{h,L}(A_j)$; then $f = \sum_{j=0}^{\infty} \lambda_j a_j$. Since $F \in T_2^2(M)$, we have $\|\pi_{h,L}(F)\|_{L^2(M)} \le C\|F\|_{T_2^2(M)}$. Thus, we learn from [30, Lemma 4.12] that the sum also converges in $L^2(M)$.

We claim that a_j, $j = 0, 1, \dots$, are $(1, 2, \varepsilon)$-molecules up to multiplication by some uniform constant.

Indeed, note that $a_j = L^K b_j$, where

$$b_j = C \int_0^{\infty} \rho^K(t) e^{-\rho(t)L} A_j(\cdot, t) \frac{\rho'(t)\,dt}{\rho(t)}.$$

Now we estimate the norm $\|(\rho(r_{B_j})L)^k b_j\|_{L^2(C_i(B))}$, where r_{B_j} is the radius of B_j. For simplicity, we ignore the index j. Consider any function $g \in L^2(C_i(B))$ with $\|g\|_{L^2(C_i(B))} = 1$; then, for $k = 0, 1, \dots, K$,

$$\begin{aligned}
&\left|\int_M (\rho(r_B)L)^k b(x) g(x)\,d\mu(x)\right| \\
&\quad\lesssim \left|\int_M \left(\int_0^{\infty} (\rho(r_B)L)^k \rho^K(t) e^{-\rho(t)L}(A_j(\cdot,t))(x)\frac{\rho'(t)\,dt}{\rho(t)}\right) g(x)\,d\mu(x)\right| \\
&\quad= \left|\int_{\widehat{B}} \left(\frac{\rho(r_B)}{\rho(t)}\right)^k \rho^K(t) A_j(x,t)(\rho(t)L)^k e^{-\rho(t)L} g(x)\,d\mu(x)\frac{\rho'(t)\,dt}{\rho(t)}\right| \\
&\quad\lesssim \left(\int_{\widehat{B}} |A_j(x,t)|^2\,d\mu(x)\frac{dt}{t}\right)^{1/2}\left(\int_{\widehat{B}} \left|\left(\frac{\rho(r_B)}{\rho(t)}\right)^k \rho^K(t)(\rho(t)L)^k e^{-\rho(t)L} g(x)\right|^2 d\mu(x)\frac{dt}{t}\right)^{1/2}.
\end{aligned}$$

In the last inequality, we apply the Hölder inequality as well as (1.8).

We continue to estimate by using the definition of T_2^1-atoms and the off-diagonal estimates of the heat kernel.

For $i = 0, 1$, the above quantity is dominated by

$$\mu^{-1/2}(B)\rho(r_B)\left(\int_{\widehat{B}} |(\rho(t)L)^k e^{-\rho(t)L} g(x)|^2\,d\mu(x)\frac{dt}{t}\right)^{1/2} \lesssim \mu^{-1/2}(B)\rho(r_B).$$

Next, for $i \ge 2$, the above estimate is controlled:

$$\begin{aligned}
&\mu^{-1/2}(B)\left(\int_0^{r_B} \left(\frac{\rho(r_B)}{\rho(t)}\right)^{2k} \rho^{2K}(t)\|(\rho(t)L)^k e^{-\rho(t)L} g\|_{L^2(B)}^2 \frac{dt}{t}\right)^{1/2} \\
&\quad\lesssim \mu^{-1/2}(B)\left(\int_0^{r_B} \left(\frac{\rho(r_B)}{\rho(t)}\right)^{2k} \rho^{2K}(t) \exp\left(-c\left(\frac{2^i r_B}{t}\right)^{\tau}\right)\frac{dt}{t}\right)^{1/2} \\
&\quad\lesssim \mu^{-1/2}(B)\left(\int_0^{r_B} \left(\frac{\rho(r_B)}{\rho(t)}\right)^{2k} \rho^{2K}(t)\left(\frac{t}{2^i r_B}\right)^{\varepsilon+\nu}\frac{dt}{t}\right)^{1/2} \\
&\quad\lesssim \mu^{-1/2}(2^i B)\rho^K(r_B)2^{-i\varepsilon}.
\end{aligned}$$

In the first inequality, we use Lemma 2.5. Since $k = 0, 1, \dots, K$, the last inequality always holds for any $\varepsilon > 0$.

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Therefore,

$$\|(\rho(r_B)L)^k b\|_{L^2(C_i(B))} = \sup_{\|g\|_{L^2(C_i(B))}=1} \left|\int_M (\rho(r_B)L)^k b(x)g(x)\,d\mu(x)\right| \lesssim \mu^{-1/2}(2^i B)\rho^K(r_B)2^{-i\varepsilon}. \qquad \square$$

4. Comparison of Hardy spaces and Lebesgue spaces

In this section, we will study the relations between $L^p(M)$, $H^p_{L,S_h^\rho}(M)$ and $H^p_{L,S_h}(M)$ under the assumptions of (D) and ($DG^{p_0}_\rho$). We first show that $L^p(M)$ and $H^p_{L,S_h^\rho}(M)$ are equivalent. Next we give some examples such that $L^p(M)$ and $H^p_{L,S_h}(M)$ are not equivalent. More precisely, the inclusion $L^p \subset H^p_{L,S_h}$ may be false for $1 < p < 2$.

4.1. Equivalence of $L^p(M)$ and $H^p_{L,S_h^\rho}(M)$ for $p_0 < p < p_0'$. We will prove Theorem 1.6. That is, if M satisfies (D) and ($DG^{p_0}_\rho$), then $H^p_{L,S_h^\rho}(M) = L^p(M)$ for $p_0 < p < p_0'$.

We first note that if M satisfies (D) and ($DG^{p_0}_\rho$) for some $1 \le p_0 < 2$, then L is injective. A similar result can be found in [15].

LEMMA 4.1. *If M satisfies (D) and ($DG^{p_0}_\rho$) for some $1 \le p_0 < 2$, then the operator L is injective on $L^2(M)$. Consequently, $H^2(M) = L^2(M)$.*

PROOF. For any $f \in N(L)$, that is, $Lf = 0$,

$$e^{-\rho(t)L}f - f = \int_0^{\rho(t)} \frac{\partial}{\partial s} e^{-sL} f\,ds = -\int_0^{\rho(t)} Le^{-sL} f\,ds = 0.$$

As a consequence of Lemma 2.3, we have that for all $x \in M$ and $t \ge 0$,

$$\left(\int_{B(x,t)} |f|^{p_0'}\right)^{1/p_0'} = \|e^{-\rho(t)L}f\|_{L^{p_0'}(B(x,t))} \lesssim V(x,t)^{1/p_0'-1/2}\|f\|_{L^2(B(x,t))}.$$

Now, letting $t \to \infty$, $V(x,t) \to \infty$ because of the doubling volume property. Thus, we obtain that $f = 0$.

Due to the self-adjointness of L in $L^2(M)$, we get $L^2(M) = \overline{R(L)} \bigoplus N(L)$, where the sum is orthogonal. Hence, $N(L) = 0$ implies that $H^2(M) = L^2(M)$. $\square$

Our main tool is the Calderón–Zygmund decomposition (see, for example, [17, Corollaire 2.3]).

THEOREM 4.2. *Let (M, d, μ) be a metric measure space satisfying the doubling volume property. Let $1 \le q \le \infty$ and $f \in L^q$. Let $\lambda > 0$. Then there exists a decomposition of f, $f = g + b = g + \sum_i b_i$, so that:*

(1) *$|g(x)| \le C\lambda$ for almost all $x \in M$;*

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(2) *there exists a sequence of balls* $B_i = B(x_i, r_i)$ *so that each* b_i *is supported in* B_i,

$$\int |b_i(x)|^q \, d\mu(x) \leq C\lambda^q \mu(B_i);$$

(3) $\sum_i \mu(B_i) \leq (C/\lambda^q) \int |f(x)|^q \, d\mu(x)$;

(4) $\|b\|_q \leq C\|f\|_q$ *and* $\|g\|_q \leq C\|f\|_q$;

(5) *there exists* $k \in \mathbb{N}^*$ *such that each* $x \in M$ *is contained in at most* k *balls* B_i.

Now we are ready to prove Theorem 1.6.

Proof of Theorem 1.6. By Lemma 4.1, it suffices to prove that for any $f \in R(L) \cap L^p(M)$ with $p_0 < p < p_0'$,

$$\|S_h^\rho f\|_{L^p} \lesssim \|f\|_{L^p}. \tag{4.1}$$

With this fact at hand, we can obtain by duality that $\|f\|_{L^p} \leq C\|S_h^\rho f\|_{L^p}$ for $p_0 < p < p_0'$.

Indeed, for $f \in R(L)$, write the identity

$$f = C \int_0^\infty (\rho(t)L)^2 e^{-2\rho(t)L} f \frac{\rho'(t)\, dt}{\rho(t)},$$

where the integral $C \int_\varepsilon^{1/\varepsilon} (\rho(t)L)^2 e^{-2\rho(t)L} f(\rho'(t)\, dt/\rho(t))$ converges to f in $L^2(M)$ as $\varepsilon \to 0$.

Then, for $f \in R(L) \cap L^p(M)$,

$$\begin{aligned}
\|f\|_{L^p} &= \sup_{\|g\|_{L^{p'}} \leq 1} |\langle f, g\rangle| \simeq \sup_{\|g\|_{L^{p'}} \leq 1} \left| \iint_{M \times (0,\infty)} F(y,t)G(y,t)\, d\mu(y) \frac{\rho'(t)\, dt}{\rho(t)} \right| \\
&\simeq \sup_{\|g\|_{L^{p'}} \leq 1} \left| \int_M \iint_{\Gamma(x)} F(y,t)G(y,t) \frac{d\mu(y)}{V(x,t)} \frac{\rho'(t)\, dt}{\rho(t)}\, d\mu(x) \right| \\
&\lesssim \sup_{\|g\|_{L^{p'}} \leq 1} \|F\|_{T_2^p} \|G\|_{T_2^{p'}} \simeq \sup_{\|g\|_{L^{p'}} \leq 1} \|S_h f\|_{L^p} \|S_h g\|_{L^{p'}} \\
&\lesssim \sup_{\|g\|_{L^{p'}} \leq 1} \|S_h f\|_{L^p} \|g\|_{L^{p'}} = \|S_h f\|_{L^p}.
\end{aligned}$$

Here $F(y,t) = \rho(t)Le^{-\rho(t)L} f(y)$ and $G(y,t) = \rho(t)Le^{-\rho(t)L} g(y)$. The second line's equivalence is due to the doubling volume property.

By an approximation process, the above argument holds for $f \in L^p(M)$.

For $p > 2$, the L^p norm of the conical square function is controlled by its vertical analogue (for a reference, see [3], where the proof can be adapted to the homogenous setting), which is always L^p bounded for $p_0 < p < p_0'$ by adapting the proofs in [12] and [21] (if $\{e^{-tL}\}_{t>0}$ is a symmetric Markov semigroup, then it is L^p bounded for $1 < p < \infty$, according to [36]). Hence, (4.1) holds.

It remains to show (4.1) for $p_0 < p < 2$.

In the following, we will prove the weak (p_0, p_0) boundedness of S_h^ρ by using the Calderón–Zygmund decomposition. Since S_h^ρ is also L^2 bounded as shown in

(1.11), then, by interpolation, (4.1) holds for every $p_0 < p < 2$. The proof is similar to [2, Proposition 6.8] and [3, Theorem 3.1], which originally comes from [22].

We take the Calderón–Zygmund decomposition of f at height λ, that is, $f = g + \sum b_i$, with $\operatorname{supp} b_i \subset B_i$. Since S_h^ρ is a sublinear operator, write

$$\begin{aligned} S_h^\rho\Big(\sum_i b_i\Big) &= S_h^\rho\Big(\sum_i (I - (I - e^{-\rho(r_i)})^N + (I - e^{-\rho(r_i)L})^N)b_i\Big) \\ &\le S_h^\rho\Big(\sum_i (I - (I - e^{-\rho(r_i)})^N)b_i\Big) + S_h^\rho\Big(\sum_i (I - e^{-\rho(r_i)})^N b_i\Big). \end{aligned}$$

Here $N \in \mathbb{N}$ is chosen to be larger than $2\nu/\beta_1$, where ν is as in (1.5).

Then it is enough to prove that

$$\begin{aligned} \mu(\{x \in M : S_h^\rho(f)(x) > \lambda\}) &\le \mu\Big(\Big\{x \in M : S_h^\rho(g)(x) > \frac{\lambda}{3}\Big\}\Big) \\ &\quad + \mu\Big(\Big\{x \in M : S_h^\rho\Big(\sum_i (I - (I - e^{-\rho(r_i)})^N)b_i\Big)(x) > \frac{\lambda}{3}\Big\}\Big) \\ &\quad + \mu\Big(\Big\{x \in M : S_h^\rho\Big(\sum_i (I - e^{-\rho(r_i)L})^N b_i\Big)(x) > \frac{\lambda}{3}\Big\}\Big) \\ &\lesssim \frac{1}{\lambda^{p_0}} \int |f(x)|^{p_0}\, d\mu(x). \end{aligned}$$

We treat g in a routine way. Since S_h^ρ is L^2 bounded as shown in (1.11), then

$$\mu\Big(\Big\{x \in M : S_h^\rho(g)(x) > \frac{\lambda}{3}\Big\}\Big) \lesssim \lambda^{-2}\|g\|_2^2 \lesssim \lambda^{-p_0}\|g\|_{p_0} \lesssim \lambda^{-p_0}\|f\|_{p_0}.$$

Now for the second term. Note that $I - (I - e^{-\rho(r_i)L})^N = \sum_{k=1}^N (-1)^{k+1}\binom{N}{k} e^{-k\rho(r_i)L}$; it is enough to show that for every $1 \le k \le N$,

$$\mu\Big(\Big\{x \in M : S_h^\rho\Big(\sum_i e^{-k\rho(r_i)L} b_i\Big)(x) > \frac{\lambda}{3N}\Big\}\Big) \lesssim \frac{1}{\lambda^{p_0}} \int |f(x)|^{p_0}\, d\mu(x). \tag{4.2}$$

Note the following slight improvement of (2.1): for every $1 \le k \le N$ and for every $j \ge 1$,

$$\|e^{-k\rho(r_i)L} b_i\|_{L^2(C_j(B_i))} \lesssim \frac{2^{j\nu}}{\mu^{1/p_0 - 1/2}(B_i)} e^{-c_k 2^{j\tau(k\rho(r_i))}} \|b_i\|_{L^{p_0}(B_i)}. \tag{4.3}$$

Here $\tau(r) = \beta_1/(\beta_1 - 1)$ if $0 < r < 1$, otherwise $\tau(r) = \beta_2/(\beta_2 - 1)$. Indeed, it is obvious for $r_i \ge 1$ and $0 < r_i < k^{-1/\beta_1}$. For $k^{-1/\beta_1} \le r_i < 1$, that is, $k\rho(r_i) \ge 1$,

$$\Big(\frac{(2^j r_i)^{\beta_2}}{k\rho(r_i)}\Big)^{1/(\beta_2 - 1)} \simeq 2^{j(\beta_2/(\beta_2 - 1))} = 2^{j\tau(k\rho(r_i))}.$$

With the above preparations, we can show (4.2) now. Write

$$\mu\Big(\Big\{x : \Big|S_h^\rho\Big(\sum_i e^{-k\rho(r_i)L} b_i\Big)(x)\Big| > \frac{\lambda}{3N}\Big\}\Big) \lesssim \frac{1}{\lambda^2}\Big\|\sum_i e^{-k\rho(r_i)L} b_i\Big\|_2^2.$$

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By a duality argument,

$$\Big\|\sum_i e^{-k\rho(r_i)L}b_i\Big\|_2 = \sup_{\|\phi\|_2=1}\int_M\Big|\sum_i e^{-k\rho(r_i)L}b_i\Big||\phi|\,d\mu \le \sup_{\|\phi\|_2=1}\sum_i\sum_{j=1}^{\infty}\int_{C_j(B_i)}|e^{-k\rho(r_i)L}b_i||\phi|\,d\mu$$
$$=: \sup_{\|\phi\|_2=1}\sum_i\sum_{j=1}^{\infty}A_{ij}.$$

Applying the Cauchy–Schwarz inequality, (4.3) and (1.5),

$$\begin{aligned}A_{ij} &\le \|e^{-k\rho(r_i)L}b_i\|_{L^2(C_j(B_i))}\|\phi\|_{L^2(C_j(B_i))}\\ &\lesssim 2^{3j\nu/2}e^{-c2^{j\tau(k\rho(r_i))}}\mu(B_i)\Big(\frac{1}{\mu(B_i)}\int_{B_i}|b_i|^{p_0}\,d\mu\Big)^{1/p_0}\inf_{y\in B_i}(\mathcal{M}(|\phi|^2)(y))^{1/2}\\ &\lesssim e^{-c2^{j\tau(k\rho(r_i))}}\mu(B_i)\inf_{y\in B_i}(\mathcal{M}(|\phi|^2)(y))^{1/2}.\end{aligned}$$

Here $\mathcal{M}$ denotes the Hardy–Littlewood maximal operator

$$\mathcal{M}f(x) = \sup_{B\ni x}\frac{1}{\mu(B)}\int_B|f(x)|\,d\mu(x),$$

where B ranges over all balls containing x.

Then

$$\begin{aligned}\Big\|\sum_i e^{-k\rho(r_i)L}b_i\Big\|_2 &\lesssim \lambda\sup_{\|\phi\|_2=1}\sum_i\sum_{j=1}^{\infty}e^{-c2^{j\tau(k\rho(r_i))}}\mu(B_i)\inf_{y\in B_i}(M(|\phi|^2)(y))^{1/2}\\ &\lesssim \lambda\sup_{\|\phi\|_2=1}\int\sum_i\mathbb{1}_{B_i}(y)(\mathcal{M}(|\phi|^2)(y))^{1/2}\,d\mu(y)\\ &\lesssim \lambda\sup_{\|\phi\|_2=1}\int_{\bigcup_i B_i}(\mathcal{M}(|\phi|^2)(y))^{1/2}\,d\mu(y)\\ &\lesssim \lambda\mu^{1/2}\Big(\bigcup_i B_i\Big)\lesssim \lambda^{1-p_0/2}\Big(\int|f|^{p_0}\,d\mu\Big)^{1/2}.\end{aligned}$$

The third inequality is due to the finite overlap of the Calderón–Zygmund decomposition. In the last line, for the first inequality, we use Kolmogorov's inequality (see, for example, [26, page 91]).

Therefore,

$$\mu\Big(\Big\{x : \Big|S_h^{\rho}\Big(\sum_i e^{-k\rho(r_i)L}b_i\Big)(x)\Big| > \frac{\lambda}{3N}\Big\}\Big) \lesssim \frac{1}{\lambda^{p_0}}\int|f|^{p_0}\,d\mu.$$

For the third term,

$$\mu\Big(\Big\{x\in M : S_h^{\rho}\Big(\sum_i(I-e^{-\rho(r_i)L})^N b_i\Big)(x) > \frac{\lambda}{3}\Big\}\Big)$$
$$\le \mu\Big(\bigcup_j 4B_j\Big) + \mu\Big(\Big\{x\in M\setminus\bigcup_j 4B_j : S_h^{\rho}\Big(\sum_i(I-e^{-\rho(r_i)L})^N b_i\Big)(x) > \frac{\lambda}{3}\Big\}\Big).$$

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From the Calderón–Zygmund decomposition and doubling volume property,

$$\mu\Big(\bigcup_j 4B_j\Big) \le \sum_j \mu(4B_j) \lesssim \sum_j \mu(B_j) \lesssim \frac{1}{\lambda^{p_0}}\|f\|_{p_0}.$$

It remains to show that

$$\begin{aligned}\Lambda &:= \mu\Big(\Big\{x \in M \Big\backslash \bigcup_j 4B_j : S_h^\rho\Big(\sum_i (I - e^{-\rho(r_i)L})^N b_i\Big)(x) > \frac{\lambda}{3}\Big\}\Big)\\ &\lesssim \frac{1}{\lambda^{p_0}} \int |f(x)|^{p_0}\, d\mu(x).\end{aligned}$$

As a consequence of the Chebichev inequality, Λ is dominated by

$$\begin{aligned}&\frac{9}{\lambda^2}\int_{M\setminus\bigcup_j 4B_j}\Big(S_h^\rho\Big(\sum_i (I - e^{-\rho(r_i)L})^N b_i\Big)(x)\Big)^2\, d\mu(x)\\ &\le \frac{9}{\lambda^2}\int_{M\setminus\bigcup_j 4B_j}\iint_{\Gamma(x)}\Big(\sum_i \rho(t)Le^{-\rho(t)L}(I - e^{-\rho(r_i)L})^N b_i(y)\Big)^2 \frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\, d\mu(x)\\ &\le \frac{18}{\lambda^2}\int_{M\setminus\bigcup_j 4B_j}\iint_{\Gamma(x)}\Big(\sum_i \mathbb{1}_{2B_i}(y)\rho(t)Le^{-\rho(t)L}(I - e^{-\rho(r_i)L})^N b_i(y)\Big)^2 \frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\, d\mu(x)\\ &\quad + \frac{18}{\lambda^2}\int_{M\setminus\bigcup_j 4B_j}\iint_{\Gamma(x)}\Big(\sum_i \mathbb{1}_{M\setminus 2B_i}(y)\rho(t)Le^{-\rho(t)L}(I - e^{-\rho(r_i)L})^N b_i(y)\Big)^2 \frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\, d\mu(x)\\ &=: \frac{18}{\lambda^2}(\Lambda_{\mathrm{loc}} + \Lambda_{\mathrm{glob}}).\end{aligned}$$

Now for the estimate of Λ_{loc}. Due to the bounded overlap of $2B_i$, we can put the sum of i out of the square up to a multiplicative constant. That is,

$$\begin{aligned}\Lambda_{\mathrm{loc}} &\lesssim \sum_i \int_{M\setminus\bigcup_j 4B_j}\int_0^\infty\int_{B(x,t)} (\mathbb{1}_{2B_i}(y)\rho(t)Le^{-\rho(t)L}(I - e^{-\rho(r_i)L})^N b_i(y))^2 \frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\, d\mu(x)\\ &\lesssim \sum_i \int_{M\setminus\bigcup_j 4B_j}\int_{2r_i}^\infty\int_{B(x,t)} (\mathbb{1}_{2B_i}(y)\rho(t)Le^{-\rho(t)L}(I - e^{-\rho(r_i)L})^N b_i(y))^2 \frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\, d\mu(x)\\ &\lesssim \sum_i \int_{2r_i}^\infty\int_M\Big(\int_{B(y,t)}\frac{d\mu(x)}{V(x,t)}\Big)(\mathbb{1}_{2B_i}(y)\rho(t)Le^{-\rho(t)L}(I - e^{-\rho(r_i)L})^N b_i(y))^2\, d\mu(y)\frac{dt}{t}\\ &\lesssim \sum_i \int_{2r_i}^\infty\int_{2B_i} (\rho(t)Le^{-\rho(t)L}(I - e^{-\rho(r_i)L})^N b_i(y))^2\, d\mu(y)\frac{dt}{t}.\end{aligned}$$

For the second inequality, note that for every i, $x \in M \setminus \bigcup_j 4B_j$ means that $x \notin 4B_i$. Then $y \in 2B_i$ and $d(x,y) < t$ imply that $t \ge 2r_i$. Thus, the integral is zero for every i if $0 < t < 2r_i$. We obtain the third inequality by using the Fubini theorem and (1.6).

Then, by using (4.3), it follows that

$$\begin{aligned}
\Lambda_{\mathrm{loc}} &\lesssim \sum_i \int_{2r_i}^{\infty}\int_{2B_i}\Big(\frac{\mu^{1/p_0-1/2}(B_i)}{V^{1/p_0-1/2}(y,t)}\,\frac{V^{1/p_0-1/2}(y,t)}{\mu^{1/p_0-1/2}(B_i)}\rho(t)Le^{-\rho(t)L}(I-e^{-\rho(r_i)L})^N b_i(y)\Big)^2\,d\mu(y)\frac{dt}{t}\\
&\lesssim \sum_i \int_{2r_i}^{\infty}\int_{2B_i}\Big(\frac{V^{1/p_0-1/2}(y,4r_i)}{V^{1/p_0-1/2}(y,t)}\,\frac{V^{1/p_0-1/2}(y,t)}{\mu^{1/p_0-1/2}(B_i)}\rho(t)Le^{-\rho(t)L}(I-e^{-\rho(r_i)L})^N b_i(y)\Big)^2\,d\mu(y)\frac{dt}{t}\\
&\lesssim \mu^{1-2/p_0}(B_i)\sum_i \int_{2r_i}^{\infty}\Big(\frac{4r_i}{t}\Big)^{\nu'(2/p_0-1)}\|V^{1/p_0-1/2}(\cdot,t)\rho(t)Le^{-\rho(t)L}(I-e^{-\rho(r_i)L})^N b_i\|_2^2\frac{dt}{t}\\
&\lesssim \mu^{1-2/p_0}(B_i)\sum_i \|(I-e^{-\rho(r_i)L})^N b_i\|_{p_0}^2\\
&\lesssim \mu^{1-2/p_0}(B_i)\sum_i \|b_i\|_{p_0}^2 \lesssim \lambda^2\sum_i \mu(B_i)\lesssim \lambda^{2-p_0}\int |f|^{p_0}\,d\mu.
\end{aligned}$$

For the second inequality, we use the reverse doubling property (1.7). The third inequality follows from the $L^{p_0}-L^2$ boundedness of the operator $V^{1/p_0-1/2}(\cdot,t)$ $\rho(t)Le^{-\rho(t)L}$ (see Lemma 2.5). Then, by using the L^{p_0} boundedness of the heat semigroup, we get the fourth inequality.

Now for the global part. We split the integral into annuli, that is,

$$\begin{aligned}
\Lambda_{\mathrm{glob}} &\le \int_M \iint_{\Gamma(x)}\Big(\sum_i \mathbb{1}_{M\setminus 2B_i}(y)\rho(t)Le^{-\rho(t)L}(I-e^{-\rho(r_i)L})^N b_i(y)\Big)^2\frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\,d\mu(x)\\
&\le \int_0^{\infty}\int_M\int_{B(y,t)}\Big(\sum_i \mathbb{1}_{M\setminus 2B_i}(y)\rho(t)Le^{-\rho(t)L}(I-e^{-\rho(r_i)L})^N b_i(y)\Big)^2\frac{d\mu(x)}{V(x,t)}\,d\mu(y)\frac{dt}{t}\\
&\le \int_0^{\infty}\int_M\Big(\sum_i \mathbb{1}_{M\setminus 2B_i}(y)\rho(t)Le^{-\rho(t)L}(I-e^{-\rho(r_i)L})^N b_i(y)\Big)^2\,d\mu(y)\frac{dt}{t}.
\end{aligned}$$

In order to estimate the above L^2 norm, we use an argument of dualization. Take the supremum of all functions $h(y,t)\in L^2(M\times(0,\infty), d\mu\,dt/t)$ with norm 1; then

$$\begin{aligned}
\Lambda_{\mathrm{glob}}^{1/2} &\le \Big(\int_0^{\infty}\int_M\Big(\sum_i \mathbb{1}_{M\setminus 2B_i}(y)\rho(t)Le^{-\rho(t)L}(I-e^{-\rho(r_i)L})^N b_i(y)\Big)^2\,d\mu(y)\frac{dt}{t}\Big)^{1/2}\\
&= \sup_h \iint_{M\times(0,\infty)}\Big|\sum_i \mathbb{1}_{M\setminus 2B_i}(y)\rho(t)Le^{-\rho(t)L}(I-e^{-\rho(r_i)L})^N b_i(y)\Big||h(y,t)|\frac{d\mu(y)\,dt}{t}\\
&\le \sup_h \sum_i\sum_{j\ge 2}\int_0^{\infty}\int_{C_j(B_i)}|\rho(t)Le^{-\rho(t)L}(I-e^{-\rho(r_i)L})^N b_i(y)||h(y,t)|\frac{d\mu(y)\,dt}{t}\\
&\le \sup_h \sum_i\sum_{j\ge 2}\Big(\int_0^{\infty}\int_{C_j(B_i)}|\rho(t)Le^{-\rho(t)L}(I-e^{-\rho(r_i)L})^N b_i(y)|^2\frac{d\mu(y)\,dt}{t}\Big)^{1/2}\\
&\qquad\times\Big(\int_0^{\infty}\int_{C_j(B_i)}|h(y,t)|^2\frac{d\mu(y)\,dt}{t}\Big)^{1/2}.
\end{aligned}$$

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Denote

$$I_{ij} = \Big(\int_0^\infty \int_{C_j(B_i)} |\rho(t)Le^{-\rho(t)L}(I - e^{-\rho(r_i)L})^N b_i(y)|^2 \, d\mu(y)\, dt/t\Big)^{1/2}.$$

Let $H_{t,r}(\zeta) = \rho(t)\zeta e^{-\rho(t)\zeta}(1 - e^{-\rho(r)\zeta})^N$. Then

$$I_{ij} = \Big(\int_0^\infty \|H_{t,r_i}(L)b_i\|^2_{L^2(C_j(B_i))} \frac{dt}{t}\Big)^{1/2}. \tag{4.4}$$

We will estimate $\|H_{t,r_i}(L)b_i\|_{L^2(C_j(B_i))}$ by functional calculus. The notation is mainly taken from [2, Section 2.2].

For any fixed t and r, $H_{t,r}$ is a holomorphic function satisfying

$$|H_{t,r}(\zeta)| \lesssim |\zeta|^{N+1}(1 + |\zeta|)^{-2(N+1)}$$

for all $\zeta \in \Sigma = \{z \in \mathbb{C}^* : |\arg z| < \xi\}$ with any $\xi \in (0, \pi/2)$.

Since L is a nonnegative self-adjoint operator or equivalently L is a bisectorial operator of type 0, we can express $H_{t,r}(L)$ by functional calculus. Let $0 < \theta < \omega < \xi < \pi/2$; then

$$H_{t,r}(L) = \int_{\Gamma_+} e^{-zL}\eta_+(z)\, dz + \int_{\Gamma_-} e^{-zL}\eta_-(z)\, dz,$$

where $\Gamma_\pm$ is the half-ray $\mathbb{R}^+ e^{\pm i(\pi/2-\theta)}$ and

$$\eta_\pm(z) = \int_{\gamma_\pm} e^{\zeta z} H_{t,r}(\zeta)\, d\zeta, \quad \forall z \in \Gamma_\pm,$$

with $\gamma_\pm$ being the half-ray $\mathbb{R}^\pm e^{\pm i\omega}$.

Then, for any $z \in \Gamma_\pm$,

$$\begin{aligned}
|\eta_\pm(z)| &= \Big|\int_{\gamma_\pm} e^{\zeta z}\rho(t)\zeta e^{-\rho(t)\zeta}(1 - e^{-\rho(r)\zeta})^N \, d\zeta\Big| \\
&\le \int_{\gamma_\pm} |e^{\zeta z - \rho(t)\zeta}|\rho(t)|\zeta||1 - e^{-\rho(r)\zeta}|^N |\, d\zeta| \\
&\le \int_{\gamma_\pm} e^{-c|\zeta|(|z|+\rho(t))}\rho(t)|\zeta||1 - e^{-\rho(r)\zeta}|^N |\, d\zeta| \\
&\lesssim \int_0^\infty e^{-cs(|z|+\rho(t))}\rho(t)\rho^N(r)s^{N+1}\, ds \le \frac{C\rho(t)\rho^N(r)}{(|z| + \rho(t))^{N+2}}.
\end{aligned}$$

In the second inequality, the constant $c > 0$ depends on θ and ω. Indeed, $\Re(\zeta z) = |\zeta||z|\Re e^{\pm i(\pi/2-\theta+\omega)}$. Since $\theta < \omega$, $\pi/2 < \pi/2 - \theta + \omega < \pi$ and $|e^{\zeta z}| = e^{-c_1|\zeta||z|}$ with $c_1 = -\cos(\pi/2 - \theta + \omega)$. Also, it is obvious to see that $|e^{\rho(t)\zeta}| = e^{-c_2\rho(t)|\zeta|}$. Thus, the second inequality follows. In the third inequality, let $\zeta = se^{\pm i\omega}$; we have $|d\zeta| = ds$. In addition, we dominate $|1 - e^{-\rho(r)\zeta}|^N$ by $(\rho(r)\zeta)^N$.

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We choose θ appropriately such that $|z| \sim \Re z$ for $z \in \Gamma_\pm$; then, for any $j \geq 2$ fixed,

$$\begin{aligned}\|H_{t,r_i}(L)b_i\|_{L^2(C_j(B_i))} &\lesssim \Big(\int_{\Gamma_+} + \int_{\Gamma_-}\Big)\|e^{-\Re zL}b_i\|_{L^2(C_j(B_i))}\frac{\rho(t)}{(|z|+\rho(t))^2}\frac{\rho^N(r_i)}{(|z|+\rho(t))^N}|dz| \\ &\lesssim \int_0^\infty \|e^{-sL}b_i\|_{L^2(C_j(B_i))}\frac{\rho(t)\rho^N(r_i)}{(s+\rho(t))^{N+2}}\,ds.\end{aligned}$$

Applying Lemma 2.5,

$$\begin{aligned}\|H_{t,r_i}(L)b_i\|_{L^2(C_j(B_i))} &\lesssim \frac{2^{j\nu}\|b_i\|_{p_0}}{\mu^{1/2-1/p_0}(B_i)}\int_0^\infty e^{-c(2^jr_i/\rho^{-1}(s))^{\tau(s)}}\frac{\rho(t)\rho^N(r_i)}{(s+\rho(t))^{N+2}}\,ds \\ &\lesssim \frac{2^{j\nu}\|b_i\|_{p_0}}{\mu^{1/2-1/p_0}(B_i)}\Big(\int_0^{\rho(t)} + \int_{\rho(t)}^\infty\Big)e^{-c(2^jr_i/\sigma(s))^{\tau(s)}}\frac{\rho(t)\rho^N(r_i)}{(s+\rho(t))^{N+2}}\,ds \\ &=: \frac{2^{j\nu}\|b_i\|_{p_0}}{\mu^{1/2-1/p_0}(B_i)}(H_1(t,r_i,j)+H_2(t,r_i,j)).\end{aligned} \tag{4.5}$$

In the second and the third lines, $\tau(s)$ is originally defined in (4.3). In fact, it should be $\tau(\rho^{-1}(s))$. Since $\rho^{-1}(s)$ and s are unanimously larger or smaller than one, we always have $\tau(s) = \tau(\rho^{-1}(s))$.

Hence, by the Minkowski inequality, we get from (4.4) and (4.5) that

$$I_{ij} \lesssim \frac{2^{j\nu}\|b_i\|_{p_0}}{\mu^{1/2-1/p_0}(B_i)}\Big(\Big(\int_0^\infty H_1^2(t,r_i,j)\frac{dt}{t}\Big)^{1/2} + \Big(\int_0^\infty H_2^2(t,r_i,j)\frac{dt}{t}\Big)^{1/2}\Big). \tag{4.6}$$

It remains to estimate the two integrals $\int_0^\infty H_1^2(t,r_i,j)(dt/t)$ and $\int_0^\infty H_2^2(t,r_i,j)(dt/t)$. We claim that

$$\int_0^\infty H_1^2(t,r_i,j)\frac{dt}{t},\quad \int_0^\infty H_2^2(t,r_i,j)\frac{dt}{t} \lesssim 2^{-2\beta_1 Nj}. \tag{4.7}$$

We estimate first $\int_0^\infty H_1^2(t,r_i,j)(dt/t)$. Since $\rho(t)\rho^N(r_i)/(s+\rho(t))^{N+2} \leq \rho^N(r_i)/\rho(t)^{N+1}$,

$$H_1(t,r_i,j) \leq \int_0^{\rho(t)} e^{-c(2^jr_i/\sigma(s))^{\beta_2/(\beta_2-1)}}\frac{\rho^N(r_i)}{\rho^{N+1}(t)}\,ds \lesssim e^{-c(2^jr_i/t)^{\beta_2/(\beta_2-1)}}\frac{\rho^N(r_i)}{\rho^N(t)}.$$

It follows that

$$\begin{aligned}\int_0^\infty H_1^2(t,r_i,j)\frac{dt}{t} &\lesssim \int_0^\infty e^{-2c(2^jr_i/t)^{\beta_2/(\beta_2-1)}}\frac{\rho^{2N}(r_i)}{\rho^{2N}(t)}\frac{dt}{t} \\ &\lesssim \int_0^{2^jr_i}\Big(\frac{t}{2^jr_i}\Big)^c\frac{\rho^{2N}(r_i)}{\rho^{2N}(t)}\frac{dt}{t} + \int_{2^jr_i}^\infty \frac{\rho^{2N}(r_i)}{\rho^{2N}(t)}\frac{dt}{t} \\ &\lesssim \frac{\rho^{2N}(r_i)}{\rho^{2N}(2^jr_i)} \lesssim 2^{-2\beta_1 Nj}.\end{aligned}$$

In the first inequality, we dominate the exponential term by a polynomial one for the first integral, where c in the second line is chosen to be larger than $2\beta_2 N$.

We now estimate $\int_0^\infty H_2^2(t, r_i, j)(dt/t)$. Write $\rho(t)\rho^N(r_i)/(s+\rho(t))^{N+2} \le \rho(t)\rho^N(r_i)/s^{N+2}$. On the one hand,

$$H_2(t, r_i, j) = \int_{\rho(t)}^\infty e^{-c(2^j r_i/\sigma(s))^{\tau(s)}} \frac{\rho(t)\rho^N(r_i)}{(s+\rho(t))^{N+2}}\, ds \le \int_{\rho(t)}^\infty \frac{\rho(t)\rho^N(r_i)}{s^{N+2}}\, ds = C\frac{\rho^N(r_i)}{\rho^N(t)}. \tag{4.8}$$

On the other hand,

$$H_2(t, r_i, j) \lesssim 2^{-\beta_1 N j} \frac{\rho(t)}{\rho(2^j r_i)}. \tag{4.9}$$

In fact,

$$\begin{aligned} H_2(t, r_i, j) &\le \int_{\rho(t)}^\infty e^{-c(2^j r_i/\sigma(s))^{\beta_2/(\beta_2-1)}} \frac{\rho(t)\rho^N(r_i)}{s^{N+1}}\frac{ds}{s} \\ &\lesssim 2^{-\beta_1 N j}\frac{\rho(t)}{\rho(2^j r_i)} \int_{\rho(t)}^\infty e^{-c(2^j r_i/\sigma(s))^{\beta_2/(\beta_2-1)}} \frac{\rho^{N+1}(2^j r_i)}{s^{N+1}}\frac{ds}{s} \\ &\lesssim 2^{-\beta_1 N j}\frac{\rho(t)}{\rho(2^j r_i)}. \end{aligned}$$

Now we split the integral into two parts in the same way and control them by using (4.8) and (4.9) separately. Then

$$\begin{aligned} \int_0^\infty H_2^2(t, r_i, j)\frac{dt}{t} &\lesssim \int_0^{2^j r_i} 2^{-2\beta_1 N j}\frac{\rho^2(t)}{\rho^2(2^j r_i)}\frac{dt}{t} + \int_{2^j r_i}^\infty \frac{\rho^{2N}(r_i)}{\rho^{2N}(t)}\frac{dt}{t} \\ &\lesssim 2^{-2\beta_1 N j}. \end{aligned}$$

Therefore, it follows from (4.6) and (4.7) that

$$I_{ij} \lesssim \frac{\mu^{1/2}(2^j B_i)\|b_i\|_{p_0}}{\mu^{1/p_0}(B_i)} 2^{-\beta_1 N j}. \tag{4.10}$$

Now for the integral $(\int_0^\infty \int_{C_j(B_i)} |h(y, t)|^2 (d\mu(y)\, dt/t))^{1/2}$. Take $\tilde{h}(y) = \int_0^\infty |h(y, t)|^2 (dt/t)$; then

$$\left(\int_0^\infty \int_{C_j(B_i)} |h(y, t)|^2 \frac{d\mu(y)\, dt}{t}\right)^{1/2} \le \mu^{1/2}(2^{j+1}B_i) \inf_{z\in B_i} \mathcal{M}^{1/2}\tilde{h}(z), \tag{4.11}$$

where $\mathcal{M}$ is the Hardy–Littlewood maximal function.

Following the route for the proof of (4.2), we get from (4.10) and (4.11) that

$$\begin{aligned} \Lambda_{\text{glob}}^{1/2} &\lesssim \sup_h \sum_i \sum_{j\ge 2} \frac{2^{j\nu}\|b_i\|_{p_0}}{\mu^{1/2-1/p_0}(B_i)} 2^{-\beta_1 N j}\mu^{1/2}(2^{j+1}B_i) \inf_{z\in B_i} \mathcal{M}^{1/2}\tilde{h}(z) \\ &\lesssim \lambda \sup_h \int_M \sum_i \mathbb{1}_{B_i}(y)\mathcal{M}^{1/2}\tilde{h}(y)\, d\mu(y) \end{aligned}$$

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$$\lesssim \lambda \sup_h \int_{\bigcup_i B_i} \mathcal{M}^{1/2}\tilde{h}(y)\, d\mu(y)$$
$$\lesssim \lambda \mu\Big(\bigcup_i B_i\Big)^{1/2} \lesssim \lambda^{1-p_0/2} \int |f|^{p_0}\, d\mu.$$

Here the supremum is taken over all the functions h with $\|h\|_{L^2(d\mu\, dt/t)} = 1$. Since $N > 2\nu/\beta_1$, the sum $\sum_{j\geq 2} 2^{-\beta_1 N j + 3\nu j/2}$ converges and we get the second inequality. The fourth one is a result of Kolmogorov's inequality.

Thus, we have shown that $\Lambda_{\text{glob}} \lesssim \lambda^{2-p_0} \int |f|^{p_0}\, d\mu$. □

4.2. Counterexamples to $H^p_{L,S_h}(M) = L^p(M)$. Before moving forward to the proof of Theorem 1.8, let us recall the following two theorems about the Sobolev inequality and the Green operator.

THEOREM 4.3 [19]. *Let (M, μ) be a σ-finite measure space. Let T_t be a semigroup on L^s, $1 \leq s \leq \infty$, with infinitesimal generator $-L$. Assume that T_t is equicontinuous on L^1 and L^∞. Then the following two conditions are equivalent.*

(1) *There exists $C > 0$ such that $\|T_t\|_{1\to\infty} \leq C t^{-D/2}$ for all $t \geq 1$.*

(2) *T_1 is from L^1 to L^∞ and, for $q > 1$, there exists C such that*

$$\|f\|_p \leq C(\|L^{\alpha/2} f\|_q + \|L^{\alpha/2} f\|_p), \quad f \in \mathcal{D}(L^{\alpha/2}),$$

where $0 < \alpha q < D$ and $1/p = 1/q - \alpha/D$.

THEOREM 4.4. *Let M be a complete noncompact manifold. Then there exists a Green's function $G(x, y)$ which is smooth on $(M \times M)\backslash D$ satisfying*

$$\Delta_x \int_M G(x, y) f(y)\, d\mu(y) = f(x), \quad \forall f \in C_0^\infty(M).$$

For a proof, see, for example, [34].

We also observe the following useful lemma.

LEMMA 4.5. *Let M be a Riemannian manifold satisfying the polynomial volume growth (1.12) and the two-sided sub-Gaussian heat kernel estimate ($HK_{2,m}$). Let B be an arbitrary ball with radius $r \geq 4$. Then there exists a constant $c > 0$ depending on d and m such that for all t with $r^m/2 \leq t \leq r^m$,*

$$\int_B p_t(x, y)\, d\mu(y) \geq c, \quad \forall x \in B.$$

PROOF. Note that for any $x, y \in B$, we have $t \geq r^m/2 \geq 2r \geq d(x, y)$. Then ($HK_{2,m}$) yields

$$\int_B p_t(x, y)\, d\mu(y) \geq \int_B \frac{c}{t^{d/m}} \exp\Big(-C\Big(\frac{d^m(x, y)}{t}\Big)^{1/(m-1)}\Big)\, d\mu(y)$$
$$\geq \frac{c\mu(B)}{t^{d/m}} \exp\Big(-C\Big(\frac{r^m}{t}\Big)^{1/(m-1)}\Big) \geq c. \qquad □$$

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With these preparations, we can prove Theorem 1.8.

PROOF OF THEOREM 1.8. Let $\phi_n \in C_0^\infty(M)$ be a cut-off function as follows: $0 \leq \phi_n \leq 1$ and, for some $x_0 \in M$,

$$\phi_n(x) = \begin{cases} 1, & x \in B(x_0, n), \\ 0, & x \in M \backslash B(x_0, 2n). \end{cases}$$

For simplicity, we denote $B(x_0, n)$ by B_n.

Taking $f_n = G\phi_n$, Theorem 4.4 says that $\Delta f_n = \phi_n$.

On the one hand, we apply Theorem 4.3 by choosing $T_t = e^{-t\Delta}$. Indeed, $e^{-t\Delta}$ is Markov and hence bounded on L^p, equicontinuous on L^1, L^∞ and satisfies

$$\|e^{-t\Delta}\|_{1\to\infty} = \sup_{x,y\in M} p_t(x,y) \leq Ct^{-D/2},$$

where $D = 2d/m > 2$. Then, taking $\alpha = 2$ and $p > D/(D-2)$, it follows that

$$\|f_n\|_p \leq C(\|\Delta f_n\|_q + \|\Delta f_n\|_p),$$

where $1/p = 1/q - \alpha/D$, that is, $q = Dp/(D+2p) = dp/(d+mp)$.

Using the facts that $\Delta f_n = \phi_n$ and $\phi_n \leq \mathbb{1}_{B(x_0,2n)}$,

$$\begin{aligned} \|f_n\|_p &\lesssim (\|\phi_n\|_{dp/(d+mp)} + \|\phi_n\|_p) \lesssim (V^{(d+mp)/dp}(x_0, 2n) + V^{1/p}(x_0, 2n)) \\ &\lesssim (n^{m+d/p} + n^{d/p}) \lesssim n^{m+d/p}. \end{aligned} \tag{4.12}$$

In particular, $\|f_n\|_2 \lesssim n^{m+d/2}$.

On the other hand,

$$\begin{aligned} \|S_h f_n\|_p^p &= \int_M \bigg(\iint_{\Gamma(x)} |t^2 \Delta e^{-t^2\Delta} f_n(y)|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \bigg)^{p/2} d\mu(x) \\ &= \int_M \bigg(\iint_{\Gamma(x)} |t^2 e^{-t^2\Delta} \phi_n(y)|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \bigg)^{p/2} d\mu(x). \end{aligned}$$

Since $\phi_n \geq \mathbb{1}_{B_n} \geq 0$, it follows from the Markovian property of the heat semigroup that

$$\|S_h f_n\|_p^p \geq \int_M \bigg(\iint_{\Gamma(x)} |t^2 e^{-t^2 L} \mathbb{1}_{B_n}(y)|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \bigg)^{p/2} d\mu(x).$$

By using Lemma 4.5, we have $e^{-t^2 L}\mathbb{1}_{B_{n/2}} \geq c$ if $n^{m/2}/2 \leq t \leq n^{m/2}$. Then

$$\|S_h f_n\|_p^p \gtrsim \int_{B(x_0, n^{m/2}/4)} \bigg(\int_{n^{m/2}/2}^{n^{m/2}} \int_{B(x,t)\cap B_{n/2}} \frac{t^3}{V(x,t)} \, d\mu(y) \, dt \bigg)^{p/2} d\mu(x).$$

Observe also that, for $t > n^{m/2}/2$ and $x \in B(x_0, n^{m/2}/4)$, we have $B_n \subset B(x,t)$ as long as n is large enough. Then the volume growth (1.12) gives us a lower bound in terms of n. That is,

$$\|S_h f_n\|_p^p \gtrsim \int_{B(x_0, n^{m/2}/4)} \bigg(\int_{n^{m/2}/2}^{n^{m/2}} \frac{\mu(B_n) t^3}{V(x, n^{m/2})} \, dt \bigg)^{p/2} d\mu(x) \gtrsim n^{(md/2)(1-p/2)} n^{mp+dp/2}.$$

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Comparing the upper bound of $\|f_n\|_p$ in (4.12) for $p > D/(D-2)$,

$$\|S_h f_n\|_p \gtrsim n^{(md/2)(1/p-1/2)+m+d/2} \gtrsim n^{d(m/2-1)(1/p-1/2)}\|f_n\|_p, \tag{4.13}$$

where $p > D/(D-2)$.

Assume that $D > 4$, that is, $m < d/2$; we have $D/(D-2) < 2$. Then, for $D/(D-2) < p < 2$, since $m > 2$,

$$n^{d(m/2-1)(1/p-1/2)} \to \infty \quad \text{as } n \to \infty.$$

Thus, (4.13) implies that $L^p \subset H^p_{S_h}$ is not true for $p \in (D/(D-2), 2)$, that is, $p \in (d/(d-m), 2)$, where $2 < m < d/2$.

Our conclusion is: for any fixed $p \in (d/(d-m), 2)$, according to (4.12) and (4.13), there exists a family of functions $\{g_n = f_n/n^{m+d/p}\}_{n\geq 1}$ such that $\|g_n\|_p \leq C$, $\|g_n\|_2 \leq n^{d/2-d/p} \to 0$ and $\|S_h g_n\|_p \geq n^{d(m/2-1)(1/p-1/2)} \to +\infty$ as n goes to infinity. Therefore, S_h is not L^p bounded for $p \in (d/(d-m), 2)$ and the inclusion $L^p \subset H^p_{S_h^{m'}}$ does not hold for $p \in (d/(d-m), 2)$. □

More generally, a slight adaption of Theorem 1.8 plus Theorem 1.6 yields the following result.

Corollary 4.6. *Let M be a Riemannian manifold satisfying (1.12) and $(HK_{2,m})$ as above. Let $p \in (d/(d-m), 2)$. Then, for any $0 < m' \leq m$, $L^p(M) = H^p_{S_h^{m'}}(M)$ if and only if $m' = m$.*

Proof. If $m' = m$, Theorem 1.6 says that $L^p \subset H^p_{S_h^m}$.

Conversely, by doing a slight adjustment for the above proof, we can show that $L^p \subset H^p_{S_h^{m'}}$ is false for $p \in (d/(d-m), 2)$, where $2 < m < d/2$ and $m' < m$. □

5. The $H^1 - L^1$ boundedness of Riesz transforms on fractal manifolds

This section is devoted to an application of the Hardy space theory we introduced above.

Let (M, d, μ) be a Riemannian manifold satisfying the doubling volume property (D) and the sub-Gaussian estimate $(UE_{2,m})$. Note that we could as well consider a metric measure Dirichlet space which admits a 'carré du champ' (see, for example, [6, 28]).

Recall that the Riesz transform $\nabla\Delta^{-1/2}$ is of weak type $(1, 1)$ on M, as follows.

Theorem 5.1 [14]. *Let M be a manifold satisfying the doubling volume property (D) and the heat kernel estimate $(UE_{2,m})$, $m > 2$. Then the Riesz transform is weak $(1, 1)$ bounded and bounded on L^p for $1 < p \leq 2$.*

The proof depends on the following integrated estimate for the gradient of the heat kernel.

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LEMMA 5.2 [14]. *Let M be as above. Then, for all $y \in M$ and all $r, t > 0$,*

$$\int_{M \setminus B(y,r)} |\nabla_x h_t(x,y)|\, d\mu(x) \lesssim \frac{1}{\sqrt{t}} \exp\Big(-c\Big(\frac{\rho(r)}{t}\Big)^{1/(m-1)}\Big),$$

where ρ is defined in (1.4).

Our aim here is to prove Theorem 1.9. More specifically, we will show that the Riesz transform is $H^1_{\Delta,m,\mathrm{mol}}(M) - L^1(M)$ bounded. Due to Theorem 1.5, it is $H^1_{\Delta,m}(M) - L^1(M)$ bounded. The method we use is similar to that in [32, Theorem 3.2]. Note that the pointwise assumption (UE_ρ) simplifies the proof below.

Note first the following lemma, which is crucial in our proof.

LEMMA 5.3. *Let M be as above and let $p \in (1,2)$. Then, for any $E, F \subset M$ and for any $n \in \mathbb{N}$,*

$$\| |\nabla \Delta^n e^{-t\Delta} f| \|_{L^p(F)} \lesssim \begin{cases} \dfrac{1}{t^{n+1/2}} e^{-c(d^2(E,F)/t)} \|f\|_{L^p(E)} & \text{if } 0 < t < 1, \\ \dfrac{1}{t^{n+1/2}} e^{-c(d^m(E,F)/t)^{1/(m-1)}} \|f\|_{L^p(E)} & \text{if } t \geq 1, \end{cases} \tag{5.1}$$

where $f \in L^p(M)$ is supported in E. Consequently,

$$\| |\nabla \Delta^n e^{-t\Delta} f| \|_{L^p(F)} \lesssim \frac{1}{t^{n+1/2}} e^{-c(\rho(d(E,F))/t)^{1/(m-1)}} \|f\|_{L^p(E)}. \tag{5.2}$$

REMARK 5.4. To prove the lemma, it is enough to show the following two estimates:

$$\| |\nabla e^{-t\Delta} f| \|_{L^p(F)} \lesssim \begin{cases} e^{-c(d^2(E,F)/t)} \|f\|_{L^p(E)} & \text{if } 0 < t < 1, \\ e^{-c(d^m(E,F)/t)^{1/(m-1)}} \|f\|_{L^p(E)} & \text{if } t \geq 1 \end{cases}$$

and

$$\|(t\Delta)^n e^{-t\Delta} f\|_{L^p(F)} \lesssim \begin{cases} e^{-c(d^2(E,F)/t)} \|f\|_{L^p(E)} & \text{if } 0 < t < 1, \\ e^{-c(d^m(E,F)/t)^{1/(m-1)}} \|f\|_{L^p(E)} & \text{if } t \geq 1. \end{cases}$$

Then (5.1) follows by adapting the proof of [31, Lemma 2.3]. Note that the first estimate can be obtained by using Stein's approach, similarly to the proof of Lemma 5.2. The second estimate is a direct consequence of (UE_ρ) and the analyticity of the heat semigroup (see [24] for its discrete analogue). We omit the details of the proof here.

REMARK 5.5. Note that (5.1) implies (5.2) (see [14, Corollary 2.4]), which may simplify the calculation in the subsequent proofs.

Finally we will give the proof for Theorem 1.9.

PROOF OF THEOREM 1.9. Denote $T := \nabla \Delta^{-1/2}$. It suffices to show that, for any $(1, 2, \varepsilon)$-molecule a associated to a function b and a ball B with radius r_B, there exists a constant C such that $\|Ta\|_{L^1(M)} \leq C$.

Write

$$Ta = Te^{-\rho(r_B)\Delta} a + T(I - e^{-\rho(r_B)\Delta})a.$$

Then

$$\|Ta\|_{L^1(M)} \leq \|T(I - e^{-\rho(r_B)\Delta})a\|_{L^1(M)} + \|Te^{-\rho(r_B)\Delta} a\|_{L^1(M)} =: I + II.$$

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We first estimate I. We have

$$\begin{aligned} I &\leq \sum_{i\geq 1} \|T(I - e^{-\rho(r_B)\Delta})\mathbb{1}_{C_i(B)} a\|_{L^1(M)} \\ &\leq \sum_{i\geq 1} (\|T(I - e^{-\rho(r_B)\Delta})\mathbb{1}_{C_i(B)} a\|_{L^1(M\setminus 2^{i+2}B)} + \|T(I - e^{-\rho(r_B)\Delta})\mathbb{1}_{C_i(B)} a\|_{L^1(2^{i+2}B)}). \end{aligned}$$

Using the Cauchy–Schwarz inequality and the L^2 boundedness of T and $e^{-\rho(r_B)\Delta}$, it follows that

$$\|T(I - e^{-\rho(r_B)\Delta})\mathbb{1}_{C_i(B)} a\|_{L^1(2^{i+2}B)} \lesssim V(2^{i+2}B)\|a\|_{L^2(C_i(B))} \lesssim 2^{-i\varepsilon}. \tag{5.3}$$

Now we claim that

$$\|T(I - e^{-\rho(r_B)\Delta})\mathbb{1}_{C_i(B)} a\|_{L^1(M\setminus 2^{i+2}B)} \lesssim 2^{-i\varepsilon}. \tag{5.4}$$

Combining (5.3) and (5.4), we obtain that I is bounded.

In order to prove (5.4), we adapt the trick in [14]. For the sake of completeness, we write it down. First note that the spectral theorem gives us $\Delta^{-1/2} f = c\int_0^\infty e^{-s\Delta} f(ds/\sqrt{s})$. Therefore,

$$\begin{aligned} \Delta^{-1/2}(I - e^{-t\Delta})a &= c\int_0^\infty (e^{-s\Delta} - e^{-(s+\rho(r_B))\Delta})a\frac{ds}{\sqrt{s}} \\ &= c\int_0^\infty \Big(\frac{1}{\sqrt{s}} - \frac{\chi_{\{s>\rho(r_B)\}}}{\sqrt{s-\rho(r_B)}}\Big)e^{-s\Delta} a\, ds. \end{aligned}$$

Set

$$k_{\rho(r_B)}(x,y) = \int_0^\infty \Big|\frac{1}{\sqrt{s}} - \frac{\chi_{\{s>\rho(r_B)\}}}{\sqrt{s-\rho(r_B)}}\Big| |\nabla_x h_s(x,y)|\, ds.$$

Then

$$\begin{aligned} \|T(I - e^{-\rho(r_B)\Delta})\mathbb{1}_{C_i(B)} a\|_{L^1(M\setminus 2^{i+2}B)} &\lesssim \int_{M\setminus 2^{i+2}B)} \int_{C_i(B)} k_{\rho(r_B)}(x,y)|a(y)|\, d\mu(y)\, d\mu(x) \\ &\lesssim \int_{C_i(B)} |a(y)| \int_{d(x,y)\geq 2^i r} k_{\rho(r_B)}(x,y)\, d\mu(x)\, d\mu(y). \end{aligned}$$

It remains to show that $\int_{d(x,y)\geq 2^i r} k_{\rho(r_B)}(x,y)\, d\mu(x)$ converges uniformly. Indeed, Lemma 5.2 yields

$$\begin{aligned} \int_{d(x,y)\geq 2^i r} k_{\rho(r_B)}(x,y)\, d\mu(x) &= \int_0^\infty \Big|\frac{1}{\sqrt{s}} - \frac{\chi_{\{s>\rho(r_B)\}}}{\sqrt{s-\rho(r_B)}}\Big| \int_{d(x,y)\geq 2^i r} |\nabla_x h_s(x,y)|\, d\mu(x)\, ds \\ &\lesssim \int_0^\infty \Big|\frac{1}{\sqrt{s}} - \frac{\chi_{\{s>\rho(r_B)\}}}{\sqrt{s-\rho(r_B)}}\Big| \frac{1}{\sqrt{s}} \exp\Big(-c\Big(\frac{\rho(2^i r)}{s}\Big)^{1/(m-1)}\Big)\, ds \\ &\lesssim 1. \end{aligned}$$

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We now turn to estimate II. We have

$$\begin{aligned}II &= \Big\| c\int_0^\infty \nabla e^{-(s+\rho(r_B))\Delta} a\frac{ds}{\sqrt{s}}\Big\|_{L^1(M)}\\ &\lesssim \int_0^{\rho(r_B)} \| |\nabla e^{-(s+\rho(r_B))\Delta} a| \|_{L^1(M)}\frac{ds}{\sqrt{s}} + \int_{\rho(r_B)}^\infty \| |\nabla e^{-(s+\rho(r_B))\Delta}\Delta^K b| \|_{L^1(M)}\frac{ds}{\sqrt{s}}\\ &=: II_1 + II_2.\end{aligned}$$

We estimate II_1 as follows:

$$II_1 \le \sum_{i\ge1}\int_0^{\rho(r_B)} (\| |\nabla e^{-(s+\rho(r_B))\Delta}\mathbb{1}_{C_i(B)} a| \|_{L^1(2^{i+2}B)} + \| |\nabla e^{-(s+\rho(r_B))\Delta}\mathbb{1}_{C_i(B)} a| \|_{L^1(M\setminus 2^{i+2}B)})\frac{ds}{\sqrt{s}}.$$

We estimate the first term inside the sum by Cauchy–Schwarz and the fact that $\|e^{-t\Delta}\|_{2\to2}\lesssim 1/\sqrt{t}$. Then

$$\begin{aligned}\int_0^{\rho(r_B)} \| |\nabla e^{-(s+\rho(r_B))\Delta}\mathbb{1}_{C_i(B)} a| \|_{L^1(2^{i+2}B)}\frac{ds}{\sqrt{s}} &\lesssim \int_0^{\rho(r_B)} V^{1/2}(2^{i+2}B)\|a\|_{L^2(C_i(B))}\frac{ds}{\sqrt{s+\rho(r_B)}\sqrt{s}}\\ &\lesssim 2^{-i\varepsilon}\int_0^{\rho(r_B)}\frac{ds}{\rho(r_B)\sqrt{s}}\\ &\lesssim 2^{-i\varepsilon}.\end{aligned}$$

For the second term inside the sum, we use Lemma 5.2 again. Then

$$\begin{aligned}&\int_0^{\rho(r_B)} \| |\nabla e^{-(s+\rho(r_B))\Delta}\mathbb{1}_{C_i(B)} a| \|_{L^1(M\setminus 2^{i+2}B)}\frac{ds}{\sqrt{s}}\\ &\quad\lesssim \int_0^{\rho(r_B)}\int_{M\setminus 2^{i+2}B}\int_{C_i(B)} |\nabla p_{s+\rho(r_B)}(x,y)a(y)|\,d\mu(y)\,d\mu(x)\frac{ds}{\sqrt{s}}\\ &\quad\lesssim \int_0^{\rho(r_B)}\int_{C_i(B)}\int_{d(x,y)\ge 2^{i+1}B} |\nabla p_{s+\rho(r_B)}(x,y)|\,d\mu(x)|a(y)|\,d\mu(y)\frac{ds}{\sqrt{s}}\\ &\quad\lesssim \|a\|_{L^1(C_i(B))}\int_0^{\rho(r_B)}\frac{ds}{\sqrt{s+\rho(r_B)}\sqrt{s}}\\ &\quad\lesssim 2^{-i\varepsilon}.\end{aligned}$$

It remains to estimate II_2. Using the same method as for II_1,

$$\begin{aligned}II_2 \le \sum_{i\ge1}\int_{\rho(r_B)}^\infty &(\| |\nabla e^{-(s+\rho(r_B))\Delta}\Delta^K\mathbb{1}_{C_i(B)} b| \|_{L^1(2^{i+2}B)} + \| |\nabla e^{-(s+\rho(r_B))\Delta}\Delta^K\mathbb{1}_{C_i(B)} b| \|_{L^1(M\setminus 2^{i+2}B)})\\ &\times\frac{ds}{\sqrt{s}}.\end{aligned}$$

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For the first term inside the sum, we estimate by using the Cauchy–Schwartz inequality and spectral theory. Then

$$\begin{aligned}
&\int_{\rho(r_B)}^{\infty} \| |\nabla \Delta^K e^{-(s+\rho(r_B))\Delta} \mathbb{1}_{C_i(B)} b| \|_{L^1(2^{i+2}B)} \frac{ds}{\sqrt{s}} \\
&\quad \lesssim \int_{\rho(r_B)}^{\infty} \mu^{1/2}(2^{i+2}B) \| |\nabla \Delta^K e^{-(s+\rho(r_B))\Delta} \mathbb{1}_{C_i(B)} b| \|_{L^2(M)} \frac{ds}{\sqrt{s}} \\
&\quad \lesssim \int_{\rho(r_B)}^{\infty} \mu^{1/2}(2^{i+2}B) \|b\|_{L^2(C_i(B))} \frac{ds}{(s+\rho(r_B))^{K+1/2}\sqrt{s}} \\
&\quad \lesssim 2^{-i\varepsilon} \rho^K(r_B) \int_{\rho(r_B)}^{\infty} \frac{ds}{s^{K+1}} \lesssim 2^{-i\varepsilon}.
\end{aligned}$$

For the second term inside the sum, we use Lemma 5.3; then

$$\begin{aligned}
&\int_{\rho(r_B)}^{\infty} \| |\nabla \Delta^K e^{-(s+\rho(r_B))\Delta} \mathbb{1}_{C_i(B)} b| \|_{L^1(M \setminus 2^{i+2}B)} \frac{ds}{\sqrt{s}} \\
&\lesssim \sum_{l=i+2}^{\infty} \int_{\rho(r_B)}^{\infty} \mu^{1/p'}(2^{l+1}B) \| |\nabla \Delta^K e^{-(s+\rho(r_B))\Delta} \mathbb{1}_{C_i(B)} b| \|_{L^p(C_l(B)} \frac{ds}{\sqrt{s}} \\
&\lesssim \sum_{l=i+2}^{\infty} \int_{\rho(r_B)}^{\infty} \mu^{1/p'}(2^{l+1}B) \exp\left(-c\left(\frac{\rho(d(C_l(B), C_i(B)))}{s+\rho(r_B)}\right)^{1/(m-1)}\right) \frac{\|b\|_{L^p(C_i(B)} ds}{\sqrt{s}(s+\rho(r_B))^{K+1/2}} \\
&\lesssim \sum_{l=i+2}^{\infty} 2^{-i\varepsilon} \rho^K(r_B) \left(\frac{\mu(2^{l+1}B)}{\mu(2^i B)}\right)^{1/p'} \int_{\rho(r_B)}^{\infty} \exp\left(-c\left(\frac{\rho(2^l r_B)}{s+\rho(r_B)}\right)^{1/(m-1)}\right) \frac{ds}{\sqrt{s}(s+\rho(r_B))^{K+1/2}} \\
&\lesssim \sum_{l=i+2}^{\infty} 2^{-i\varepsilon} \rho^K(r_B) 2^{(l-i)\nu/p'} \int_{\rho(r_B)}^{\infty} \left(\frac{s}{\rho(2^l r_B)}\right)^c \frac{ds}{s^{K+1}} \\
&\lesssim \sum_{l=i+2}^{\infty} 2^{-i\varepsilon} \rho^K(r_B) 2^{(l-i)\nu/p'} \frac{1}{\rho^c(2^l r_B) \rho^{K-c}(r_B)} \\
&\lesssim 2^{-i\varepsilon}.
\end{aligned}$$

This finishes the proof. □

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