INDECOMPOSABLE 1-FACTORIZATIONS OF THE COMPLETE MULTIGRAPH

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Abstract

The existence of 1-factorizations of the complete multigraph \( \lambda K_n \) which cannot be decomposed into 1-factorizations with smaller \( \lambda \) is studied.

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1. Introduction

Any 1-factorization of the complete graph \( K_{2n} \) provides a schedule for the \( 2n - 1 \) rounds of a simple round robin tournament for \( 2n \) teams, with each team meeting each other team exactly once. If each team is to meet each other team exactly \( \lambda \) times, one schedule for such a multiple round robin tournament is obtained by combining any \( \lambda \) schedules (whether identical or not) for a single round robin tournament. In graph-theoretic terms, combining any \( \lambda \) 1-factorizations of \( K_{2n} \) yields a 1-factorization of \( \lambda K_{2n} \).

One might ask the converse question: given a 1-factorization of \( \lambda K_{2n}, \lambda > 1 \), is it the union of \( \lambda \) 1-factorizations of \( K_{2n} \)? It is readily seen that the answer can be "no"; it suffices to take the 15 distinct 1-factors of \( K_6 \), remove the 5 1-factors of the unique 1-factorization, and observe that the remaining 10 1-factors cannot be partitioned into two 1-factorizations of \( K_6 \). A more general question would be as
follows: given a 1-factorization of $\lambda K_{2n}$, $\lambda > 1$, can it be written as a union of 1-factorizations of $\lambda' K_{2n}$ and $\lambda'' K_{2n}$ for some $\lambda', \lambda'' < \lambda$, for which $\lambda' + \lambda'' = \lambda$? If a 1-factorization cannot be written in this way, we call it indecomposable. We examine the existence of indecomposable 1-factorizations of $K_{2n}$ in this paper, and show that there exist indecomposable 1-factorizations of $\lambda K_{2n}$ for arbitrarily high values of $\lambda$. We also settle existence of indecomposable 1-factorizations for $2 \leq \lambda \leq 6$, leaving a few small open cases.

2. Main results

A 1-factorization of the complete multigraph $\lambda K_{2n}$ is a pair $(V, F)$ where $V$ is the vertex set of $K_{2n}$, and $F$ is a collection of $\lambda(2n - 1)$ 1-factors. A comprehensive survey of research on 1-factorizations of complete graphs is given in [3]. If no two members of $F$ are identical as 1-factors (i.e., no 1-factors are "repeated"), the 1-factorization is said to be simple. We denote a 1-factorization of $\lambda K_{2n}$ by $OF(2n, \lambda)$; when it is indecomposable, we denote it by $IOF(2n, \lambda)$.

In what follows we need an auxiliary result on the existence of 1-factorizations of certain graphs. For $x \in \mathbb{Z}_n$, define $|x|$ as $x$ if $0 < x < \lfloor n/2 \rfloor$, and $-x$ if $\lfloor n/2 \rfloor < x < n$. For $n \geq 2$ and $L \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, let $G(n, L)$ be the regular graph with vertex set $\mathbb{Z}_n$ and edge set $E$ given by $\{x, y\} \in E$ if and only if $|x - y| \in L$.

**Lemma 1.** Let $n$ be an even positive integer, and let $\emptyset \neq L \subseteq \{1, 2, \ldots, n/2\}$. Then $G(n, L)$ has a 1-factorization if and only if $n/\gcd(j, n)$ is even for at least one $j \in L$.

**Proof.** See [2, 4].

Our first result shows that there are indecomposable 1-factorizations with arbitrarily high index $\lambda$.

**Theorem 2.** There exists a simple $IOF(2n, n - 1)$ whenever $2n - 1$ is a prime.

**Proof.** Let $V = \mathbb{Z}_{2n-1} \cup \{\infty\}$, and let $\theta$ be a generator of $\mathbb{Z}_{2n-1}$. Let $F$ be the 1-factor $\{(2i - 1, 2i) \mid 1 \leq i < n\} \cup \{(0, \infty)\}$. Let $F_i = \theta^i F$, $0 \leq i \leq n - 2$, and let $FF = \{F_i \mid 0 \leq i \leq n - 2\} \mod 2n - 1$. Then $(V, FF)$ is an $OF(2n, n - 1)$, which by construction is simple. Let us show that it is also indecomposable. Assume that there exists an $OF(2n, \lambda)(V, F')$ with $F' \subseteq FF$, $\lambda < n - 1$. Consider all of the pairs $\{x, y\}$, $x, y \in \mathbb{Z}_{2n-1}$ having $|x - y| = 1$. There are $2n - 1$
such pairs, each of which is contained in exactly \( \lambda \) factors of \( F' \). On the other hand, \( F' \) contains \( m \) 1-factors for some \( m < 2n - 1 \) whose edges \( \{x, y\} \) are such that \(|x - y| = 1\), and each of these 1-factors contains \( 2n - 2 \) such edges. Thus we have \( \lambda(2n - 1) = m(2n - 2) \), which is a contradiction.

Any nonempty set of edges of a 1-factor \( F \) is a subfactor of \( F \). An \( OF(2n, \lambda) \) \((V, F)\) is said to be a sub-OF of an \( OF(2s, \lambda) \) \((W, G)\) if \( V \subseteq W \), and for each \( f \in F \) there is a \( g \in G \) such that \( f \) is a subfactor of \( g \). We also say that \((V, F)\) is embedded in \((W, G)\).

**Theorem 3.** Any \( OF(2n, \lambda) \) can be embedded in a simple \( OF(2s, \lambda) \) for \( s \geq 2n \) provided \( \lambda \leq 2n - 1 \).

**Proof.** Let \((V, F)\) be an \( OF(2n, \lambda) \) with \( V = \{v_1, v_2, \ldots, v_{2n}\} \) and \( F = \{F_{i,j} \mid 1 \leq i \leq 2n - 1, 1 \leq j \leq \lambda\} \). Note that \((V, F)\) is not required to be simple. However, we may assume without loss of generality that if \((V, F)\) contains repeated 1-factors whenever \( F_{i,j} \) and \( F_{k,j} \) are identical as 1-factors, then \( i \neq k \).

Let \( w = s - n \), and consider the complete graph \( K_{2w} \) with vertex set \( Z_{2w} \) (we assume here that \( V \cap Z_{2w} = \emptyset \)). The graph \( G(2w, \{w - n + 1, w - n + 2, \ldots, w\}) \) is regular of degree \( 2n - 1 \), and by Lemma 1 has a 1-factorization. Let \( H_i, 1 \leq i \leq 2n - 1 \) be the 1-factors of such a 1-factorization. We construct a set \( K \) of 1-factors on the \( 2s \) vertices \( V \cup Z_{2w} \), taking \( K = \{K_{i,j} = F_{i,j} \cup H_i \mid 1 \leq i \leq 2n - 1, 1 \leq j \leq \lambda\} \). \( K \) is a set of \( \lambda(2n - 1) \) distinct 1-factors.

The remaining 1-factors involve edges between \( V \) and \( Z_{2w} \), and are constructed as follows. Let \( A = \{A_r \mid 1 \leq r \leq w - n\} \) be a set of \( w - n \) disjoint pairs: \( A_r = \{a_r, b_r\}, a_r, b_r \in Z_{2w}, |a_r - b_r| = r, A_r \cap A_q = \emptyset \) for \( r \neq q \). Such a set \( A \) always exists and is easy to construct by taking a Skolem or hooked Skolem \( (w - 1) \)-sequence and omitting from it the \( n - 1 \) pairs with largest differences \( w - n + 1, \ldots, w - 1 \).

Let \( Y = \{y_1, \ldots, y_{2n}\} = Z_{2w} - \bigcup_{r=1}^{w-n} A_r \). Define, for \( i \in Z_{2w}, M_i = \{(v_t, y_i + i) \mid 1 \leq t \leq 2n\} \cup \{(a_r + i, b_r + i) \mid 1 \leq r \leq w - n\} \). Clearly \( M_i \) is a 1-factor of \( K_{2s} \) on \( V \cup Z_{2w} \). Now let \( C \) be any \( \lambda \times 2n \) Latin rectangle, and let \( b_j \) be the permutation given by the \( j \)th row of \( C \). Let \( M_{i,j} = \{(v_t, y_{b_j} + i) \mid 1 \leq t \leq 2n\} \cup \{(a_r + i, b_r + i) \mid 1 \leq r \leq w - n\} \). It is straightforward to verify that \( M = \{M_{i,j} \mid i \in Z_{2w}, 1 \leq j \leq \lambda\} \) is a set of \( 2w\lambda \) distinct 1-factors, and further that \((V \cup Z_{2w}, K \cup M)\) is a simple \( OF(2s, \lambda) \) containing the (not necessarily simple) \( OF(2n, \lambda)(V, F) \).

**Corollary 4.** If there exists an \( IOF(2n, \lambda) \) with \( \lambda \leq 2n - 1 \), there exists a simple \( IOF(2s, \lambda) \) for \( s \geq 2n \).
Before proceeding further, we observe that any \( OF(4, \lambda) \), \( \lambda > 1 \), is trivially decomposable. Thus if an \( IOF(2n, \lambda) \) exists for \( \lambda > 1 \), then \( n \geq 3 \).

**Theorem 5.** A simple \( IOF(2n, 2) \) exists if and only if \( n \geq 3 \).

**Proof.** An \( IOF(6, 2) \) exists by Theorem 2 (and also by the remarks in the introduction). Theorem 3 then gives a simple \( IOF(2n, 2) \) for all \( n \geq 6 \). It remains only to exhibit solutions for \( n = 4 \) and 5. One simple \( IOF(8, 2) \) has \( V = \mathbb{Z}_7 \cup \{\infty\} \), and \( F = F' \cup F'' \) where \( F' = \{\{0, \infty\}, \{1, 6\}, \{2, 3\}, \{4, 5\} \mod 7\} \) and \( F'' = \{\{0, \infty\}, \{1, 5\}, \{2, 4\}, \{3, 6\} \mod 7\} \).

An \( IOF(10, 2) \) is developed similarly with \( F' = \{\{0, \infty\}, \{1, 4\}, \{2, 6\}, \{3, 7\}, \{5, 8\} \mod 9\} \) and \( F'' = \{\{0, \infty\}, \{1, 3\}, \{2, 4\}, \{5, 6\}, \{7, 8\} \mod 9\} \).

**Theorem 6.** A simple \( IOF(2n, 3) \) exists if and only if \( n \geq 4 \).

**Proof.** An exhaustive search easily verifies that there is no \( IOF(6, 3) \), whether simple or not. Theorem 2 yields a simple \( IOF(8, 3) \), and then Theorem 3 gives a simple \( IOF(2n, 3) \) for every \( n \geq 8 \). It remains only to give simple \( IOF(2n, 3) \) for \( n = 5, 6, \) and 7; these are given in the appendix.

**Theorem 7.** A simple \( IOF(2n, 4) \) exists if and only if \( n \geq 4 \).

**Proof.** Necessity is obvious. For sufficiency, Theorem 3 together with a simple \( IOF(2n, 4) \) for \( n = 4, 5, 6, \) and 7 is enough; these \( IOFs \) are given in the appendix.

**Theorem 8.** A simple \( IOF(2n, 5) \) exists for \( n = 5, 6, 7 \) and all \( n \geq 10 \).

**Proof.** A simple \( IOF(12, 5) \) exists by Theorem 2, and a simple \( IOF(10, 5) \) and \( IOF(14, 5) \) are given in the appendix; Theorem 3 then gives simple \( IOF(2n, 5) \) for all \( n \geq 10 \).

**Theorem 9.** A simple \( IOF(2n, 6) \) exists for all \( n \geq 6 \).

**Proof.** A simple \( IOF(14, 6) \) exists by Theorem 2, and a simple \( IOF(12, 6) \) is given in the appendix. Theorem 3, together with a nonsimple \( IOF(8, 6) \) given in the appendix, give simple \( IOF(2n, 6) \) for all \( n \geq 8 \).

Of course, the application of the techniques developed does not merely apply to small values of \( \lambda \); for example, we have

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THEOREM 10. (i) A simple IOF(2n, \( \lambda \)) for \( \lambda = 8 \) or 9 exists for \( n = 6, 7 \) and all \( n \geq 12 \).
(ii) A simple IOF(2n, 10) exists for \( n = 7 \) and all \( n \geq 14 \).
(iii) A simple IOF(2n, 12) exists for all \( n \geq 16 \).

PROOF. Simple IOF(12, 8), IOF(14, 8), IOF(12, 9), IOF(14, 9) and IOF(14, 10) and a nonsimple IOF(16, 12) are given in the appendix. The rest follows from Theorem 3.

3. Conclusions and open problems

There are exactly three nonisomorphic OF(6, 2)'s of which exactly one is indecomposable [5]. There exists no indecomposable IOF(6, 3), whether simple or not. This can be determined by exhaustive search. Virtually nothing else is known about the enumeration of OF(2n, \( \lambda \))'s for \( \lambda > 1 \).

One might ask what is the maximum \( \lambda = \lambda(2n) \) such that there exists a simple IOF(2n, \( \lambda \)). Taking all distinct 1-factors of \( K_{2n} \) obviously produces a simple OF(2n, (2n – 3)!!), where \( n!! \) is the product of all odd numbers up to \( n \). Thus \( \lambda(2n) \leq (2n – 3)!! – 1 \). One has \( \lambda(6) = 2 \), but nothing else seems to be known about \( \lambda(2n) \).

Let us mention one other (undoubtedly difficult) problem concerning 1-factorizations of \( \lambda K_{2n} \). Suppose \( P = (p_1, p_2, \ldots, p_k) \) is a partition of the number (2n – 3)!!. Is it possible to partition the 1-factors of \( K_{2n} \) on \( V \) into subsets \( F_1, \ldots, F_k \) such that each \((V, F_i)\) is an IOF(2n, \( p_i \))? Let us call \( P \) admissible if the answer is yes. It is easily seen that (1, 2) is the only admissible partition for \( n = 3 \). Cameron [1] has shown that for \( n = 4 \), the partition (1*15) is admissible but it follows from Theorems 5–7 that many other partitions are admissible for \( n = 4 \).

Acknowledgments

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Appendix

We list here the “base” 1-factors for the IOF(2n, \( \lambda \))’s referred to in Section 2. These were produced using a straightforward backtracking algorithm by computer. The vertex set is always taken to be \( Z_{2n-1} \cup \{\infty\} \).
Simple $\text{IOF}(10, 3)$

\[
\begin{align*}
\{0, \infty\} & \{1, 8\} \{2, 3\} \{4, 5\} \{6, 7\} \\
\{0, \infty\} & \{1, 3\} \{2, 7\} \{4, 6\} \{5, 8\} \\
\{0, \infty\} & \{1, 6\} \{2, 5\} \{3, 8\} \{4, 7\}
\end{align*}
\]

Simple $\text{IOF}(12, 3)$

\[
\begin{align*}
\{0, \infty\} & \{1, 7\} \{2, 3\} \{4, 5\} \{6, 10\} \{8, 9\} \\
\{0, \infty\} & \{1, 7\} \{2, 9\} \{3, 5\} \{4, 6\} \{8, 10\} \\
\{0, \infty\} & \{1, 4\} \{2, 7\} \{3, 10\} \{5, 8\} \{6, 9\}
\end{align*}
\]

Simple $\text{IOF}(14, 3)$

\[
\begin{align*}
\{0, \infty\} & \{1, 12\} \{2, 3\} \{4, 5\} \{6, 7\} \{8, 10\} \{9, 11\} \\
\{0, \infty\} & \{1, 10\} \{2, 11\} \{3, 12\} \{4, 7\} \{5, 8\} \{6, 9\} \\
\{0, \infty\} & \{1, 7\} \{2, 10\} \{3, 8\} \{4, 9\} \{5, 11\} \{6, 12\}
\end{align*}
\]

Simple $\text{IOF}(8, 4)$

\[
\begin{align*}
\{0, \infty\} & \{1, 2\} \{3, 6\} \{4, 5\} \\
\{0, \infty\} & \{1, 4\} \{2, 3\} \{5, 6\} \\
\{0, \infty\} & \{1, 6\} \{2, 4\} \{3, 5\} \\
\{0, \infty\} & \{1, 5\} \{2, 4\} \{3, 6\}
\end{align*}
\]

Simple $\text{IOF}(10, 4)$

\[
\begin{align*}
\{0, \infty\} & \{1, 2\} \{3, 4\} \{5, 6\} \{7, 8\} \\
\{0, \infty\} & \{1, 4\} \{2, 7\} \{3, 5\} \{6, 8\} \\
\{0, \infty\} & \{1, 7\} \{2, 6\} \{3, 5\} \{4, 8\} \\
\{0, \infty\} & \{1, 3\} \{2, 6\} \{4, 7\} \{5, 8\}
\end{align*}
\]

Simple $\text{IOF}(12, 4)$

\[
\begin{align*}
\{0, \infty\} & \{1, 2\} \{3, 4\} \{5, 10\} \{6, 7\} \{8, 9\} \\
\{0, \infty\} & \{1, 10\} \{2, 8\} \{3, 5\} \{4, 6\} \{7, 9\} \\
\{0, \infty\} & \{1, 4\} \{2, 5\} \{3, 8\} \{6, 9\} \{7, 10\} \\
\{0, \infty\} & \{1, 7\} \{2, 6\} \{3, 10\} \{4, 8\} \{5, 9\}
\end{align*}
\]
### Simple $IOF(14, 4)$

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### Nonsimple $IOF(8, 6)$

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- $\{0, \infty\}$ $\{1, 4\}$ $\{2, 5\}$ $\{3, 6\}$
- $\{0, \infty\}$ $\{1, 3\}$ $\{2, 5\}$ $\{4, 6\}$ three times

### Simple $IOF(12, 6)$

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### Simple $IOF(14, 8)$

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### Simple $IOF(12, 9)$

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Simple $IOF(14, 9)$

\[
\begin{align*}
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(0, \infty) & \quad \{0, \infty\} & \quad \{1, 2\} & \quad \{2, 6\} & \quad \{3, 5\} & \quad \{4, 10\} & \quad \{5, 8\} & \quad \{6, 12\} & \quad \{7, 11\} \\
(0, \infty) & \quad \{0, \infty\} & \quad \{1, 9\} & \quad \{2, 10\} & \quad \{3, 6\} & \quad \{4, 8\} & \quad \{5, 12\} & \quad \{6, 12\} & \quad \{7, 11\} \\
(0, \infty) & \quad \{0, \infty\} & \quad \{1, 9\} & \quad \{2, 11\} & \quad \{3, 7\} & \quad \{4, 10\} & \quad \{5, 9\} & \quad \{6, 12\} & \quad \{7, 11\} \\
(0, \infty) & \quad \{0, \infty\} & \quad \{1, 6\} & \quad \{2, 10\} & \quad \{3, 9\} & \quad \{4, 8\} & \quad \{5, 12\} & \quad \{6, 12\} & \quad \{7, 11\} \\
(0, \infty) & \quad \{0, \infty\} & \quad \{1, 8\} & \quad \{2, 7\} & \quad \{3, 11\} & \quad \{4, 10\} & \quad \{5, 9\} & \quad \{6, 12\} & \quad \{7, 12\} \\
\end{align*}
\]

Simple $IOF(14, 10)$

\[
\begin{align*}
(0, \infty) & \quad \{0, \infty\} & \quad \{1, 12\} & \quad \{2, 3\} & \quad \{3, 6\} & \quad \{4, 5\} & \quad \{5, 7\} & \quad \{6, 9\} & \quad \{10, 11\} \\
(0, \infty) & \quad \{0, \infty\} & \quad \{1, 12\} & \quad \{2, 6\} & \quad \{3, 5\} & \quad \{4, 7\} & \quad \{5, 10\} & \quad \{6, 12\} & \quad \{7, 11\} \\
(0, \infty) & \quad \{0, \infty\} & \quad \{1, 6\} & \quad \{2, 5\} & \quad \{3, 4\} & \quad \{4, 10\} & \quad \{5, 8\} & \quad \{6, 12\} & \quad \{7, 11\} \\
(0, \infty) & \quad \{0, \infty\} & \quad \{1, 9\} & \quad \{2, 10\} & \quad \{3, 6\} & \quad \{4, 8\} & \quad \{5, 12\} & \quad \{6, 12\} & \quad \{7, 11\} \\
(0, \infty) & \quad \{0, \infty\} & \quad \{1, 10\} & \quad \{2, 6\} & \quad \{3, 12\} & \quad \{4, 8\} & \quad \{7, 11\} & \quad \{8, 9\} & \quad \{9, 10\} \\
(0, \infty) & \quad \{0, \infty\} & \quad \{1, 9\} & \quad \{2, 7\} & \quad \{3, 8\} & \quad \{4, 12\} & \quad \{5, 10\} & \quad \{6, 12\} & \quad \{7, 11\} \\
(0, \infty) & \quad \{0, \infty\} & \quad \{1, 7\} & \quad \{2, 8\} & \quad \{3, 9\} & \quad \{4, 10\} & \quad \{5, 11\} & \quad \{6, 12\} & \quad \{7, 11\} \\
(0, \infty) & \quad \{0, \infty\} & \quad \{1, 8\} & \quad \{2, 11\} & \quad \{3, 7\} & \quad \{4, 9\} & \quad \{5, 10\} & \quad \{6, 12\} & \quad \{7, 12\} \\
\end{align*}
\]

Nonsimple $IOF(16, 12)$

\[
\begin{align*}
(0, \infty) & \quad \{0, \infty\} & \quad \{1, 14\} & \quad \{2, 3\} & \quad \{3, 14\} & \quad \{4, 5\} & \quad \{5, 13\} & \quad \{6, 7\} & \quad \{8, 9\} & \quad \{10, 11\} & \quad \{12, 13\} & \quad \text{twice} \\
(0, \infty) & \quad \{0, \infty\} & \quad \{1, 4\} & \quad \{2, 7\} & \quad \{3, 14\} & \quad \{4, 10\} & \quad \{5, 13\} & \quad \{6, 12\} & \quad \{8, 10\} & \quad \{9, 11\} & \quad \{9, 12\} & \quad \text{five times} \\
(0, \infty) & \quad \{0, \infty\} & \quad \{1, 6\} & \quad \{2, 14\} & \quad \{3, 7\} & \quad \{4, 10\} & \quad \{5, 13\} & \quad \{8, 11\} & \quad \{9, 11\} & \quad \{9, 12\} \\
(0, \infty) & \quad \{0, \infty\} & \quad \{1, 10\} & \quad \{2, 5\} & \quad \{3, 14\} & \quad \{4, 9\} & \quad \{6, 13\} & \quad \{7, 11\} & \quad \{8, 12\} \\
(0, \infty) & \quad \{0, \infty\} & \quad \{1, 7\} & \quad \{2, 12\} & \quad \{3, 14\} & \quad \{4, 9\} & \quad \{5, 10\} & \quad \{6, 13\} & \quad \{8, 11\} \\
(0, \infty) & \quad \{0, \infty\} & \quad \{1, 11\} & \quad \{2, 8\} & \quad \{3, 14\} & \quad \{4, 10\} & \quad \{5, 12\} & \quad \{6, 9\} & \quad \{7, 13\} \\
(0, \infty) & \quad \{0, \infty\} & \quad \{1, 9\} & \quad \{2, 10\} & \quad \{3, 7\} & \quad \{4, 14\} & \quad \{5, 13\} & \quad \{6, 12\} & \quad \{8, 11\} \\
\end{align*}
\]
References