AN ARITHMETICAL EXCURSION VIA STONEHAM NUMBERS

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Abstract

Let $p$ be a prime and $b$ a primitive root of $p^2$. In this paper, we give an explicit formula for the number of times a value in $\{0, 1, \ldots, b-1\}$ occurs in the periodic part of the base-$b$ expansion of $1/p^m$. As a consequence of this result, we prove two recent conjectures of Aragón Artacho et al. [‘Walking on real numbers’, Math. Intelligencer 35(1) (2013), 42–60] concerning the base-$b$ expansion of Stoneham numbers.

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1. Introduction

Let $b \geq 2$ be an integer. A real number $\alpha \in (0, 1)$ is called $b$-normal if in the base-$b$ expansion of $\alpha$ the asymptotic frequency of the occurrence of any word $w \in \{0, 1, \ldots, b-1\}^*$ of length $n$ is $1/b^n$. A canonical example of such a number is Champernowne’s number,

$$C_{10} := 0.123456789101112131415161718192021 \cdots,$$

which, given here in base 10, is the size-ordered concatenation of $\mathbb{N}$ (each number written in base 10) proceeded by a decimal point. Champernowne’s number was shown to be 10-normal by Champernowne [5] in 1933 and transcendental by Mahler [9] in 1937.

In 1973, Stoneham [12] defined the following class of numbers. Let $b, c \geq 2$ be relatively prime integers. The Stoneham number $\alpha_{b,c}$ is given by

$$\alpha_{b,c} := \sum_{n \geq 1} \frac{1}{c^n b^n}.$$
Stoneham [12] showed that $\alpha_{2,3}$ is 2-normal. A new proof of this result was given by Bailey and Misiurewicz [4], and finally, in 2002, Bailey and Crandall [3] proved that $\alpha_{b,c}$ is $b$-normal for all coprime integers $b, c \geq 2$; see also Bailey and Borwein [2]. Transcendence of $\alpha_{b,c}$ follows easily by Mahler’s method; the interested reader can see the details Appendix A.

Recently Aragón Artacho et al. [1] made two conjectures concerning properties of the base-4 expansion of the Stoneham number $\alpha_{2,3}$ and the base-3 expansion of $\alpha_{3,5}$, respectively. In this paper, we prove their conjectures, and as such they are stated here as theorems (we have fixed a few small typos in their published conjectures).

**Theorem 1.1.** Let the base-4 expansion of $\alpha_{2,3}$ be given by $\alpha_{2,3} := \sum_{k \geq 1} d_k 4^{-k}$, with $d_k \in \{0, 1, 2, 3\}$. Then, for all $n \geq 0$:

(i) $\sum_{k \geq 1} \frac{1}{3^{n+1}+3^n-1} (e^{\pi i/2})^{d_k} = \begin{cases} i & \text{if } n \text{ is odd}, \\ 1 & \text{if } n \text{ is even}; \end{cases}$

(ii) $d_k = d_{3^n+k} = d_{2 \cdot 3^n+k}$ for $k = \frac{3}{2} (3^n + 1), \frac{3}{2} (3^n + 1) + 1, \ldots, \frac{3}{2} (3^n + 1) + 3^n - 1$.

**Theorem 1.2.** Let the base-3 expansion of $\alpha_{3,5}$ be given by $\alpha_{3,5} := \sum_{k \geq 1} a_k 3^{-k}$, with $a_k \in \{0, 1, 2\}$. Then, for all $n \geq 0$:

(i) $\sum_{k=1}^{5^n+1} (e^{\pi i/3})^{a_k} = (-1)^n e^{\pi i/3}$;

(ii) $a_k = a_{4 \cdot 5^n+k} = a_{8 \cdot 5^n+k} = a_{12 \cdot 5^n+k} = a_{16 \cdot 5^n+k}$ for $k = 5^n+1 + j$, with $j = 1, \ldots, 4 \cdot 5^n$.

We note here that the Stoneham numbers $\alpha_{b,c}$ are in some ways very similar to Champernowne’s numbers. They are not concatenations of consecutive integers, but the concatenation of periods of certain rational numbers. Let $b, c \geq 2$ be coprime integers and let $w_n$ be the word $w \in \{0, 1, \ldots, b-1\}^*$ of minimal length such that

$$
\left(\frac{1}{c^n}\right)_b = 0.\overline{w}_n,
$$

where $(x)_b$ denotes the base-$b$ expansion of the real number $x$ and $\overline{w}$ denotes the infinitely repeated word $w$. Then the Stoneham numbers are similar to the numbers

$$0, w_1 w_2 w_3 w_4 w_5 \cdots w_n \cdots,$$

which are given by concatenating the words $w_n$. Indeed, the Stoneham number has this structure, but with the $w_j$ repeated and cyclicly shifted.

**Remark.** While we will be considering the base-4 expansion of $\alpha_{2,3}$ we are still dealing with a normal number; $\alpha_{2,3}$ is also 4-normal. This is given by a result of Schmidt [11] who proved in 1960 that the $r$-normal real number $x$ is $s$-normal if $\log r / \log s \in \mathbb{Q}$.

## 2. Base-$b$ expansions of rationals

To prove the above theorems in as much generality as possible we will need to consider how we write a reduced fraction $a/k$ in the base $b$. Such an algorithm is well known, but we remind the reader here, as it will be useful to have the general
**Base-b Algorithm for \(a/k < 1\).**

Let \(b, k \geq 2\) be integers and \(a \geq 1\) be an integer coprime to \(k\). Set \(r_0 = a\) and write

\[
egin{align*}
    r_0b &= q_1k + r_1 \\
    r_1b &= q_2k + r_2 \\
    &\vdots \\
    r_{j-1}b &= q_jk + r_j \\
    &\vdots
\end{align*}
\]

where \(q_j \in \{0, 1, \ldots, b - 1\}\) and \(r_j \in \{0, 1, \ldots, k - 1\}\) for each \(j\). Stop when \(r_n = r_0\). Then

\[
\left(\frac{a}{k}\right)_b = 0.q_1q_2\cdots q_n.
\]

**Figure 1.** The base-\(b\) algorithm for the reduced rational \(a/k < 1\).

framework for the proofs of Theorems 1.1 and 1.2. To write \(a/k\) in the base \(b\), we use a sort of modified division algorithm; see Figure 1.

We record here facts about the base-\(b\) algorithm which we will need.

**Lemma 2.1.** Suppose that \(b, k \geq 2\) are coprime, and that \(r_j\) and \(q_j\) are defined by the base-\(b\) algorithm for \(a/k\). Then \(\gcd(r_i, k) = 1\).

**Proof.** Suppose that \(p|m\), and proceed by induction on \(i\). Firstly, \(r_0 = a\) and by assumption \(\gcd(r_0, k) = \gcd(a, k) = 1\).

Now suppose that \(\gcd(r_i, k) = 1\), so that also \(\gcd(r_i b, k) = 1\). Then

\[
    r_{i+1} = r_i b - q_i k \equiv r_i b \not\equiv 0 \mod p,
\]

since \(\gcd(b, k) = 1\). Thus \(\gcd(r_{i+1}, k) = 1\). \(\square\)

Also, we have that equivalent \(r_j\) give equal \(q_j\).

**Lemma 2.2.** Suppose \(b, k \geq 2\) are coprime, and that \(r_j\) and \(q_j\) are defined by the base-\(b\) algorithm for the reduced fraction \(a/k\). Then \(r_i \equiv r_j \mod b\) if and only if \(q_i = q_j\).

**Proof.** Suppose that \(r_i \equiv r_j \mod b\). By considering the difference between \(r_{i-1} = q_i k + r_i\) and \(r_{j-1} = q_j k + r_j\) modulo \(b\), we see that \(b(q_i - q_j)k\), so that since \(\gcd(b, k) = 1\), we have that \(b(q_i - q_j)\). Since \(q_i, q_j \in \{0, 1, \ldots, b - 1\}\), we thus have that \(q_i = q_j\).

Conversely, suppose that \(q_i = q_j\). Here, again, we can consider the difference between the defining equations for \(q_i\) and \(q_j\) modulo \(b\); this gives the desired result. \(\square\)
Indeed, the value of $q_j$ is determined by the residue class of $r_j$ modulo $b$ and the value of $k^{-1}$ modulo $b$.

**Lemma 2.3.** Suppose that $b, k \geq 2$ are coprime, and that $r_j$ and $q_j$ are defined by the base-$b$ algorithm for the reduced fraction $a/k$. Then $r_i \equiv j \pmod{b}$ if and only if $q_i \equiv -jk^{-1} \pmod{b}$, where $q_i \in \{0, 1, \ldots, b-1\}$.

**Proof.** If $r_i \equiv j \pmod{b}$, then the equation $r_i - 1 = q_i k + r_i$ gives $q_i k \equiv -j \pmod{b}$, which in turn gives that $q_i \equiv -jk^{-1} \pmod{b}$. Since $q_i \in \{0, 1, \ldots, b-1\}$ we are done with this direction of proof.

Conversely, suppose that $q_i = (-jk^{-1} \pmod{b})$. Then surely $q_i \equiv -jk^{-1} \pmod{b}$ and so $q_i k \equiv -j \pmod{b}$. Thus, again using $r_{i-1} b = q_i k + r_i$, we have that $r_i \equiv j \pmod{b}$. □

The following lemma is a direct corollary of Lemma 2.3.

**Lemma 2.4.** Suppose that $b, k \geq 2$ are coprime, and that $r_j$ and $q_j$ are defined by the base-$b$ algorithm for the reduced fraction $a/k$. Then $r_i \equiv 0 \pmod{b}$ if and only if $q_i = 0$.

**Proof.** Apply Lemma 2.3 with $j = 0$. □

We will use the following classical theorem (see [10, Theorem 12.4]) and lemma.

**Theorem 2.5.** Let $b$ be a positive integer. Then the base-$b$ expansion of a rational number either terminates or is periodic. Further, if $r, s \in \mathbb{Z}$ with $0 < r/s < 1$ where $\gcd(r, s) = 1$ and $s = TU$, where every prime factor of $T$ divides $b$ and $\gcd(U, b) = 1$, then the period length of the base-$b$ expansion of $r/s$ is the order of $b$ modulo $U$, and the preperiod length is $N$, where $N$ is the smallest positive integer such that $T \mid b^N$.

Theorem 2.5 tells us that the base-$b$ expansion of $a/k$ is purely periodic (recall that $\gcd(b, k) = 1$), and that the minimal period is $\text{ord}_k b$, which divides $\varphi(k)$, so that this also is a period. This result can be exploited using the following number-theoretic result, a proof of which can be found in most elementary number theory texts; for example, see [10, Theorem 9.10].

**Lemma 2.6.** A primitive root of $p^2$ is a primitive root of $p^k$ for any integer $k \geq 2$.

Applying Lemma 2.6 gives the following result.

**Lemma 2.7.** Let $0 < a/p^m < 1$ be a rational number in lowest terms and let $b \geq 2$ be an integer that is a primitive root of $p^2$. Suppose that $(1/p^m)_b = \overline{q_1 q_2 \cdots q_n}$ is given by the base-$b$ algorithm. Then

$$
\left( \frac{a}{p^m} \right)_b = \overline{q_{\sigma(1)} q_{\sigma(2)} \cdots q_{\sigma(n)}}
$$

where $\sigma$ is a cyclic shift on $n$ letters.

**Proof.** This is a direct consequence of the base-$b$ algorithm. □

As a consequence of the above lemmas we are able to provide the following characterisation of certain base-$b$ expansions.
\textbf{Proposition 2.8.} Let \( m \geq 1 \) be an integer, \( p \) be an odd prime, \( b \geq 2 \) be an integer coprime to \( p \), and \( q_j \) and \( r_j \) be given by the base-\( b \) algorithm for the reduced fraction \( a/p^m \). If \( b \) is a primitive root of \( p \) and \( p^2 \), then period\( (a/p^m) = \varphi(p^m) \) and

\[ \# \{ j \mid q_j = 0 \} = \left\lfloor \frac{p^m}{b} \right\rfloor - \left\lfloor \frac{p^{m-1}}{b} \right\rfloor. \]

\textbf{Proof.} The fact that period\( (a/p^m)_b = \varphi(p^m) \) follows directly from \( b \) being a primitive root of \( p \) and \( p^2 \) (Lemma 2.6 and Theorem 2.5). This further implies that the \( \varphi(p^m) \) values of \( r_i \) given by the base-\( b \) algorithm for \( a/p^m \) are distinct. Applying Lemma 2.1 gives that

\[ \{r_1, r_2, \ldots, r_{\varphi(p^m)}\} = \{i \leq p^m : \text{gcd}(i, p) = 1\}. \quad (2.1) \]

Also recall that

\[ \left( \frac{a}{p^m} \right)_b = .q_1q_2 \cdots q_{\varphi(p^m)}, \]

and that by Lemma 2.4, \( q_i = 0 \) if and only if \( r_i \equiv 0 \pmod{b} \). Note that there are exactly

\[ \left\lfloor \frac{p^m}{b} \right\rfloor - \left\lfloor \frac{p^{m-1}}{b} \right\rfloor = \left\lfloor \frac{p^m}{b} \right\rfloor - \left\lfloor \frac{p^{m-1}}{b} \right\rfloor \]

elements of \( \{i \leq p^m : \text{gcd}(i, p) = 1\} \) which are divisible by \( b \). Thus using the set equality (2.1), we have that there are exactly \( \left\lfloor \frac{p^m}{b} \right\rfloor - \left\lfloor \frac{p^{m-1}}{b} \right\rfloor \) elements of \( \{r_1, r_2, \ldots, r_{\varphi(p^m)}\} \) divisible by \( b \). Appealing to Lemma 2.4, we then have that there are \( \left\lfloor \frac{p^m}{b} \right\rfloor - \left\lfloor \frac{p^{m-1}}{b} \right\rfloor \) of \( q_1, q_2, \ldots, q_{\varphi(p^m)} \) such that \( q_j = 0 \).

Note that while we record the \( q_i = 0 \) case because of its simplicity, the method can be applied to count any value of \( q_i \) that is desired by using the appropriate case of Lemma 2.3. In fact, we will do this in a few special cases to prove Theorems 1.1.

3. The base-\( b \) expansion of the Stoneham number \( \alpha_{b,p} \)

We will need properties for both the base-\( b \) and base-\( b^2 \) expansions of the Stoneham number \( \alpha_{b,p} \).

\textbf{Proposition 3.1.} Let \( b, p \geq 2 \) be coprime integers with \( p \) a prime. Denote the base-\( b \) expansion of \( \alpha_{b,p} \) as

\[ \alpha_{b,p} = \sum_{j \geq 1} \frac{1}{p^j b^{m_j}} = \sum_{k \geq 1} \frac{a_k}{b^{k}}, \]

where \( a_k \in \{0, 1, \ldots, b - 1\} \), and write

\[ \left( \sum_{j=0}^{m-1} \frac{p^j}{p^m} \right)_b = .q_1q_2 \cdots q_n, \]

where \( q_i \) is determined by the base-\( b \) algorithm, for each \( i \), so \( n = \text{ord}_{p^m} b \). Then \( q_i = a_{p^m+j+1} \) for each \( i \in \{1, 2, \ldots, n\} \) and each \( j \in \{0, 1, 2, \ldots, p \cdot \varphi(p^m)/\text{ord}_{p^m} b - 1\} \).

It is worth noting that Proposition 3.1 is the full generalisation of Theorem 1.1(ii).
We require the following lemma.

**Lemma 3.2.** Let $b, c \geq 2$ be coprime. Then, for any $m \geq 1$,

$$
\alpha_{b,c} = \sum_{n=1}^{m} \frac{1}{c^n b^c} < \frac{1}{b^{c^{m+1}}}.
$$

That is, the base-$b$ expansion of $\alpha_{b,c}$ agrees with the $b$-ary expansion of its $m$th partial sum up to the $c^{m+1}$th place.

**Proof.** Let $m \geq 1$ and note that

$$
\sum_{n \geq m+1} \frac{1}{c^n} = \frac{1}{c^{m+1}} - \frac{1}{c^{m}} < 1.
$$

Using this fact, we have that

$$
\alpha_{b,c} = \sum_{n=1}^{m} \frac{1}{c^n b^c} = \sum_{n \geq m+1} \frac{1}{c^n b^c} < \frac{1}{b^{c^{m+1}}} \sum_{n \geq m+1} \frac{1}{c^n} < \frac{1}{b^{c^{m+1}}},
$$

which is the desired result. 

**Proof of Proposition 3.1.** Let $m \geq 1$, $s_m = p^m b^{p^m}$, and define the positive integer $r_m$ by

$$
r_m = \sum_{n=1}^{m} \frac{1}{p^n b^p}.
$$

Then

$$
\gcd(r_m, s_m) = \gcd(r_m, p^m b^{p^m}) = \gcd(r_m, pb) = 1.
$$

We apply Theorem 2.5 with $b = b$, $r = r_m$, $s = s_m$, $T = b^{p^m}$, and $U = p^m$ to give that the period length of the base-$b$ expansion of $r_m/s_m$ is the order of $b$ modulo $p^m$, which we will write as

$$
\text{period}(r_m/s_m) = \text{ord}_{p^m} b,
$$

and the preperiod length of $r_m/s_m$ is $p^m$, which we will write as

$$
\text{preperiod}(r_m/s_m) = p^m.
$$

Combining the observations of the previous paragraph with Lemma 3.2 gives that

$$
a_{p^m+1} a_{p^m+2} \cdots a_{p^m+1} = \underbrace{\text{WWW} \cdots \text{W}}_{(p \cdot \varphi(p^m)/\text{ord}_{p^m} b) \text{ times}},
$$

where $w = q_1 q_2 \cdots q_{\text{ord}_{p^m} b}$ is a word on the alphabet $\{0, 1, \ldots, b\}$ with length $\text{ord}_{p^m} b$.

To finish the proof of this proposition, it is enough to appeal to Lemma 3.2 to show that

$$
\left( \sum_{j=0}^{p^m-1} p^j \right)_{b} = \overline{w}
$$

where $w$ is as defined in the previous sentence, which follows directly from the definition of $\alpha_{b,p}$. 

\[\square\]
Theorem 1.1 concerns a base-$b^2$ expansion; we will provide some specialised results for this case only when $b = 2$, in order to specifically prove Theorem 1.1, as the more interesting case for generalisations is the base-$b$ case.

**Lemma 3.3.** Let $b, c \geq 2$ be coprime. Then, for any $m \geq 1$, \[
\alpha_{b,c} = \sum_{n=1}^{m} \frac{1}{c^n b^n} < \frac{1}{(b^2)^{m+1/2}}.
\]
That is, the base-$b^2$ expansion of $\alpha_{b,c}$ agrees with the base-$b^2$ expansion of its $m$th partial sum up to the $\lceil c^{m+1/2} \rceil$th place.

**Proof.** This is a direct consequence of Lemma 3.2. \hfill \Box

**Proposition 3.4.** Let $p$ be an odd prime such that 2 is a primitive root of $p$ and $p^2$. Denote the base-$4$ expansion of $\alpha_{2,p}$ as \[
\alpha_{2,p} = \sum_{j \geq 1} \frac{1}{p^j / 2^{m_j}} = \sum_{k \geq 1} \frac{d_k}{4^k},
\]
where $d_k \in \{0, 1, \ldots, 3\}$, and write \[
\left(\sum_{j=0}^{m-1} \frac{p^j}{p^m}\right)_4 = q_1 q_2 \cdots q_n,
\]
where the $q_i$s are determined by the base-$4$ algorithm, so $n = \text{ord}_{p^m} 4 = \varphi(p^m) / 2$. Then $q_i = \hat{d}_{(p^{m+1})/2 + m_j + 1}$ for each $i \in \{1, \ldots, n\}$ and each $j \in \{0, 1, 2, \ldots, p-1\}$.

**Proof.** This proposition follows as a corollary of Proposition 3.1. Indeed, by Proposition 3.1, we have a prefix $u$ of odd length $p$ and words $w_m$ of even length $\varphi(p^m)$ such that \[
(\alpha_{2,p})_2 = u w_1 w_1 \cdots w_1 \underbrace{w_2 w_2 \cdots w_2}_{p \text{ times}} \cdots \underbrace{w_m w_m \cdots w_m}_{p \text{ times}} \cdots.
\]
Now the word $w_m$ is the minimal repeated word given by the base-$2$ expansion of $(\sum_{j=0}^{m-1} p^j) / p^m$. But \[
0 < \frac{\sum_{j=0}^{m-1} p^j}{p^m} = \frac{p^m - 1}{p^m} < \frac{1}{p} \leq \frac{1}{2},
\]
and so the first letter of $w_m$, for each $m$, is necessarily 0. Define the word $v_m$ by $w_m = 0 v_m$. Then \[
(\alpha_{2,p})_2 = u w_1 w_1 \cdots w_1 \underbrace{w_2 w_2 \cdots w_2}_{p \text{ times}} \cdots \underbrace{w_m w_m \cdots w_m}_{p \text{ times}} \cdots
\]
\[
= u 0 v_1 0 v_1 \cdots 0 v_1 0 v_2 0 v_2 \cdots 0 v_2 \cdots 0 v_m 0 v_m \cdots 0 v_m \cdots,
\]
where the word $u 0$ is of even length $p + 1$ and the word $v_m 0$ is of even length $\varphi(p^m)$.
As in the statement of Proposition 3.1, let \(a_k\) be the \(k\)th letter in the base-2 expansion of \(\alpha_{2,p}\), and as in the statement of the current proposition, let \(d_k\) be the \(k\)th letter in the base-4 expansion of \(\alpha_{2,p}\). Then

\[
d_k = 2a_{2k-1} + a_{2k}.
\]

Using this fact, it is an immediate consequence of (3.1) that there are words \(U\) of length \((p + 1)/2\) and \(W_m\) of length \(\varphi(p^m)/2\) such that

\[
(\alpha_{2,p})_4 = .U \underbrace{W_1 W_1 \cdots W_1}_{p\text{ times}} \underbrace{W_2 W_2 \cdots W_2}_{p\text{ times}} \cdots \underbrace{W_m W_m \cdots W_m}_{p\text{ times}} \cdots
\]

As in Proposition 3.1, to finish the proof of this proposition, it is enough to apply Lemma 3.3 to show that

\[
(W_0, W_m, \ldots, W_m, \ldots)
\]

where \(W_m\) is as defined in the previous sentence, which follows directly from the definition of \(\alpha_{2,p}\).

\[\square\]

4. The Aragon, Bailey, Borwein and Borwein conjectures

In this section, we apply the results of Section 3 to prove Theorems 1.1 and 1.2. As it turns out, the proof of Theorem 1.2 is a bit more straightforward, so we present its proof first.

**Proof of Theorem 1.2** For convenience let us write \(\omega := e^{\pi i/3}\) and let \(r_i\) and \(q_i\) be given by the base-3 algorithm for \(1/5^n\). Note that, by Proposition 3.1,

\[
\sum_{k=1}^{1+5^n+1} \omega^k = \sum_{j=0}^{2} \#\{i \leq \varphi(5^n+1) : q_i = j\} \cdot \omega^j.
\]

Now \(\#\{i \leq \varphi(5^n) : q_i = j\}\) can be given by looking at where the number \(5^n\) lies modulo 15. Since, for every 15 consecutive numbers, 12 of them are coprime to 5, and these 12 fall into the three equivalence classes modulo 3 with an equal frequency of 4 times each, we need only look at the remainder of \(5^n\) modulo 15. An easy calculation gives that

\[
5^n \equiv \begin{cases} 
5 \pmod{15} & \text{if } n \text{ is odd,} \\
10 \pmod{15} & \text{if } n \text{ is even.}
\end{cases}
\]

This allows us to give that

\[
\#\{i \leq \varphi(5^n) : r_i \equiv 0 \pmod{3}\} = \begin{cases} 
4 \cdot \left\lfloor \frac{5^n}{15} \right\rfloor + 1 & \text{if } n \text{ is odd,} \\
4 \cdot \left\lfloor \frac{5^n}{15} \right\rfloor + 3 & \text{if } n \text{ is even,}
\end{cases}
\]
An arithmetical excursion via Stoneham numbers

The number of $i \leq \varphi(5^n) : r_i \equiv 1 \pmod{3}$ is
$$
\# \{ i \leq \varphi(5^n) : r_i \equiv 1 \pmod{3} \} =
\begin{cases} 
4 \cdot \left\lfloor \frac{5^n}{15} \right\rfloor + 2 & \text{if } n \text{ is odd}, \\
4 \cdot \left\lfloor \frac{5^n}{15} \right\rfloor + 3 & \text{if } n \text{ is even}, 
\end{cases}
$$
and
$$
\# \{ i \leq \varphi(5^n) : r_i \equiv 2 \pmod{3} \} =
\begin{cases} 
4 \cdot \left\lfloor \frac{5^n}{15} \right\rfloor + 1 & \text{if } n \text{ is odd}, \\
4 \cdot \left\lfloor \frac{5^n}{15} \right\rfloor + 2 & \text{if } n \text{ is even}. 
\end{cases}
$$

Applying Lemma 2.3 to the preceding equalities gives that
$$
\# \{ i \leq \varphi(5^n) : q_i = 0 \} =
\begin{cases} 
4 \cdot \left\lfloor \frac{5^n}{15} \right\rfloor + 1 & \text{if } n \text{ is odd}, \\
4 \cdot \left\lfloor \frac{5^n}{15} \right\rfloor + 3 & \text{if } n \text{ is even}, 
\end{cases}
$$
$$
\# \{ i \leq \varphi(5^n) : q_i = 1 \} =
\begin{cases} 
4 \cdot \left\lfloor \frac{5^n}{15} \right\rfloor + 2 & \text{if } n \text{ is odd}, \\
4 \cdot \left\lfloor \frac{5^n}{15} \right\rfloor + 2 & \text{if } n \text{ is even}, 
\end{cases}
$$
and
$$
\# \{ i \leq \varphi(5^n) : q_i = 2 \} =
\begin{cases} 
4 \cdot \left\lfloor \frac{5^n}{15} \right\rfloor + 1 & \text{if } n \text{ is odd}, \\
4 \cdot \left\lfloor \frac{5^n}{15} \right\rfloor + 3 & \text{if } n \text{ is even}. 
\end{cases}
$$

Thus, since $1 + \omega + \omega^2 = 0$,
$$
\sum_{k=1+5n+1}^{1+5n+4+5n} \omega^{|i|} = \sum_{j=0}^{2} \# \{ i \leq \varphi(5^{n+1}) : q_i = j \} \cdot \omega^j
= \begin{cases} 
\omega & \text{if } n + 1 \text{ is odd}, \\
-\omega & \text{if } n + 1 \text{ is even}, 
\end{cases}
= (-1)^n \omega,
$$
which proves part (i).

Part (ii) follows directly from Proposition 3.1 with $b = 3$ and $p = 5$. □

**Proof of Theorem 1.1.** Note that
$$
\frac{1}{3^n 2^3} = \frac{8}{3^n} \cdot \frac{1}{4^{\frac{3(n+1)}{2}}}
$$

Let $r_i$ and $q_i$ be given by the base 4 algorithm for $8/3^n$. We will use the fact that each of these $r_i$ is equivalent to 2 modulo 3. This is easily seen as we have for
each \( i \) that \( r_{i-1} = q_i 3^n + r_i \), so that, taking this equality modulo 3, we have that \( r_{i-1} \equiv r_i \pmod{3} \). Recalling that \( r_0 = 8 \) shows that indeed \( r_i \equiv 2 \pmod{3} \) for each \( i \).

Since \( \text{ord}_3 4 = 3^{n-1} \), we have, by Proposition 3.4, that

\[
\frac{3}{2} (3^n + 3^{n-1}) (e^{\pi i/2})^{a_i} = \sum_{j=0}^{3} \#\{i \leq \varphi(3^{n+1})/2 : q_i = j\} \cdot (e^{\pi i/2})^j.
\]

Now \( \#\{i \leq 3^n : q_i = j\} \) can be given by looking at where the number \( 3^n \) lies modulo 12. Since, for every 12 consecutive numbers, four of them are equivalent to 2 modulo 3, and these four fall into the four distinct equivalence classes modulo 3, we must consider the remainder of \( 3^n \) modulo 12. We have that

\[
3^n \equiv \begin{cases} 
3 & \text{if } n \text{ is odd}, \\
9 & \text{if } n \text{ is even}.
\end{cases}
\]

Thus

\[
\#\{i \leq \varphi(3^n)/2 : r_i \equiv 0 \pmod{4}\} = \begin{cases} 
\left\lfloor \frac{3^n}{12} \right\rfloor & \text{if } n \text{ is odd,} \\
\left\lfloor \frac{3^n}{12} \right\rfloor + 1 & \text{if } n \text{ is even,}
\end{cases}
\]

\[
\#\{i \leq \varphi(3^n)/2 : r_i \equiv 1 \pmod{4}\} = \begin{cases} 
\left\lfloor \frac{3^n}{12} \right\rfloor & \text{if } n \text{ is odd,} \\
\left\lfloor \frac{3^n}{12} \right\rfloor + 1 & \text{if } n \text{ is even,}
\end{cases}
\]

\[
\#\{i \leq \varphi(3^n)/2 : r_i \equiv 2 \pmod{4}\} = \begin{cases} 
\left\lfloor \frac{3^n}{12} \right\rfloor + 1 & \text{if } n \text{ is odd,} \\
\left\lfloor \frac{3^n}{12} \right\rfloor & \text{if } n \text{ is even,}
\end{cases}
\]

and

\[
\#\{i \leq \varphi(3^n)/2 : r_i \equiv 3 \pmod{4}\} = \begin{cases} 
\left\lfloor \frac{3^n}{12} \right\rfloor & \text{if } n \text{ is odd,} \\
\left\lfloor \frac{3^n}{12} \right\rfloor & \text{if } n \text{ is even.}
\end{cases}
\]

By Lemma 2.3, we have that

\[
\#\{i \leq \varphi(3^n)/2 : q_i = 0\} = \begin{cases} 
\left\lfloor \frac{3^n}{12} \right\rfloor & \text{if } n \text{ is odd,} \\
\left\lfloor \frac{3^n}{12} \right\rfloor + 1 & \text{if } n \text{ is even,}
\end{cases}
\]

\[
\#\{i \leq \varphi(3^n)/2 : q_i = 1\} = \begin{cases} 
\left\lfloor \frac{3^n}{12} \right\rfloor & \text{if } n \text{ is odd,} \\
\left\lfloor \frac{3^n}{12} \right\rfloor & \text{if } n \text{ is even,}
\end{cases}
\]
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\[
\#\{i \leq \varphi(3^n)/2 : q_i = 2\} = \begin{cases} 
\frac{3^n}{12} + 1 & \text{if } n \text{ is odd}, \\
\frac{3^n}{12} & \text{if } n \text{ is even},
\end{cases}
\]

and

\[
\#\{i \leq \varphi(3^n)/2 : q_i = 3\} = \begin{cases} 
\frac{3^n}{12} & \text{if } n \text{ is odd}, \\
\frac{3^n}{12} + 1 & \text{if } n \text{ is even}.
\end{cases}
\]

Since \(1 + (e^{\pi i/2}) + (e^{\pi i/2})^2 + (e^{\pi i/2})^3 = 0\), we thus have that

\[
\sum_{k=\frac{1}{2}(3^n+1)}^{\frac{3}{2}(3^n+3^n-1)} (e^{\pi i/2})^a_k = \sum_{j=0}^{3} \#\{i \leq \varphi(3^{n+1})/2 : q_i = j\} \cdot (e^{\pi i/2})^j
\]

\[
= \begin{cases} 
-1 & \text{if } n + 1 \text{ is odd}, \\
-i & \text{if } n + 1 \text{ is even},
\end{cases}
\]

\[
= -\begin{cases} 
i & \text{if } n \text{ is odd}, \\
1 & \text{if } n \text{ is even},
\end{cases}
\]

which proves part (i).

Part (ii) follows directly from Proposition 3.4 with \(b = 2\) and \(p = 3\).

\[\square\]

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**Appendix A. Transcendence of Stoneham numbers**

In this appendix, we give details of the transcendence of the Stoneham number \(\alpha_{b,c}\) for any choice of integers \(b, c \geq 2\). In fact, Mahler’s method gives much stronger results, which imply this desired conclusion.

We start out by letting \(c \geq 2\) be an integer and define

\[
F_c(x) := \sum_{n \geq 1} \frac{x^n}{c^n}.
\]

Notice that \(F_c(x)\) satisfies the Mahler functional equation

\[
F_c(x^c) = cF_c(x) - x^c. \tag{A.1}
\]
Now suppose that $F_c(x) \in \mathbb{C}(x)$. Then there are polynomials $a(x), b(x) \in \mathbb{C}[x]$ such that

$$F_c(x) - \frac{a(x)}{b(x)} = 0.$$ 

Since $F_c(x) \in \mathbb{C}[[x]]$ is not a polynomial, we may assume, without loss of generality, that $\gcd(a(x), b(x)) = 1$ and $b(0) \neq 0$ and $b(x) \not\in \mathbb{C}$. Sending $x \to x^c$ and applying the functional equation, we thus have that

$$F_c(x) - \frac{a(x)}{b(x)} = 0 = F_c(x^c) - \frac{a(x^c)}{b(x^c)} = F_c(x) - \left( \frac{x^c}{c} + a(x^c) \right),$$

so that

$$\frac{x^c}{c} + \frac{a(x^c)}{b(x^c)} = \frac{a(x)}{b(x)}.$$ \hspace{1cm} (A.2)

Now as functions, the right- and left-hand sides of (A.2) must have the same singularities. But $b(x^c)$ will have more zeros (counting multiplicity if needed) than $b(x)$ unless $b(x)$ is a constant, which is a contradiction. Thus $F_c(x)$ does not represent a rational function. In fact, we can now appeal to the following theorem, to give that $F_c(x)$ is transcendental over $\mathbb{C}(x)$.

**Theorem A.1 (Nishioka [6]).** Suppose that $F(x) \in \mathbb{C}[[x]]$ satisfies one of the following for an integer $d > 1$:

(i) $F(x^d) = \phi(x, F(x))$,

(ii) $F(x) = \phi(x, F(x^d))$,

where $\phi(x, u)$ is a rational function in $x, u$ over $\mathbb{C}$. If $F(x)$ is algebraic over $\mathbb{C}(x)$, then $F(x) \in \mathbb{C}(x)$.

To prove the transcendence of the Stoneham numbers, we appeal to a classical result of Mahler [8]. We record it here as taken from Nishioka’s monograph [7].

**Theorem A.2 (Mahler [8]).** Let $\mathbf{I}$ be the set of algebraic integers over $\mathbb{Q}$, $K$ be an algebraic number field, $\mathbf{I}_K = K \cap \mathbf{I}$, $f(x) \in K[[x]]$ with radius of convergence $R > 0$ satisfying the functional equation for an integer $d > 1$,

$$f(x^d) = \sum_{i=0}^{m} a_i(x) f(x)^i, \quad m < d, \quad a_i(x), b_i(x) \in \mathbf{I}_K[x],$$

and $\Delta(x) := \text{Res}(A, B)$ be the resultant of $A(u) = \sum_{i=0}^{m} a_i(x) u^i$ and $B(u) = \sum_{i=0}^{m} b_i(x) u^i$ as polynomials in $u$. If $f(x)$ is transcendental over $K(x)$ and $\xi$ is an algebraic number with $0 < |\xi| < \min\{1, R\}$ and $\Delta(\xi^p) \neq 0 (n \geq 0)$, then $f(\xi)$ is transcendental.

Since $F_c(x)$ is transcendental over $\mathbb{C}(x)$, $F_c(x)$ satisfies the functional equation (A.1), and $\text{Res}(cu - x^{c}, 1) \neq 0$ for all $x$, we have the following corollary to Mahler’s theorem.

**Corollary A.3.** Let $c \geq 2$ be an integer. The number $\sum_{n \geq 1} (1/c^n) \xi^{c^n}$ is transcendental for all algebraic numbers $\xi$ with $0 < |\xi| < 1$. In particular, for all $b, c \geq 2$, the Stoneham number $\alpha_{b,c}$ is transcendental.
References


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