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ATTRACTING INVARIANT CURVES IN PLANAR DISCRETE DYNAMICAL SYSTSMS

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We study the properties of an invariant attracting curve passing through an attracting fixed point of a planar discrete dynamical system. We compare these properties to the corresponding properties of the invariant repelling curve studied in [**3**] in order to determine the dynamic behaviour of the system near the fixed point.

1. INTRODUCTION

We continue in this paper the study started in [**3**] of invariant curves passing through fixed or periodic attracting points of a two dimensional map such that the eigenvalues a and b of the differential of the map at the fixed point are both real and their absolute values are different and smaller than one. Aronson, Chory, Hall and McGehee showed in [**1**] the role played by these curves in the changes in smoothness and even in the total break up of invariant circles born in a Hopf bifurcation. They showed the importance of determining the precise behaviour of orbits near the attracting fixed point and, in particular, made a conjecture about the relationship between condition $|a| < |b|^k$ and the smoothness of the invariant circle. Although they considered this conjecture, the question of the relative behaviour of invariant curves still retains its interest.

In this paper we give some results parallel to those in [**3**]. Here we prove the existence and then study the properties of an invariant curve tangent to the principal direction associated to the eigenvalue with largest absolute value. This curve, h_s, will be attracting in a sense that will be made more precise later.

The existence and regularity of this curve have already been considered in the literature, see for instance [4]. We provide simplified (and better suited to the context) proofs for these facts in the general case $|a| < 1$, $|a| < |b|$, study the dependence of this curve with respect to the map generating the dynamics and, in the last section, use these results together with the results of [**3**] to describe the precise attracting behaviour of this curve near the fixed point. We consider then the suitation proposed by Aronson et al., $|a| < |b|^k < 1$.

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2. Existence and properties of h_s

Proposition 1. *Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (ax + f(x, y), by + g(x, y))$, be a continuous map satisfying*

(1.1) $f(0, 0) = g(0, 0) = 0$,

and let $\delta > 0$ and $\varepsilon > 0$ be such that

(1.2) $f(x, y) = g(x, y) = 0$, *for every* $y \notin [-\delta, \delta]$,

(1.3) $|f(x_1, y_2) - f(x_1, y_1)| \leqslant \varepsilon \|(x_2, y_2) - (x_1, y_1)\|$,
$|g(x_1; y_2) - g(x_1, y_1)| \leqslant \varepsilon \|(x_2, y_2) - (x_1, y_1)\|$, *for every* $x_1, x_2 \in [-\delta, \delta]$ *and every* $y_1, y_2 \in \mathbb{R}$,

(1.4) $\varepsilon \leqslant (|b| - |a|)/2$ *or, equivalently,* $M = (|a| + \varepsilon)/(|b| - \varepsilon) \leqslant 1$,

(1.5) $k = (|a| + \varepsilon)(|b| - \varepsilon)/(|b| - 2\varepsilon) < 1$.

Under these conditions, there exists a unique continuous map $h_s\colon \mathbb{R} \to \mathbb{R}$ such that

(1.a) $h_x(0) = 0$,

(1.b) $|h_s(y)| \leqslant \delta$, *for every* $y \in \mathbb{R}$,

(1.c) $|h_s(y_2) - h_s(y_1)| \leqslant M\,|y_2 - y_1|$ *(h_s is Lipschitz),*

(1.d) *the curve $x = h_s(y)$ is invariant and locally attracting under T. More precisely, for every $(x_n, y_n) = T^n(x_0, y_0)$ with $|x_0| \leqslant \delta$, $|x_n - h_s(y_n)| \leqslant k^n\,|x_0 - h_s(y_0)|$ holds for every $n \geqslant 0$.*

Remark. Notice that (1.4) implies $|a| < |b|$ and (1.5) implies $|a| < 1$.

This proof that follows and the proof of Theorem 3 are simplified versions of the proof of the centre manifold theorem as stated in [4].

Proof: Consider, for fixed $\delta > 0$ and $M \leqslant 1$, the set of continuous maps from $\mathbb{R}$ into itself satisfying,

(H.1) $h(0) = 0$,

(H.2) $|h(y)| \leqslant \delta$, for all $y \in \mathbb{R}$,

(H.3) $|h(y_2) - h(y_1)| \leqslant M\,|y_2 - y_1|$ for every $y_1, y_2 \in \mathbb{R}$,

and denote it by $C^0_{\delta, M}$. This is a closed (and therefore complete) subset of the class of all bounded continuous maps from $\mathbb{R}$ into itself, $C_b(\mathbb{R})$, in the topology induced by the distance $d_\infty(h_1, h_2) = \sup\{|h_1(y) - h_2(y)|\,/y \in \mathbb{R}\}$.

For every $h \in C^0_{\delta, M}$, the map $G_h\colon \mathbb{R} \to \mathbb{R}$ defined by the formula $G_h(y) = by + g(h(y), y)$ is clearly continuous. We shall see that it is also invertible and that G_h^{-1} is Lipschitz continuous with constant $1/(|b| - \varepsilon) > 0$.

For this purpose, for a fixed $\overline{y} \in \mathbb{R}$, consider the map $F\colon \mathbb{R} \to \mathbb{R}$ given by $F(y) =$

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$\overline{y}/b - (1/b)g(h(y), y)$. We have that

$$
\begin{aligned}
|F(y_2) - F(y_1)| &= \frac{1}{b}\,|g(h(y_2), y_2) - g(h(y_1), y_1)| \\
&\leqslant \frac{\varepsilon}{b}\,\|(h(y_2), y_2) - (h(y_1), y_1)\| \leqslant \frac{\varepsilon}{b}\,|y_2 - y_1|.
\end{aligned}
$$

Thus, if $\varepsilon/b < 1$ then F is a contractive map and has therefore a unique fixed point. That is, for every $\overline{y} \in \mathbb{R}$ there exists a unique $y \in \mathbb{R}$ such that $\overline{y} = by + g(h(y), y)$ or, equivalently, $y = G_h^{-1}(\overline{y})$. Moreover,

$$
\begin{aligned}
\left|G_h^{-1}(\overline{y}_2) - G_h^{-1}(\overline{y}_1)\right| &= |y_2 - y_1| \\
&\leqslant \frac{|\overline{y}_2 - \overline{y}_1|}{|b|} + \frac{1}{|b|}\,|g(h(y_2), y_2) - g(h(y_1), y_1)| \\
&\leqslant \frac{|\overline{y}_2 - \overline{y}_1|}{|b|} + \frac{\varepsilon}{|b|}\,|y_2 - y_1|
\end{aligned}
$$

and hence

$$
|y_2 - y_1| \leqslant \frac{1}{|b| - \varepsilon}\,|\overline{y}_2 - \overline{y}_1|.
$$

We now define the map

$$
\begin{aligned}
\mathcal{F}\colon C^0_{\delta, M} &\longrightarrow C^0_{\delta, M} \\
h &\longmapsto \mathcal{F}h\colon \mathbb{R} \longrightarrow \mathbb{R} \\
\mathcal{F}h(\overline{y}) &= a\,h\big(G_h^{-1}(\overline{y})\big) + f\big(h\big(G_h^{-1}(\overline{y})\big), G_h^{-1}(\overline{y})\big)
\end{aligned}
$$

or, in a shorter form, $\mathcal{F}h(\overline{y}) = ah(y) + f(h(y), y)$, where $\overline{y} = by + g(h(y), y)$. $\mathcal{F}$ is well defined since

(H.1) $\mathcal{F}h(0) = 0$,

(H.2) $|\mathcal{F}h(\overline{y})| \leqslant |a|\,|h(y)| + |f(h(y), y)| \leqslant (|a| + \varepsilon)\delta \leqslant \delta$,

(H.3) $|\mathcal{F}(\overline{y}_2) - \mathcal{F}h(\overline{y}_1)| \leqslant |a|\,|h(y_2) - h(y_1)| + |f(h(y_2), y_2) - f(h(y_1), y_1)|$
$\leqslant |a|\,|y_2 - y_1| + \varepsilon\,\|(h(y_2), y_2) - (h(y_1), y_1)\| \leqslant (|a| + \varepsilon)\,|y_2 - y_1|$
$\leqslant ((|a| - \varepsilon)/(|b| + \varepsilon))\,|\overline{y}_2 - \overline{y}_1| = M\,|\overline{y}_2 - \overline{y}_1|$.

We claim that $\mathcal{F}$ is a contractive map. First notice that

$$
\begin{aligned}
&|\mathcal{F}h_2(\overline{y}) - \mathcal{F}h_1(\overline{y})| \\
&\quad\leqslant |a|\,|h_2(y_2) - h_1(y_1)| + |f(h_2(y_2), y_2) - h(h_1(y_1), y_1)| \\
&\quad\leqslant |a|\,|h_2(y_2) - h_1(y_1)| + \varepsilon\,\|(h_2(y_2), y_2) - (h_1(y_1), y_1)\| \\
&\quad\leqslant (|a| + \varepsilon)(|h_2(y_2) - h_1(y_1)| + |y_2 - y_1|),
\end{aligned}
$$

where $\overline{y} = by_1 + g(h_1(y_1), y_1) = by_2 + g(h_2(y_2), y_2)$, and

(1.i) $|h_2(y_2) - h_1(y_1)| \leqslant |h_2(y_2) - h_2(y_1)| + |h_2(y_1) - h_1(y_1)|$
$\leqslant M\,|y_2 - y_1| + |h_2(y_1) - h_1(y_1)|\,|y_2 - y_1| + |h_2(y_1) - h_1(y_1)|$.

Thus,

$$\text{(1.ii)}\quad |y_2 - y_1| \leqslant (1/|b|)\,|g(h_2(y_2), y_2) - g(h_1(y_1), y_1)| \leqslant (\varepsilon/|b|)(|h_2(y_1) - h_1(y_1)| + |y_2 - y_1|)$$

(here we use (1.i)), and then, $|y_2 - y_1| \leqslant (\varepsilon/(|b|-\varepsilon))\,|h_2(y_1) - h_1(y_1)|$. From all this we obtain

$$\begin{aligned}|\mathcal{F}h_2(\overline{y}) - \mathcal{F}h_1(\overline{y})| &\leqslant (|a|+\varepsilon)\left(1 + \frac{\varepsilon}{|b|-\varepsilon}\right)|h_1(y_2) - h_1(y_1)| \\ &= \frac{|b|}{|b|-\varepsilon}(|a|+\varepsilon)\,|h_2(y_1) - h_1(y_1)|\end{aligned}$$

and hence, $\|\mathcal{F}h_2 - \mathcal{F}h_1\|_\infty \leqslant (|b|/(|b|-\varepsilon))(|a|+\varepsilon)\,\|h_2 - h_1\|_\infty$. Therefore, since $|b|/(|b|-\varepsilon) < (|b|-\varepsilon)/(|b|-2\varepsilon)$, hypothesis (1.5) yields that $\mathcal{F}$ is contractive.

$\mathcal{F}$ has then a unique fixed point in $C^0_{\delta,M}$ that we denote by h_s. The curve $x = h_s(y)$ is invariant under T since $h_s(\overline{y}) = ah_s(y) + f(h_s(y), y)$, with $y = G^{-1}_{h_s}(\overline{y})$, or, equivalently, $h_s(by + g(h_s(y), y)) = ah_s(y) + f(h_s(y), y)$.

Finally, let (x_0, y_0) be a point such that $\|x_0\| \leqslant \delta$, and set $(x_1, y_1) = T(x_0, y_0)$. Consider $\overline{y} = G^{-1}_{h_s}(y_1)$. We have that $T(h_s(\overline{y}), \overline{y}) = (h_s(y_1), y_1)$, where $y_1 = b\overline{y} + g(h_s(\overline{y}), \overline{y}) = by_0 + g(x_0, y_0)$, $h_s(y_1) = ah_s(\overline{y}) + f(h_s(\overline{y}), \overline{y})$, and $x_1 = ax_0 + f(x_0, y_0)$. From this,

$$|(h_s(y_1), y_1)| \leqslant |a|\,|h_s(\overline{y}) - x_0| + \varepsilon\,\|(h_s(\overline{y}), \overline{y}) - (x_0, y_0)\|.$$

Since $|b|\,|\overline{y} - y_0| \leqslant \varepsilon\,\|(h_s(\overline{y}), \overline{y}) - (x_0, y_0)\|$, then

$$|\overline{y} - y_0| \leqslant \frac{\varepsilon}{|b|-\varepsilon}\,|h_s(\overline{y}) - x_0| \leqslant |h_s(\overline{y}) - x_0|$$

and so

$$|h_s(y_1) - x_1| \leqslant (|a|+\varepsilon)\,|h_s(\overline{y}) - x_0|.$$

On the other hand,

$$\begin{aligned}|h_s(\overline{y}) - x_0| &\leqslant |h_s(\overline{y}) - h_s(y_0)| + |h_s(y_0) - x_0| \\ &\leqslant |\overline{y} - y_0| + |h_s(y_0) - x_0| \\ &\leqslant \frac{\varepsilon}{|b|-\varepsilon}\,|h_s(\overline{y}) - x_0| + |h_s(y_0) - x_0|.\end{aligned}$$

Hence $|h_s(\overline{y}) - x_0| \leqslant ((|b|-\varepsilon)/(|b|-2\varepsilon))\,|h_s(y_0) - x_0|$.

From all this, $|h_s(y_1) - x_1| \leqslant ((|a|+\varepsilon)(|b|-\varepsilon)/(|b|-2\varepsilon))\,|h_s(y_0) - x_0|$. Now, (1.ii) and the fact that $|x_0| \leqslant \delta$, give (iterating this reasoning) (1.d). ∎

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3. Regularity of h_s

In the case $|a| < |b|^2$, additional hypotheses about the regularity of T give a better regularity of h_s. We shall make use of the following result.

Lemma 2. *Let $\overline{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \mathbb{R}^{n+1}$ and $M > 0$. Define $C^n_{\alpha,M}$ to be the class of C^n maps $h\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that*

(H1) $h(0) = h'(0) = \ldots = h^{(n)}(0) = 0$,
(H2) $\left|h^{(j)}(y)\right| \leqslant \alpha_j$ *for every* $j = 0, 1, \ldots, n$,
(H3) $\left|h^{(n)}(y_2) - h^{(n)}(y_1)\right| \leqslant M\,|y_2 - y_1|$, *for every* y_1 *and* $y_2 \in \mathbb{R}$.

Then, $C^n_{\alpha,M}$ is a closed subset of $C^0_{\alpha_0,M}$ with the topology induced by the distance $d_\infty(h_1, h_2) = \sup\{|h_2(y) - h_1(y)| \,/ y \in \mathbb{R}\}$.

Proof: We shall proceed by induction on n. For $n = 1$ and $\alpha = (\alpha_0, \alpha_1)$, we only need to show that if $(h_k)_k$ converges to h in the distance d_∞ and $h_k \in C^1_{\alpha,M'}$, for all k, then

(2.i) h has derivative at every point,
(2.ii) $h'(0) = 0$,
(2.iii) $|h'(y)| \leqslant \alpha_1$,
(2.iv) $|h'(y_2) - h'(y_1)| \leqslant M\,|y_2 - y_1|$, for every y_1 and $y_2 \in \mathbb{R}$.

Consider $(h'_k)_k \subset \left(C^0_{\alpha_1,M}(\mathbb{R}), d_\infty\right)$ and, for fixed $a \in \mathbb{R}$ and $\varepsilon > 0$, $\overline{h}'_k = h'_k/_{[a-\varepsilon,a+\varepsilon]}$. $\left(\overline{h}'_k\right)_k$ is, by (H3), an equicontinuous set and, by (H2), it is bounded at every point. Thus, by Ascoli's theorem, $\left(\overline{h}'_k\right)_k$ is a relatively compact set. Therefore, there exist $\left(\overline{h}_{k_j}\right) \subseteq (h_k)$ and $g \in C^0_{\alpha_1,M}(\mathbb{R})$ such that $\left(\overline{h}'_{k_j}\right)$ converges uniformly to g. Taking limits in the expression

$$\frac{\overline{h}_{k_j}(x) - \overline{h}_{k_j}(a)}{x - a} - g(a) = \overline{h}'_{k_j}(\xi_j) - g(a),$$

where $|\xi_j - a| \leqslant |x - a| \leqslant \varepsilon$ (and possibly taking a subsequence such that $\xi_j \longrightarrow \xi \in [a - \varepsilon, a + \varepsilon]$), we obtain $\left(\overline{h}(x) - \overline{h}(a)\right)/(x - a) - g(a) = g(\xi) - g(a)$. Since g is continuous, there exists $h'(a) = g(a) = \lim_j \overline{h}'_{k_j}(a) = \lim_j h'_{k_j}(a)$. This proves (2.i).

Now, $h'(0) = \lim_j \overline{h}'_{k_j}(0) = 0$ and $|h'(y)| = |\lim_j h'_{k_j}(y)| \leqslant \alpha_1$, lead to (2.ii) and (2.iii). (2.iv) is obtained by taking limits from the inequality $\left|h'_{k_j}(y_2) - h'_{k_j}(y_1)\right| \leqslant M\,|y_2 - y_1|$.

Now, assume that the reuslt holds for every $n \leqslant p$ and every $\beta \in \mathbb{R}^{p+1}$ and $N > 0$. We shall show that it also holds for $n = p + 1$. If we have $(h_k)_k \subset C^{p+1}_{\alpha,M}$, such that $(h_k)_k$ converges to h in the distance d_∞, then, in particular, $(h_k)_k \subset C^1_{\beta,\alpha_2}$, with $\beta = (\alpha_0, \alpha_1)$; this, because of $|h_k'(y_2) - h_k'(y_1)| \leqslant \left|h_k''(\xi)\right| |y_2 - y_1| \leqslant \alpha_2 |y_2 - y_1|$, $h_k(0) = 0$ and $|h_k(y)| \leqslant \alpha_0$. The case $n = 1$ gives, as before, that h has derivative at every point and that there exists $\left(h_{k_j}\right) \subset (h_k)$ such that h'_{k_j} converges uniformly to h'. Thus, since $\left(h'_{k_j}\right) \subset C^p_{\gamma, M'}$ for $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_{p+1})$, by applying the induction hypothesis to $\left(h'_{k_j}\right)$, we obtain that $h' \in C^p_{\gamma, M}$. And this, together with the continuity of h, $h(0) = 0$ and $|h(x)| \leqslant \alpha_0$, implies $h \in C^{p+1}_{\alpha, M}$. ∎

PROPOSITION 3. *Let $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T(x, y) = (ax + f(x, y), by + g(x, y))$, be a C^1 map such that there exist $1 \geqslant \delta > 0$ and $\varepsilon > 0$ for which*

(3.1) $f(0, 0) = g(0, 0) = 0$, $Df(0, 0) = Dg(0, 0) = 0$,

(3.2) $f(x, y) = g(x, y) = 0$, *for every* $y \notin [-\delta, \delta]$,

(3.3) $\|Df(x, y)\| \leqslant \varepsilon$, $\|Dg(x, y)\| \leqslant \varepsilon$, *for every* $(x, y) \in \mathbb{R}^2$,

(3.4) $\|Df(x_2, y_2) - Df(x_1, y_1)\| \leqslant \varepsilon \|(x_2, y_2) - (x_1, y_1)\|$
$\|Dg(x_2, y_2) - Dg(x_1, y_1)\| \leqslant \varepsilon \|(x_2, y_2) - (x_1, y_1)\|$
for every $x_1, x_2 \in [-\delta, \delta]$ *and every* $y_1, y_2 \in \mathbb{R}$,

(3.5) $\varepsilon \leqslant (|b| - |a|)/2$ *or, equivalently,* $M = (|a| + \varepsilon)/(|b| - \varepsilon) \leqslant 1$,

(3.6) $k = (|a| + \varepsilon)(|b| - \varepsilon)/(|b| - 2\varepsilon) < 1$

(3.7) $|a| + \delta + (|b| + |a|)/(|a| + \varepsilon)\varepsilon \leqslant (|b| - \varepsilon)^2$.

Under these conditions, there exists a unique C^1 map $h_s\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that

(3.a) $h_s(0) = 0$,

(3.b) $|h_s(y)|\,\delta$, $|h'(y)| \leqslant \delta$, *that is, for every* $y \in \mathbb{R}$,

(3.c) $|h_s'(y_2) - h_s'(y_1)| \leqslant M\,|y_2 - y_1|$ *(h' is Lipschitz),*

(3.d) *the curve $x = h_s(y)$ is invariant and locally attracting under T, that is, for every $(x_n, y_n) = T^n(x_0, y_0)$ with $|x_0| \leqslant \delta$, $|x_n - h_s(y_n)| \leqslant k^n |x_0 - h_s(y_0)|$ holds for every $n \geqslant 0$.*

REMARK. Notice that our hypotheses imply $|a| < 1$, $|a| < |b|^2$ and $Df(x, y) = Dg(x, y) = 0$, for every $|y| > \delta$.

PROOF: The proof is similar to the proof of Proposition 1. Thanks to the previous lemma, we know that the function space $C^1_{\alpha, M}$, defined as above for $\alpha = (\delta, \delta)$, is a complete metric space with the distance d_∞. For any fixed $h \in C^1_{\alpha, M}$, the mapping $G_h\colon \mathbb{R} \longrightarrow \mathbb{R}$, $G_h(y) = by + g(h(y), y)$ is a C^1 map. We claim that it is also a C^1-diffeomorphism.

Given any $\bar{y} \in \mathbb{R}$, we construct the map $F\colon \mathbb{R} \longrightarrow \mathbb{R}$, $F(y) = \bar{y}/b - (1/b)g(h(y), y)$.

F is continuous and satisfies, for every y_1 and $y_2 \in \mathbb{R}$,

$$\begin{aligned}|F(y_2) - F(y_1)| &\leqslant \frac{1}{|b|} |g(h(y_2), y_2) - g(h(y_1), y_1)| \\ &\leqslant \frac{\varepsilon}{|b|} \|(h(y_2), y_2) - (h(y_1), y_1)\| \leqslant \frac{\varepsilon}{|b|} |y_2 - y_1|\end{aligned}$$

(here we use $\delta \leqslant 1$). Since $\varepsilon/|b| < 1$, F is contractive. Thus, for every $\overline{y} \in \mathbb{R}$, there exists a unique $y = G_h^{-1}(\overline{y})$. This proves G_h is invertible. Since G_h is C^1 and $(G_n)'(y) = b + Dg(h(y), y)(h'(y), 1)$, we have that $|(G_h)'(y)| \geqslant |b| - \varepsilon > 0$, which implies $(G_h)^{-1}$ is also C^1, and

$$(G_h^{-1})'(\overline{y}) = \frac{1}{(G_h)'(y)} = \frac{1}{b + Dg(h(y), y)(h'(y), 1)},$$

where $\overline{y} = by + g(h(y), y)$. Also,

$$\begin{aligned}&\left|(G_h^{-1})'(\overline{y}_2) - (G_h^{-1})'(\overline{y}_1)\right| \\ &\leqslant \frac{1}{(|b| - \varepsilon)^2} |Dg(h(y_2), y_2)(h'(y_2), 1) - Dg(h(y_1), y_1)(h'(y_1), 1)| \\ &\leqslant \frac{1}{(|b| - \varepsilon)^2} [\|Dg(h(y_2), y_2) - Dg(h(y_1), y_1)\| \, \|(h'(y_2), 1)\| \\ &\qquad + \|Dg(h(y_1), y_1)\| \, \|(h'(y_2), 1) - (h'(y_1), 1)\|] \\ &\leqslant \frac{\varepsilon(1 + M)}{(|b| - \varepsilon)^2} \|y_2 - y_1\| \leqslant \frac{\varepsilon(1 + M)}{(|b| - \varepsilon)^3} \|\overline{y}_2 - \overline{y}_1\|.\end{aligned}$$

Let us now define, denoting $\overline{y} = G_h^{-1}(y)$, the map

$$\begin{aligned}\mathcal{F}\colon C^1_{\alpha, M} &\longrightarrow C^1_{\alpha, M} \\ h &\longmapsto \mathcal{F}h\colon \ \mathbb{R} \longrightarrow \mathbb{R} \\ &\qquad\qquad y \longmapsto \mathcal{F}h(\overline{y}) = ah(y) + f(y(y), y).\end{aligned}$$

We see that $\mathcal{F}$ is well defined. $\mathcal{F}h$ is C^1 with

$$\frac{d}{d\overline{y}}(\mathcal{F}h)(\overline{y}) = (ah'(y) + DF(h(y), y)(h'(y), 1))(G_h^{-1})'(\overline{y}),$$

and satisfies

(H1) $\mathcal{F}h(0) = (\mathcal{F}h)'(0) = 0,$

(H2) $|\mathcal{F}H(y)| \leqslant |a|\,|h(y)| + |f(h(y), y)| \leqslant (|a| + \varepsilon)\delta \leqslant \delta,$

$$\left|(\mathcal{F}h)'(y)\right| \leqslant \frac{|a| + \varepsilon}{|b| - \varepsilon}\delta \leqslant \delta,$$

(H3)
$$\begin{aligned}
\left|(\mathcal{F}h)'(\overline{y}_2) - (\mathcal{F}h)'(\overline{y}_1)\right| &\leqslant |ah'(y_2) + Df(h(y_2), y_2)(h'(y_2), 1)| \\
&\circ \left|(G_h)^{-1}(\overline{y}_z) - (G_h)^{-1}(\overline{y}_1)\right| + \left|(G_h)^{-1}(\overline{y}_1)\right| [|a|\,|h'(y_2) - h'(y_1)| \\
&+ |Df(h(y_2), y_2)(-'(y_2), 1) - Df(h(y_1), y_1)(h'(y_1), 1)|] \\
&\leqslant \frac{|a| + \varepsilon}{(|b| - \varepsilon)^3}\delta\,|\overline{y}_2 - \overline{y}_1| + \frac{1}{|b| - \varepsilon}[|a|\,M + (1 + M)\varepsilon]\,|y_2 - y_1| \\
&\leqslant \frac{|a| + (1 + M)\varepsilon/M + ((|a| + \varepsilon)/(|b| - \varepsilon))(\delta/M)}{(|b| - \varepsilon)^2} M\,|\overline{y}_2 - \overline{y}_1| \\
&= \frac{|a| + \delta + (|b| + |a|)/(|a| + \varepsilon)}{(|b| - \varepsilon)^2} M\,|\overline{y}_2 - \overline{y}_1| \leqslant M\,|\overline{y}_2 - \overline{y}_1|,
\end{aligned}$$

(this because of (3.5)).

The rest of the proof is identical to that of Proposition 1, and is therefore omitted. ∎

4. Dependence with respect to T

Proposition 4. *Let $T_i\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T_i(x, y) = (ax + f_i(x, y), by + g_i(x, y))$, $i = 1, 2$, be two continuous maps satisfying (1.1) to (1.5) for common values of $\delta > 0$ and $\varepsilon > 0$. Let $h_i = h_{x_i}\colon \mathbb{R} \longrightarrow \mathbb{R}$, $i = 1, 2$, be the maps whose existence and properties are guaranteed by Proposition 1. If, additionally, $\delta < 1$ and $M\,|b| < 1$ (M as in (1.4)), then we obtain*

$$\|h_2 - h_1\| \leqslant \frac{1}{1 - M\,|b|}\alpha_f + \frac{M}{1 - M\,|b|}\alpha_g,$$

with α_f and α_g two non negative numbers such that

$$\alpha_f \leqslant \sup\{|f_2(x, y) - f_1(x, y)| \,/\, |x| \leqslant \delta, y \in \mathbb{R}\} < +\infty,$$
$$\alpha_g \leqslant \sup\{|g_2(x, y) - g_1(x, y)| \,/\, |x| \leqslant \delta, y \in \mathbb{R}\} < +\infty,$$

Proof: Using the notation of Proposition 1, $h_i(\overline{y}) = \mathcal{F}_i h_i(\overline{y}) = ah_i(y_i) + f_i(h_i(y_i), y_i)$, where $\overline{y} = by_1 + g_1(h_1(y_1), y_1) = by_2 + g_2(h_2(y_2), y_2)$. From this,

$$\begin{aligned}
|b|\,|y_2 - y_1| &= |g_2(h_2(y_2), y_2) - g_1(h_1(y_1), y_1)| \\
&\leqslant |g_2(h_2(y_2), y_2) - g_2(h_1(y_1), y_1)| + |g_2(h_1(y_1), y_1) - g_1(h_1(y_1), y_1)| \\
&\leqslant \varepsilon\,\|(h_2(y_2), y_2) - (h_1(y_1), y_1)\| + \alpha_g,
\end{aligned}$$

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where $\alpha_g = \sup\{|g_2(h_1(y), y) - g_1(h_1(y), y)| / y \in \mathbb{R}\}$. Since

$$\begin{aligned} |h_2(y_2) - h_1(y_1)| &\leqslant |h_2(y_2) - h_2(y_1)| + |h_2(y_1) - h_1(y_1)| \\ &\leqslant |h_2(y_1) - h_1(y_1)| + \delta\,|y_2 - y_1| \end{aligned}$$

and $\delta < 1$, $\|(h_2(y_2), y_2) - (h_1(y_1), y_1)\| \leqslant |h_2(y_1) - h_1(y_1)| + |y_2 - y_1|$. Using this, we obtain $|b|\,|y_2 - y_1| \leqslant \varepsilon\,[|h_2(y_1) - h_1(y_1)| + |y_2 - y_1|] + \alpha_g$, and then $|y_2 - y_1| \leqslant (\varepsilon/(|b| - \varepsilon))\,|h_2(y_1) - h_1(y_1)| + g/(|b| - \varepsilon)$. Now,

$$\begin{aligned} |h_2(\overline{y}) - h_1(\overline{y})| &\leqslant |a|\,|h_2(y_2) - h_1(y_1)| + |f_2(h_2(y_2), y_2) - f_2(h_1(y_1), y_1)| \\ &\qquad + |f_2(h_1(y_1), y_1) - f_1(h_1(y_1), y_1)| \\ &\leqslant (|a| + \varepsilon)\,[|h_2(y_2) - h_1(y_1)| + |y_2 - y_1|] + \alpha_f \\ &\leqslant \frac{|a| + \varepsilon}{|b| - \varepsilon}\,|b|\,|h_2(y_2) - h_1(y_1)| + \frac{|a| + \varepsilon}{|b| - \varepsilon}\alpha_g + \alpha_f, \end{aligned}$$

with $\alpha_f = \sup\{|f_2(h_1(y), y) - f_1(h_1(y), y)| / y \in \mathbb{R}\}$, which implies $|h_2(\overline{y}) - h_1(\overline{y})| \leqslant M\,|b|\,\|h_2 - h_1\|_\infty + M\,\alpha_g + \alpha_f$, for every $\overline{y} \in \mathbb{R}$. Taking supremes from here, we obtain the desired result. □

Remark. It is possible to state a more general result applicable to the case $T_i(x, y) = (a_i x + f_i(x, y), by + g_i(x, y))$, $a_1 \neq a_2$. However, if what is desired is a result applicable to the case in which the eigenvalues are all different, it is necessary to add the additional hypotheses $b_i > 1$. This result can be found in [**2**].

5. Application to the general case

We now modify the results to the general case of a mapping satisfying conditions (1.2) and (1.3) only locally. Theorem 6 is one of the main results of this paper and it is interesting to compare it to Theorem 5 of [**3**].

Lemma 5. *Let $T\colon U \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T(x, y) = (ax + f(x, y), by + g(x, y))$, be a C^2 map defined on an open neighbourhood of the origin. If the following conditions hold,*

(5.1) $f(0, 0) = g(0, 0) = 0$ *and* $Df(0, 0) = Dg(0, 0) = 0$,

(5.2) $\{a, b\} \cap \{0, 1\} = \emptyset$,

(5.3) $a \neq b^2$, $a^2 \neq b$,

then, there exists a local C^∞-diffeomorphism such that, in a certain neighbourhood of the origin, the map can be written as $\overline{T}(x, y) = (ax + \overline{f}(x, y), bx + \overline{g}(x, y))$, where $\overline{f}$ and $\overline{g}$ satisfy $\overline{f}(0, 0) = \overline{g}(0, 0) = 0$, $D\overline{f}(0, 0) = D\overline{g}(0, 0) = 0$, and $D^2\overline{f}(0, 0) =$

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$D^2\overline{g}(0,0) = 0$. *This change of variables depends continuously on a and b and on the coefficients of the quadratic form associated to D^2T.*

PROOF: Thanks to (5.1), f and g can be written locally as

$$f(x,y) = f_{20}x^2 + f_{11}xy + f_{02}y^2 + o\Big(\|(x,y)\|^2\Big),$$
$$g(x,y) = g_{20}x^2 + g_{11}xy + g_{02}y^2 + o\Big(\|(x,y)\|^2\Big).$$

where $o(\alpha^n)$ denotes a map such that $o(\alpha^n)/\alpha^n$ goes to zero as α goes to zero.

Consider a change of variables of the form $(\overline{x},\overline{y}) = C(x,y) = (x,y) + H(x,y)$, where each component of H is an homogeneous polynomial of degree 2 in x and y, $H(x,y) = \big(c_{20}x^2 + c_{11}xy + c_{02}y^2,\, d_{20}x^2 + d_{11}xy + d_{02}y^2\big)$. Since $DC(0,0) = Id$, the Inverse Function Theorem guarantees that C is a local C^∞-diffeomorphism. Now, $(x,y) = (\overline{x},\overline{y}) - H(x,y) = (\overline{x},\overline{y}) - H\big(C^{-1}(\overline{x},\overline{y})\big) = (\overline{x},\overline{y}) - H(\overline{x},\overline{y}) + o\Big(\|(\overline{x},\overline{y})^2\|\Big)$. Hence $\overline{T}(\overline{x},\overline{y}) = \overline{T}(C(x,y)) = C(T(x,y)) = T(x,y) + H(T(x,y)) = T(\overline{x},\overline{y}) - (a,b)\,H(\overline{x},\overline{y}) + H(a\overline{x}, b\overline{y}) + o\Big(\|(\overline{x},\overline{y})^2\|\Big)$. Equating terms, we obtain $\overline{f}_{ij} = f_{ij} - ac_{ij} + a^ib^jc_{ij}$, and $\overline{g}_{ij} = g_{ij} - bd_{ij} + a^ib^jd_{ij}$.

We can obtain $f_{ij} = g_{ij} = 0$, for every $i+j=2$, if $a - a^ib^j \neq 0$ and $b - a^ib^j \neq 0$ hold for every $i+j=2$, but this is precisely (5.2) and (5.3). The continuity and the dependence follow from the equalities $c_{ij} = (f_{ij})/\big(a - a^ib^j\big)$ and $d_{ij} = (g_{ij})/\big(b - a^ib^j\big)$. □

THEOREM 6. *Let $T\colon U \subseteq \mathbb{R}^2 \longrightarrow U$ be a C^1 map with a fixed point $p \in \mathrm{Int}\,(U)$. Let a and b be the eigenvalues, both real, of $DT(p)$, and let them satisfy $|a| < |b|$ and $|a| < 1$.*

EXISTENCE: The map T has a locally invariant continuous curve passing through the point p that can be described as the graph $x = h_s(y)$ of a Lipschitz function with constant smaller than or equal to one, using an affine change of coordinates, $\alpha_T(x,y)$. This curve is locally attracting in the sense of (1.d).

DEPENDENCE WITH RESPECT TO T: The curve depends continuously on p and DT in the following terms: the affine change of variable $\alpha_T(x,y) = L_T((x,y) - p)$, where L_T is a linear map depending continuously on $DT(p)$. Given any other C^1 map, $S\colon U \subseteq \mathbb{R}^2 \longrightarrow U$, with a fixed point $p' \in \mathrm{Int}\,(U)$ satisfying the existence conditions, after the corresponding changes of variables, α_T and α_S, have been applied, there exist two Lipschitz, locally invariant and locally attracting curves, $x = h_{s_T}(y)$ and $x = h_{s_S}(y)$, in the region $\|(x,y)\| \leqslant \delta$, for some $\delta > 0$, such that

$$\big\|h_{s_T} - h_{s_S}\big\|_\infty \leqslant K_1\,|a_2 - a_1| + K_2\,|b_2 - b_1| + K_3\alpha_f + K_4\alpha_g,$$

with K_i, $i = 1, 2$, two non negative constants and α_f and α_g such that $\lim\limits_{\beta\to 0} \max\{\alpha_f, \alpha_g\} = 0$, where β is the maximum of the quantities $\|p - p'\|$ and $\|DT(x,y) - DS(x,y)\|$, for every (x,y) such that $\|\alpha_T(x,y)\| \leqslant \delta$ and $\|\alpha_S(x,y)\| \leqslant \delta$.

DIFFERENTIABILITY: If, additionally, T is a C^2 map and $|a| < |b|^2$, $a \neq 0$ and $b \neq 1$, then there exists a curve as described that is differentiable and tangent at p to the principal direction associated to b, though an additional change of variables β_T that, in general, is not linear is required.

REMARK. Notice that the change of variables α_T used here is the same as we used in Theorem 6 of [**3**].

PROOF: EXISTENCE AND DIFFERENTIABILITY: The proof of these two statements is similar to the proof of Theorem 6 of [**3**]. We just need to find $\delta > 0$ and $\varepsilon > 0$, small enough numbers, and a new map $\overline{T}$ which coincides locally with T and that satisfies conditions (1.1) to (1.5) for the existence and, after applying Lemma 5, (3.1) to (3.7) for the differentiability. We omit this part of the proof because a similar procedure is carried out in the proof of the dependence with respect to T that we give below.

DEPENDENCE WITH RESPECT TO T: The continuous dependence with respect to p and $DT(p)$ of the changes of coordinates involved has been already justified. If we denote $T_1 = T$ and $T_2 = S$, these mapping can be written as

$$\begin{aligned} T_1(x, y) &= (a_1 x + f_1(x, y), b_1 y + g_1(x, y)), \\ T_2(x, y) &= (a_2 x + f_2(x, y), b_2 y + g_2(x, y)) \\ &= (a_1 x + (a_2 - a_1)x + f_2(x, y), b_1 y + (b_2 - b_1)y + g_2(x, y)) \\ &= \left(a_1 x + \widehat{f}_2(x, y), b_1 y + \widehat{g}_2(x, y)\right), \end{aligned}$$

where $a_1 = a$, $b_1 = b$, $a_2 = a'$, $b_2 = b'$, and $f_i(0, 0) = g_i(0, 0) = 0$, $D_i f(0, 0) = D_i g(0, 0) = 0$, for $i = 1, 2$. Notice that $\widehat{f}_2(0, 0) = \widehat{g}_2(0, 0) = 0$ and that (identifying linear maps with their associated matrices) $D\widehat{f}_2(x, y) = (a_2 - a_1, 0) + Df_2(x, y)$ and $D\widehat{g}_2(x, y) = (0, b_2 - b_1) + Dg_2(x, y)$.

Given any $\varepsilon > 0$ satisfying (1.4), (1.5) and (4.2) and given $r\colon \mathbb{R} \longrightarrow [0, 1]$ a C^∞ map satisfying $r(x) = 0$ for every $|x| \geqslant 1$ and $r(x) = 1$ for every $|x| \leqslant 1/2$, call $k = \sup\{|r'(x)| / x \in \mathbb{R}\} < +\infty$, and choose $\varepsilon' > 0$ such that $\max\{|a_2 - a_1|, |b_2 - b_1|\} + \varepsilon' \leqslant \varepsilon/(1 + 2k)$, (this is possible only if $|a_2 - a_1|$ and $|b_2 - b_1|$ are sufficiently small) and $0 < \delta(\varepsilon') < 1$ such that, by continuity of $D_i f$ and $D_i g$ at the origin, $\|Df_i(x, y)\| \leqslant \varepsilon'$ and $\|Dg_i(x, y)\| \leqslant \varepsilon'$, for every $\|(x, y)\| \leqslant \delta$ and $i = 1, 2$. Consider now $\overline{T}_1(x, y) = (a_1 x + \overline{f}_1(x, y), b_1 y + \overline{g}_1(x, y))$ and $\overline{T}_2(x, y) = (a_1 x + \overline{f}_2(x, y), b_1 y + \overline{g}_2(x, y))$, where

$$\begin{aligned} \overline{f}_1(x, y) &= r\left(\frac{x^2 + y^2}{\delta^2}\right) f_1(x, y), \ \overline{g}_1(x, y) = r\left(\frac{x^2 + y^2}{\delta^2}\right) g_1(x, y), \\ \overline{f}_2(x, y) &= r\left(\frac{x^2 + y^2}{\delta^2}\right) \widehat{f}_2(x, y), \ \overline{g}_2(x, y) = r\left(\frac{x^2 + y^2}{\delta^2}\right) \widehat{g}_2(x, y). \end{aligned}$$

$\overline{T}_1$ and $\overline{T}_2$ coincide locally with T_1 and T_2, respectively, and satisfy properties (1.1) to (1.5). For instance,

$$\begin{aligned}
|D_1\overline{f}_1(x, y)| &= \left| r'\left(\frac{x^2+y^2}{\delta^2}\right)\frac{2x}{\delta^2} f_1(x, y) + \left(\frac{x^2+y^2}{\delta^2}\right) D_1 f_1(x, y)\right| \\
&\leqslant \varepsilon'(1+2k) \leqslant \varepsilon, \\
|D_1\overline{f}_2(x, y)| &= \left| r'\left(\frac{x^2+y^2}{\delta^2}\right)\frac{2x}{\delta^2} \widehat{f}_2(x, y) + \left(\frac{x^2+y^2}{\delta^2}\right) D_1 \widehat{f}_2(x, y)\right| \\
&\leqslant (|a_2 - a_1| + \varepsilon')(1+2k) \leqslant \varepsilon,
\end{aligned}$$

and similarly for the rest, gives (1.3). We can now apply Proposition 4. Using the same notation we used there, we have that

$$\begin{aligned}
\overline{\alpha}_f &\leqslant \sup\{|\overline{f}_2(x, y) - \overline{f}_1(x, y)| \,/\, |x| \leqslant \delta,\, y \in \mathbb{R}\} \\
&\leqslant \sup\{|\overline{f}_2(x, y) - \overline{f}_1(x, y)| \,/\, \|(x, y)\| \leqslant \delta\} \\
&\leqslant \sup\{\left|\widehat{f}_2(x, y) - f_1(x, y)\right| \,/\, \|(x, y)\| \leqslant \delta\} \\
&\leqslant |a_2 - a_1|\,\delta + \sup\{|f_2(x, y) - f_1(x\,..\,y)| \,/\, \|(x, y)\| \leqslant \delta\} \\
&\leqslant |a_2 - a_1|\,\delta + \alpha_f,
\end{aligned}$$

and $\overline{\alpha}_g \leqslant |b_2 - b_1|\,\delta|\alpha_g$. Thus, we have

$$\begin{aligned}
\|h_2 - h_1\|_\infty &\leqslant \frac{1}{1 - M\,|b_1|}\overline{\alpha}_f + \frac{M}{1 - M\,|b_1|}\overline{\alpha}_g \\
&\leqslant \frac{\delta}{1 - M\,|b_1|}\,|a_2 - a_1| + \frac{M\delta}{1 - M\,|b_1|}\,|b_2 - b_1| + \frac{1}{1 - M\,|b_1|}\overline{\alpha}_f \\
&\quad + \frac{M}{1 - M\,|b_1|}\alpha_g
\end{aligned}$$

and, from here, we obtain the desired result. □

6. Dynamics near an attracting fixed point

In this section, $T\colon U \subseteq \mathbb{R}^2 \longrightarrow U$ will be an (at least) C^1 map with a fixed point $p \in \operatorname{Int}(U)$, a and b the eigenvalues, both real, of $DT(p)$. As in Theorems 6 of **[3]** and 6, we apply an affine change of coordinates α_T depending continuously on p and $DT(p)$ such that the fixed point corresponds to the origin and the principal directions associated to a and b correspond to the OX and OY axis, respectively. That is, the map can be written as $T(x, y) = (ax + f(x, y), by + g(x, y))$, where $f(0, 0) = g(0, 0) = 0$ and $Df(0, 0) = Dg(0, 0) = 0$.

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THEOREM 7. *Let $T: U \subseteq \mathbb{R}^2 \longrightarrow U$ be a C^1 map as above and let a and b satisfy $|a| < |b|^k < 1$ for some positive integer k. There exists a neighourhood V of the fixed point p such that the map has two invariant curves passing through p, that can be written as the graphs of two functions, $y = h_u(x)$ and $x = h_s(y)$, that depend continuously on p and $DT(p)$ and that are Lipschitz continuous with constant smaller than or equal to one, h_u is C^1 and $h'(0) = 0$. The only common point of these curves in V is the fixed point.*

p attracts every ordit intersecting V, that is, for every point $(x_0, y_0) \in V$, if we denote $(x_n, y_n) = T^n(x_0, y_0)$, then $\|(x_n, y_n)\| \leqslant M^n \|(x_0, y_0)\|$, for some $|b| \leqslant M < 1$. Also, every orbit in V not contained in the curve $y = h_u(x)$ satisfies $(|x_n - h_s(y_n)|)/\left(|y_n - h_u(x_n)|^k\right) \leqslant c\,N^n$, for some constant c and $0 < N < 1$.

From this, we see that for every orbit starting near p, either it is contained in the curve $y = h_u(x)$, or it separates from the curve and approaches the curve $x = h_s(y)$. We describe this phenomenon by saying that the curve given by h_u is unstable (or repelling) and the curve given by h_s is stable (or attracting). We also say that any orbit not contained in the unstable curve has a contact of order k with the stable curve.

PROOF: By proceeding as in the proofs of the cited theorems, we can find a map $\overline{T}(x, y) = \left(ax + \overline{f}(x, y), by + \overline{g}(x, y)\right)$ which coincides with T in a neighbourhood $V = B((0, 0), \delta)$, and that satisfies the existence conditions of both curves. In particular, $\left\|D\overline{f}(x, y)\right\| \leqslant \varepsilon$ and $\|D\overline{g}(x, y)\| \leqslant \varepsilon$, for every $(x, y) \in \mathbb{R}^2$ and this, for some $\varepsilon > 0$ such that $\left((|a| + \varepsilon)/(|b| - 2\varepsilon)^k\right)((|b| - \varepsilon)/(|b| - 2\varepsilon)) < 1$ and $|b| + \varepsilon < 1$. The existence of the functions h_u and h_s is then guaranteed by Theorems 6 of [**3**] and 6.

Since the graph of h_u is confined in the region $|y| \leqslant |x|$ and the graph of h_s in the region $|y| > |x|$ (this, since h_s is Lipschitz continuous with constant $(|a| + \varepsilon)/(|b| - \varepsilon) < \left((|a| + \varepsilon)/(|b| - 2\varepsilon)^k\right)((|b| - \varepsilon)/(|b| - 2\varepsilon)) < 1$), these curves only intersect at $(0, 0)$.

Now, if $(x_0, y_0) \in V$, then $\|(x_0, y_0)\| \leqslant \delta$ and

$$\begin{aligned} |x_1| &= \left|ax_0 + \overline{f}(x_0, y_0)\right| \leqslant |a|\,|x_0| + \left|D\overline{f}(\xi_1, \xi_2)\right| \|(x_0, y_0)\| \\ &\leqslant (|a| + \varepsilon)\,\|(x_0, y_0)\|, \\ |y_1| &= |by_0 + \overline{g}(x_0, y_0)| \leqslant |b|\,|y_0| + |D\overline{g}(\xi_3, \xi_4)|\,\|(x_0, y_0)\| \\ &\leqslant (|b| + \varepsilon)\,\|(x_0, y_0)\|, \end{aligned}$$

for some $\|(\xi_1, \xi_2)\| \leqslant \|(x_0, y_0)\|$ and $\|(\xi_3, \xi_4)\| \leqslant \|(x_0, y_0)\|$. From both inequalities, $\|(x_1, y_1)\| \leqslant (|b| + \varepsilon)\,\|(x_0, y_0)\| \leqslant \delta$. Iterating this, we obtain $\|(x_n, y_n)\| \leqslant (|b| + \varepsilon)^n\,\|(x_0, y_0)\|$; thus, we can take $M = |b| + \varepsilon$.

Finally, using (3.b) of [**3**] and (1.d), we obtain

$$\frac{|x_n - h_s(y_n)|}{|y_n - h_u(x_n)|^k} \leqslant \frac{[(|a| + \varepsilon)(|b| - \varepsilon)/(|b| - 2\varepsilon)]^n}{(|b| - 2\varepsilon)^{nk}}\,\frac{|x_0 - h_s(y_0)|}{|y_0 - h_u(x_0)|^k}$$

for all $y_0 \neq h_u(x_0)$, that is, for all (x_0, y_0) not on the unstable curve. Taking

$$c = \frac{|x_0 - h_s(y_0)|}{|y_0 - h_u(x_0)|^k} \text{ and } N = \frac{|a| + \varepsilon}{(|b| - 2\varepsilon)^k} \frac{|b| - \varepsilon}{|b| - 2\varepsilon} < 1,$$

we obtain the desired result. $\square$

REFERENCES

[1] D.G. Aronson, M.A. Chory, G.R. Hall and R.P. McGehee, 'Bifurcations from an invariant circle for two-parameter families of maps of the plane: A computer assisted study', *Comm. Math. Phys.* **83** (1982), 303–354.

[2] F. Esquembre, *Rotation sets for circle maps and invariant curves in planar transformations*, Ph.D. Thesis (Universidad de Murcia, 1991).

[3] F. Esquembre, 'Repelling invariant curves in planar discrete dynamical systems', *Bull. Austral. Math. Soc.* **49** (1994), 469–481.

[4] G. Iooss, *Bifurcation of Maps and Applications* (Noth Holland, 1979).

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