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ON THE MEAN VALUE OF THE ENUMERATION FUNCTION FOR MULTIPLICATIVE PARTITIONS

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Let $g(n)$ denote the number of multiplicative partitions of the natural number n. We prove that

$$\sum_{n\leqslant x} g(n) = O_\varepsilon\Big(x^2 \exp\{-\Big(\frac{1}{4}-\varepsilon\Big)\sqrt{\log x\cdot\log\log x}\}\Big),\ \varepsilon>0.$$

Consider the set $T(n)=\{(m_1,m_2,\cdots,m_s)\mid n=m_1m_2\cdots m_s\ \&\ m_i>1,\ 1\leqslant i\leqslant s\}$. where n and m_i, $1\leqslant i\leqslant s, \in \mathbb{N}$, and identify those partitions which differ only in the order of their factors. We define $g(n)=|T(n)|$, $n>1$, and $g(1)=1$. For example, $g(12)=4$, since $12=6.2=4.3=3.2.2$.

The behaviour of $g(n)$ is quite erratic, and apparently has not been previously studied in this form [3]. In 1983, Hughes and Shallit [3] proved that $g(n)\leqslant 2n^{\sqrt{2}}$. In 1987, Chen Xiao-Xia [2] proved that $g(n)\leqslant n$. It is easily seen that $g(n)=O(n^\alpha)$, $\alpha<1$ is not true.

In this paper, we discuss the mean value of $g(n)$ and prove that

$$\sum_{n\leqslant x} g(n) = O_\varepsilon\Big(x^2 \exp\{-\Big(\frac{1}{4}-\varepsilon\Big)\sqrt{\log x\cdot\log\log x}\}\Big).$$

Throughout this paper $p(n)$ is the largest prime factor of n for $n>1$, $p(1)=1$. For the proof of the result, we need the following lemmas.

Lemma 1. *Let* $\psi(x,y)=\sum_{\substack{n\leqslant x\\ p(n)\leqslant y}} 1$, *then*

$$\psi(x,y)=O\Big(x\exp\{-\frac{1}{2}\beta\log\beta\}\Big),\quad \text{for } y=x^{1/\beta}.$$

Proof: See [1]. ∎

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Lemma 2. *Let $n > 1$, then $\sum_{\alpha|\frac{n}{p(n)}} d \leqslant 1/(p_1-1)n$, where p_1 is the smallest prime factor of n.*

Proof: Let $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, $p_1 < p_2 < \cdots < p_r$. When $r = 1$, we have

$$\sum_{\alpha|\frac{n}{p(n)}} d = \frac{p_1^{\alpha_1 - 1}}{p_1 - 1} \leqslant \frac{1}{p_1-1} p_1^{\alpha_1}.$$

When $r \geqslant 2$, we have

$$\sum_{\alpha|\frac{n}{p(n)}} d = \prod_{i=1}^{r-1} \frac{p_2^{\alpha_i+1}-1}{p_i-1}\frac{p_r^{\alpha_r}-1}{p_r-1}$$

$$= \frac{1}{p_1-1}\prod_{i=1}^{r-1}\frac{p^{\alpha_i+1}-1}{p_{i+1}-1}(p_r^{\alpha_r}-1) \leqslant \frac{1}{p_1-1}\prod_{i=1}^{r-1}\frac{p_i^{\alpha_i+1}-1}{p_i}p_r^{\alpha_r} \leqslant 1/(p_1-1)n.$$

□

Lemma 3. *Let $n > 1$, then $g(n) \leqslant \sum_{\alpha|n/(p(n))} g(d)$.*

Proof: Let $n = \prod_{j=1}^{r} p_j^{\alpha_j}$, $p_1 < p_2 < \cdots < p_r$, $\alpha_j \geqslant 1$, $1 \leqslant j \leqslant r$. Consider the sets:

$$T_{j_1 j_2 \cdots j_r} = \{\left(p_r p_1^{\alpha_1 - j_1} p_2^{\alpha_2 - j_2} \cdots p_r^{(\alpha_r - 1) - j_r}, m_2, \cdots, m_s\right);$$
$$n = p_r p_1^{\alpha_1 - j_1} p_2^{\alpha_2 - j_2} \cdots p_r^{(\alpha_r-1)-j_r} m_2 \cdots m_s,\ m_i > 1,\ 2 \leqslant i \leqslant s\},$$

where $0 \leqslant j_i \leqslant \alpha_i$, for $1 \leqslant i \leqslant r-1$, and $0 \leqslant j_r \leqslant \alpha_r - 1$; and where again we identify those partitions which differ only the order of their factors.

We see easily that

$$|T_{j_1 j_2 \cdots j_r}| = g\left(p_1^{j_1} p_2^{j_2} \cdots p_r^{j_r}\right) \text{ and } T(n) = \bigcup_{j_1=0}^{\alpha_1} \bigcup_{j_2=0}^{\alpha_2} \cdots \bigcup_{j_r=0}^{\alpha_r - 1} T_{j_1 j_2 \cdots j_r}.$$

So we have

$$g(n) = |T(n)| \leqslant \sum_{j_1=0}^{\alpha_1} \sum_{j_2=0}^{\alpha_2} \cdots \sum_{j_2=0}^{\alpha_r-1} |T_{j_1 j_2 \cdots j_r}|$$
$$= \sum_{j_1=0}^{\alpha_1} \sum_{j_2=0}^{\alpha_2} \cdots \sum_{j_r=0}^{\alpha_r-1} g\left(p_1^{j_1} p_2^{j_2} \cdots p_r^{j_r}\right)$$
$$= \sum_{\alpha|\frac{n}{p(n)}} g(d).$$

∎

COROLLARY. $g(n) \leqslant n$.

PROOF: We use induction on k. When $k = 1$, we have $g(1) = 1$. Suppose $k > 1$ and $g(n) \leqslant n$, for $n \leqslant k$. By Lemma 2, we have

$$g(k+1) \leqslant \sum_{\alpha | \frac{k+1}{p(k+1)}} g(d) \leqslant \sum_{\alpha | \frac{k+1}{p(k+1)}} d \leqslant 1/(p_1 - 1)(k+1) \leqslant k+1,$$

where p_1 is the smallest prime factor of $(k+1)$. ∎

The above proof is much simpler than the proof given in [2].

PROOF OF THE MAIN RESULT: In Lemma 1, put $\beta = (\log x/(\log\log x))^{1-\delta}$, $0 < \delta < 1$ and $y = \exp\{(\log x)^{\delta}(\log\log x)^{1-\delta}\}$. Then we get

$$(1) \qquad \psi(x,y) = O\left(x \exp\{-\frac{1-\delta}{2}\left(\frac{\log x}{\log\log x}\right)^{1-\delta}(\log\log x - \log\log\log x)\}\right)$$
$$= O_\varepsilon\left(x \exp\{-\frac{(1-\delta)(1-\varepsilon)}{2}(\log x)^{1-\delta}(\log\log x)^{\delta}\}\right).$$

Since $g(n) \leqslant n$, using Lemma 3 and (1), we get

$$\sum_{n \leqslant x} g(n) = \sum_{\substack{n \leqslant x \\ p(n) \leqslant y}} g(n) + \sum_{\substack{n \leqslant x \\ p(n) > y}} g(n) = \sum_1 + \sum_2,$$

where

$$(2) \quad \sum_1 \leqslant \sum_{\substack{n \leqslant x \\ p(n) \leqslant y}} n \leqslant x\psi(x,y) = O_\varepsilon\left(x^2 \exp\{-\frac{(1-\delta)(1-\varepsilon)}{2}(\log x)^{1-\delta}(\log\log x)^{\delta}\}\right)$$

and

$$(3) \qquad \sum_2 \leqslant \sum_{\substack{n \leqslant x \\ p(m) > y}} \sum_{\alpha | \frac{n}{p(n)}} d = \sum_{\substack{n \leqslant x \\ p(n) > y}} \frac{n}{p(n)} \sum_{\alpha | \frac{n}{p(n)}} \frac{1}{d} \ll \frac{x}{y} \sum_{n \leqslant x} d(n)$$
$$= O\left(x^2 \log x \exp\{-(\log x)^{\delta}(\log\log x)^{1-\delta}\}\right).$$

Setting $1 - \delta = \delta$, we get $\delta = 1/2$. and by (2), (3), we obtain

$$\sum_{n \leqslant x} g(n) = O_\varepsilon\left(x^2 \exp\{-(1/4 - \varepsilon)\sqrt{\log x \cdot \log\log x}\}\right).$$

For any $k \geqslant 2$, $k \in \mathbb{N}$, let $(\zeta(s))^K = \sum_{n=1}^{\infty} \tau_K(n) n^{-s}$. Obviously, we have $g(n) \geqslant 1/(k!)\tau_k(n)$ and $\sum_{n \leqslant x} g(n) \geqslant 1/(K!) \sum_{n \leqslant x} \tau_K(n)$.

From $D_K(x) = \sum_{n \leqslant x} \tau_K(n) = x p_K(\log x) + O\left(x^{1-1/K} \log^{K-2} x\right)$ [4, 263–264], where p_K is a polynomial of degree $(K-1)$, and the arbitrariness of K, we obtain that for any given $A > 0$, there exists $c(A) > 0$ such that $\sum_{n \leqslant x} g(n) \geqslant c(A) x \log^A x$, when x is sufficiently large.

So, for a lower bound of the mean vlaue of $g(n)$, we conjecture that $\sum_{n \leqslant x} g(n) \geqslant c_0 x \exp\left(c_1 \log^{\lambda} x\right)$, $c_0, c_1 > 0$, $0 < \lambda \leqslant 1$, when x is sufficiently large. ∎

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