In this paper a new translation plane of order 81 is constructed. Its collineation group is solvable and acts on the line at infinity as a permutation group $K$ which is the product of a group of order 5 belonging to the center of $K$ with a group of order 48. A 2-Sylow subgroup of $K$ is the direct product of a dihedral group of order 8 with a group of order 2. $K$ admits six orbits. They have lengths 4, 6, 12, 12, 24, 24.

1. Introduction

Let $GF(q)$ be a Galois field of odd order $q$ ($q > 5$) and let $F$ be the regular spread of $PG(3, q)$. Suppose that $F$ contains a set $R$ of reguli satisfying the properties

(i) $R$ consists of $(q+3)/2$ reguli, $R_1, R_2, \ldots, R_{(q+3)/2}$,

(ii) any two reguli of $R$ have exactly two lines in common,

(iii) no three reguli of $R$ have a line in common.
Let $U$ be the partial spread consisting of all the lines of the reguli $R_i$ ($i = 1, 2, \ldots, (q+3)/2$) and let $I_R$ denote the point set covered by $U$. If $R'_i$ denotes the opposite regulus of $R_i$, the set $U'$ of all the lines of $R'_i$ is not a spread of $I_R$ because any point $P$ of $I_R$ belongs to two lines of $U'$. According to a result of Bruen [4], if a subset $V$ of $U'$ is a partial spread then $V$ contains $(q + 1)/2$ lines of each $R'_i$ such that the union of these half-reguli is $V$. If there exists such a partial spread $V$ then one can obtain a new spread $G = (F - U) \cup V$ by replacing $U$ with $V$. $G$ will be called $\beta$-derived from $F$ (with respect to $(U, V)$). We can try to obtain a new partial spread $V = \frac{1}{2}U'$ of $I_R$ by choosing only a suitable half of the lines of each opposite regulus. As for $\beta$-derived spreads the existence problem is yet unsolved except for $q = 5, 7, 11, 13$. Specific examples are given in Bruen’s paper [4] when $q = 5, 7$, in [5] and [12] when $q = 11$ and in [13] when $q = 13$.

In [4] Bruen also proved that if $q \geq 7$ the collineation group of a $\beta$-derived spread is the inherited group, that is, it is the subgroup of the collineation group of $PG(3, q)$ which leaves $F$ invariant. The collineation groups of the translation planes arising from the above mentioned examples are determined [5], [12], [13].

In this paper the first example of a $\beta$-derived spread is constructed for $q = 9$ and the collineation group of the corresponding translation plane is determined. It is a solvable group satisfying the properties listed in the summary.

2. Notation and terminology

Notation and terminology are the same as in Bruen’s paper [4]. The reader is assumed to be familiar with the theory of projective planes given in Dembowski [6] and in Hughes and Piper [7] and with the theory of permutation groups given in Wielandt [14].

Let $GF(9)$ be the quadratic extension of $GF(3) = \{0, 1, 2\}$. In particular let us consider the polynomial $x^2 + x + 2$ (which is irreducible in $GF(3)$), $i \notin GF(3)$ such that $i^2 + i + 2 = 0$ and
$3 = i$, $4 = i + 1$, $5 = i + 2$, $6 = 2i$, $7 = 1 + 2i$, $8 = 2 + 2i$; so $\text{GF}(9) = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.

The homogeneous coordinates $(x, y, z, u)$ denote points in $\text{PG}(3, q)$. The line joining the point $(x_1, y_1, z_1, u_1)$ to the point $(x_2, y_2, z_2, u_2)$ is denoted by $\langle (x_1, y_1, z_1, u_1), (x_2, y_2, z_2, u_2) \rangle$; moreover if $g, h, \ldots, w$ are elements of a group, then the subgroup generated by $g, h, \ldots, w$ is denoted by $\langle g, h, \ldots, w \rangle$.

3. Preliminaries

Before the example, the general relation between the spread $F$ and the Miquelian inversive plane $M(q)$ over $\text{GF}(q)$ is discussed briefly.

Assume that $q$ is odd and denote by $s$ a non-square element of $\text{GF}(q)$, so $\text{GF}(q^2) = \{a + bt \mid a, b \in \text{GF}(q), t^2 = s\}$. We define norm $N(a+bt)$ of $a + bt$ of $\text{GF}(q^2)$, $N(a + bt) = a^2 - sb^2$. Clearly $N(a + bt) \in \text{GF}(q)$; moreover set $N(\rho) = \{a + bt \mid N(a + bt) = \rho\}$. $M(q)$ is considered the incidence structure whose points are the elements of $\text{GF}(q^2) \cup \{\infty\}$ and whose circles are the subsets of $\text{GF}(q^2) \cup \{\infty\}$ of the following types:

(I) $a + bt + N(\rho) = \{x + yt \mid N((a-x) + (b-y)t) = \rho\}$;

(II) $C(a + bt, c + dt) = \{(a + bt) + \lambda(c + dt) \mid \lambda \in \text{GF}(q) \cup \{\infty\}\}$.

The spread $F$ is the union of all the lines $\langle (1, 0, 0, 0), (0, 1, 0, 0) \rangle, \langle (a, sb, 0, l), (b, a, l, 0) \rangle$ where $a, b$ run over $\text{GF}(q)$. By a theorem of Bruck's ([3], also [4]), the map

$\psi : \infty \rightarrow \langle (1, 0, 0, 0), (0, 1, 0, 0) \rangle$

is an incidence preserving isomorphism between $M(q)$, with its points and circles, and $F$, with its lines and reguli.

By Bruck's theorem, the problem of finding sets $R$ of reguli satisfying the properties (i), (ii), (iii) is equivalent to that of constructing chains $C$ of circles in $M(q)$ satisfying the properties.
(i)' $C$ consists of $(q+3)/2$ circles, $C_1, C_2, \ldots, C_{(q+3)/2}$.

(ii)' any two circles of $C$ have exactly two points in common,

(iii)' no three circles of $C$ have a point in common.

Let $W$ be the collineation group of $\text{PG}(3, q)$ mapping $F$ onto itself. Every $w \in W$ determines a map $w'$ of $M(q)$ onto itself defined as follows: let $P$ be any point of $M(q)$ and let $r$ be the line of $F$ which corresponds to $P$ by $\Psi$; then $w'(P)$ is the point which corresponds to $w(r)$ by $\Psi$. Therefore a map $\Phi$ is defined putting $\Phi(w) = w'$. Of course $\Phi$ is a homomorphism of $W$ into $W'$, where $W'$ denotes the automorphism group of $M(q)$. The kernel of $\Phi$ is the group $M$ of all collineations of $\text{PG}(3, q)$ fixing $F$ linewise. $M$ is a cyclic group of order $q + 1$. Moreover $W \simeq \Gamma L(2, q^2)$ and $W' \simeq \text{PGL}(2, q^2)$.

Let $H$ be the subgroup of $W$ which leaves invariant a set $R$ of reguli satisfying the properties (i), (ii), (iii). If $H'$ denotes the subgroup of $W'$ which leaves invariant the chain $C$ of circles of $M(q)$ corresponding to $R$ by $\Psi$, then $\Phi(H) = H'$. Hence $H/M \simeq H'$.

4. Two partial spreads $U$ and $V$ of $\text{PG}(3, 9)$ covering the same point set

Using the notation of Bruen's paper, our construction can be described as follows.

If $s = 3$, the norm-classes of $\text{GF}(9)$ are

- $N(0) = \{0\}$,
- $N(1) = \{1, 2, 4+4t, 8+4t, 3+5t, 6+5t, 3+7t, 6+7t, 4+8t, 8+8t\}$,
- $N(2) = \{5, 7, 4+4t, 8+4t, 3+6t, 3+3t, 6+3t, 6+6t\}$,
- $N(3) = \{5+3t, 7+3t, 3+4t, 6+4t, 5t, 5+6t, 7+6t, 7t, 3+8t, 6+8t\}$,
- $N(4) = \{3+4t, 6+4t, 5t, 3+6t, 6+6t, 6+2t, 3t, 1+5t, 2+5t, 6t, 1+7t, 2+7t\}$,
- $N(5) = \{4, 8, 1+4t, 2+4t, 5+5t, 7+5t, 1+6t, 2+6t, 5+7t, 7+7t\}$,
- $N(6) = \{t, 2t, 4+4t, 8+4t, 1+4t, 2+4t, 4+6t, 8+6t, 2+8t\}$,
- $N(7) = \{3, 6, 1+4t, 2+4t, 1+2t, 2+2t, 5+4t, 1+6t, 5+8t, 7+8t\}$,
- $N(8) = \{5+4t, 7+5t, 1+4t, 2+4t, 4+5t, 8+5t, 4+7t, 8+7t, 8t\}$.

An easy calculation shows that the following circles of $M(q)$ form a chain $C$ satisfying the properties (i)', (ii)', (iii)':
The corresponding reguli $R_1, R_2, R_3, R_4, R_5, R_6$ contained in $F$ form a set $$ satisfying the properties (i), (ii), (iii). The line $$ is denoted by $r_\infty$ and the line $$ by $r(a, b)$, then

$$
R_1 = \{r(1, 0), r(2, 0), r(1, 1), r(2, 1), r(3, 2), \\
r(6, 2), r(5, 3), r(7, 3), r(5, 7), r(7, 7)\} ,
$$

$$
R_2 = \{r(1, 0), r(2, 0), r(3, 1), r(6, 1), r(1, 2), \\
r(2, 2), r(5, 5), r(7, 5), r(5, 6), r(7, 6)\} ,
$$

$$
R_3 = \{r(5, 0), r(7, 0), r(1, 1), r(2, 1), r(1, 4), \\
r(2, 4), r(5, 5), r(7, 5), r(1, 8), r(2, 8)\} ,
$$

$$
R_4 = \{r(5, 0), r(7, 0), r(1, 2), r(2, 2), r(4, 5), \\
r(8, 5), r(5, 7), r(7, 7), r(1, 8), r(2, 8)\} ,
$$

$$
R_5 = \{r(0, 0), r(3, 1), r(6, 2), r(7, 3), r(1, 4), \\
r(4, 5), r(5, 6), r(8, 7), r(2, 8), r_\infty\} ,
$$

$$
R_6 = \{r(0, 0), r(6, 1), r(3, 2), r(5, 3), r(2, 4), \\
r(8, 5), r(7, 6), r(4, 7), r(1, 8), r_\infty\} .
$$

Therefore

$$U = \{r(1, 0), r(2, 0), r(1, 1), r(2, 1), r(3, 2), \\
r(6, 2), r(5, 3), r(7, 3), r(5, 7), r(7, 7), r(3, 1), r(6, 1), r(1, 2), \\
r(2, 2), r(5, 5), r(7, 5), r(5, 6), r(7, 6), r(5, 0), r(7, 0), r(1, 4), \\
r(2, 4), r(4, 7), r(8, 7), r(4, 5), r(8, 5), \\
r(1, 8), r(2, 8), r(0, 0), r_\infty\} .
$$

Denote by $V$ the union of the following half-reguli

$$
1_{R_1} = \{r'(i, 1), r'(i, 2), r'(i, 3), r'(i, 4), r'(i, 5)\} : \\
1_{R_1} = \{(2, 3, 0, 1), (0, 1, 1, 0), (2, 0, 0, 1), (1, 1, 1, 0), \\
(6, 6, 0, 1), (2, 3, 1, 0), (7, 8, 0, 1), (3, 5, 1, 0), \\
(7, 7, 0, 1), (7, 5, 1, 0)\} ,
$$
\[ \mathcal{R}_2' = \{<(1, 0, 0, 1), (2, 2, 1, 0)>, \langle(1, 6, 0, 1), (0, 2, 1, 0)>, \\
\langle(3, 3, 0, 1), (1, 6, 1, 0)>, \langle(5, 5, 0, 1), (5, 7, 1, 0)>, \\
\langle(5, 4, 0, 1), (6, 7, 1, 0)>)\}, \]

\[ \mathcal{R}_3' = \{<(1, 1, 0, 1), (1, 2, 1, 0)>, \langle(1, 3, 0, 1), (4, 2, 1, 0)>, \\
\langle(7, 4, 0, 1), (0, 5, 1, 0)>, \langle(7, 0, 0, 1), (5, 5, 1, 0)>, \\
\langle(8, 8, 0, 1), (7, 4, 1, 0)>)\}, \]

\[ \mathcal{R}_4' = \{<(2, 6, 0, 1), (8, 1, 1, 0)>, \langle(2, 2, 0, 1), (2, 1, 0), \\
\langle(4, 4, 0, 1), (5, 8, 1, 0)>, \langle(5, 0, 0, 1), (7, 7, 1, 0)>, \\
\langle(5, 8, 0, 1), (0, 7, 1, 0)>)\}. \]

\[ \mathcal{R}_5' = \{<(4, 1, 0, 0), (0, 0, 1, 0)>, \langle(7, 1, 0, 0), (0, 0, 1, 1)>, \\
\langle(1, 0, 0, 0), (0, 0, 1, 2)>, \langle(8, 1, 0, 0), (0, 0, 1, 3)>, \\
\langle(5, 1, 0, 0), (0, 0, 1, 6)>)\}, \]

\[ \mathcal{R}_6' = \{<(0, 1, 0, 0), (0, 0, 1, 4)>, \langle(6, 1, 0, 0), (0, 0, 1, 5)>, \\
\langle(1, 1, 0, 0), (0, 0, 1, 7)>, \langle(3, 1, 0, 0), (0, 0, 1, 8)>, \\
\langle(2, 1, 0, 0), (0, 0, 0, 1)>)\}. \]

Note that \( \mathcal{R}_L' \) is contained in the opposite regulus \( R'_L \) of \( R_L \). An explicit calculation shows that \( U \) and \( V \) cover the same subset \( I_R \) of \( \text{PG}(3, 9) \) and have no lines in common. Therefore \( G = (F-U) \cup V \) is a spread of \( \text{PG}(3, 9) \). Clearly \( G \) is a \( \beta \)-derived spread from \( F \).

5. The automorphism group of the chain \( C \)

In this section the automorphism group \( H' \) of \( M(9) \) which leaves the point set \( I \), covered by \( C \), invariant is determined. As a model of \( M(9) \) we will take the geometry \( G(Q) \) of the plane sections of an elliptic quadric \( Q \) of \( \text{PG}(3, 9) \). The passage from \( M(9) \) to \( G(Q) \) can be realized as follows. Let \( Q \) be the elliptic quadric of \( \text{PG}(3, 9) \) whose equation is

\[ Q : z u = x^2 + 6y^2. \]

As it is well known, the map

\[ \sigma : \infty \to (0, 0, 1, 0) \]

\[ \sigma : a + bt \to [a, b, a^2 + 6b^2, 1] \]
is an incidence preserving isomorphism between $M(9)$ and $Q$ with their points and circles. Moreover (cf. [6], p. 274), if $\omega'$ is an automorphism of $M(9)$, then there is a unique collineation $\tilde{\omega}$ of $PG(3, 9)$ leaving $Q$ invariant such that $\omega'$ acts on $M(9)$ as $\tilde{\omega}$ on $Q$. Thus

$$\sigma(\omega'(P)) = \tilde{\omega}(\sigma(P)) \text{ for every } P \in M(9)$$

and

$$\sigma^{-1}(\tilde{\omega}(P)) = \omega'(\sigma^{-1}(P)) \text{ for every } P \in Q.$$

Therefore the problem of determining the automorphism group $H'$ of $M(9)$ which leaves $I$ invariant is equivalent to that of determining the collineation group $\overline{H}$ of $PG(3, 9)$ which maps $Q$ onto itself and leaves $\sigma(I)$ invariant, where $\sigma(I) = \{\sigma(P) \mid P \in I\}$. First some lemmas are needed.

**Lemma 5.1.** Let $D$ be a circle of $Q$ contained in $\sigma(I)$; then $D$ coincides with a circle $\sigma(C_i)$.

**Proof.** By way of contradiction, let $D$ be distinct from any circle $\sigma(C_i)$; then for every $i = 1, 2, \ldots, 6$ we have $|D \cap \sigma(C_i)| \leq 2$ and from this

$$\sum_{i=1}^{6} |D \cap \sigma(C_i)| \leq 2 \cdot 6 = 12.$$

On the other hand each point of $D$ lies exactly on two circles $\sigma(C_i)$, so

$$\sum_{i=1}^{6} |D \cap \sigma(C_i)| = 2 \cdot 10 = 20.$$

Therefore we would have $20 \leq 12$.

**Lemma 5.2.** Let $Z$ be a collineation group of $PG(3, 9)$ which maps $Q$ onto itself leaving $\sigma(I)$ invariant; then $Z$ preserves the chain $\sigma(C)$.

**Proof.** Let $\alpha \in Z$ and $\sigma(C_i) \in \sigma(C)$; as $\alpha(\sigma(C_i))$ is a circle contained in $\sigma(I)$, the preceding Lemma 5.1 assures that $\alpha(\sigma(C_i))$ is a circle of $\sigma(C)$. This proves Lemma 5.2.
Let \( \pi_i \) \((i = 1, 2, \ldots, 6)\) be the plane of \( \text{PG}(3, 9) \) which meets \( Q \) in \( \sigma(C_i) \). Denote by \( P_i \) the pole of \( \pi_i \) with respect to \( Q \). Then, by Lemma 5.2, \( \overline{H} \) is the collineation group of \( \text{PG}(3, 9) \) which maps \( Q \) onto itself and leaves the set \( \{P_1, P_2, P_3, P_4, P_5, P_6\} \) invariant. An easy calculation shows that \( P_1(0, 2, 2, 1), P_2(0, 1, 2, 1), P_3(0, 1, 1, 1), P_4(1, 2, 2, 1), P_5(1, 1, 0, 0), P_6(2, 1, 0, 0) \).

**Lemma 5.3.** \( \overline{H} \) fixes \( Y^\infty \).

Proof. The lines joining \( P_1 \) and \( P_2, P_3 \) and \( P_4, P_5 \) and \( P_6 \) pass through \( Y^\infty \); there exist only three planes joining \( Y^\infty \) and four points among \( P_i \) \((i = 1, 2, \ldots, 6)\). So \( \overline{H} \) must fix \( Y^\infty \).

Let \( \overline{H_i} \) be the stabilizer of \( P_i \). First we note that

**Lemma 5.4.** \( \overline{H_1} = \overline{H_2}, \overline{H_3} = \overline{H_4}, \overline{H_5} = \overline{H_6} \).

**Lemma 5.5.** The identity collineation \( \text{id} \) is the only collineation of \( \overline{H} \) fixing each \( P_i \) \((i = 1, 2, \ldots, 6)\).

Proof. The only non-identity collineation which leaves each \( P_i \) invariant has equations

\[
\rho x' = x^3, \quad \rho y' = y^3, \quad \rho z' = 2u^3, \quad \rho u' = 2z^3;
\]

but this collineation does not preserve \( Q \).

**Lemma 5.6.** \( \overline{H} \) acts faithfully on \( \{P_1, P_2, \ldots, P_6\} \).

**Lemma 5.7.** \( \overline{H_6} \) is a dihedral group of order 8. \( \overline{H_6} \) is generated by the following collineations:

\[\overline{h} : \rho x' = 5x, \quad \rho y' = 5y, \quad \rho z' = 2z, \quad \rho u' = u,\]
\[\overline{s} : \rho x' = x, \quad \rho y' = y, \quad \rho z' = u, \quad \rho u' = z.\]

\( \overline{h} \) and \( \overline{s} \) act on \( \{P_1, P_2, \ldots, P_6\} \) as follows:

\[\overline{h} : (P_1 P_3 P_2 P_4) (P_5) (P_6) \quad \text{and} \quad \overline{s} : (P_1 P_2) (P_3) (P_4) (P_5) (P_6).\]

Proof. By Lemmas 5.4 and 5.6 we have that \( \overline{H_6} \) is a subgroup of the
symmetric group $S_4$ on four objects. An easy calculation shows that $\bar{h}$ and $\bar{s}$ act on \{ $P_1, P_2, \ldots, P_6$ \} as said before and $\bar{h}^4 = s^2 = (\bar{h}s)^2 = 1$. So $\langle \bar{h}, \bar{s} \rangle$ is a dihedral group of order 8 contained in $\bar{H}_6$. As $\bar{H}_1 = \bar{H}_2$ we have $\bar{H}_1 \cap \bar{H}_6 = \bar{H}_2 \cap \bar{H}_6$ and so $\bar{H}_6$ cannot contain any element of order 3. Thus $|\bar{H}_6| \leq 8$ and therefore $\bar{H}_6 = \langle \bar{h}, \bar{s} \rangle$.

As $|\bar{H}_6| = 8$ and the orbit of $P_6$ has length at most 6, the following lemma holds.

**LEMMA 5.8.** $|\bar{H}| \leq 48$.

Let us consider the collineation

$$\bar{c} : \rho x' = 5z + 7u, \; \rho y' = 5y, \; \rho z' = x + z + u, \; \rho u' = x + 2z + 2u.$$  
An easy calculation shows

**LEMMA 5.9.** $\langle \bar{c} \rangle$ has order 6 and is contained in $\bar{H}$. $\bar{c}^3$ centralizes $\bar{H}_6$.

Next we prove

**LEMMA 5.10.** $\langle \bar{c} \rangle \cap \bar{H}_6 = \{1\}$.

**Proof.** By Lemma 5.9, $|\langle \bar{c} \rangle \cap \bar{H}_6| \leq 2$. If $|\langle \bar{c} \rangle \cap \bar{H}_6| = 2$, then $\bar{c}^3 \in \bar{H}_6$ and, by Lemma 5.7, $\bar{c}^3 = \bar{h}^2$, as $\bar{h}^2$ is the central involution of $\bar{H}_6$. But one can verify that $\bar{c}^3 \neq \bar{h}^2$.

**THEOREM 5.11.** $\bar{H} = \langle \bar{h}, \bar{s}, \bar{c} \rangle$, $|\bar{H}| = 48$ and a 2-Sylow subgroup of $\bar{H}$ is the direct product of a dihedral group of order 8 with a group of order 2.

**Proof.** By Lemmas 5.7, 5.10, $\langle \bar{h}, \bar{s}, \bar{c} \rangle$ is a subgroup of order at least 48 of $\bar{H}$. By Lemma 5.8, $|\bar{H}| \leq 48$, thus $\bar{H} = \langle \bar{h}, \bar{s}, \bar{c} \rangle$ and $|\bar{H}| = 48$. The latter assertion of the theorem follows from Lemmas 5.7 and 5.9.

6. The collineation group $H$ preserving $U$

Let us denote by $1$ the identity collineation of $PG(3, 9)$ and put
(6.1) \( m_\alpha : px' = \alpha x + y \), \( py' = 3x + ay \), \( pz' = \alpha z + 3y \), \( pu' = z + \alpha u \),
where \( \alpha \in \text{GF}(9) \). Then

**LEMMA 6.1.** The cyclic group of order 10,
\[ M = \{ m_\alpha \mid \alpha \in \text{GF}(9) \} \cup \{1\} , \]
fixes \( U \) linewise; moreover \( M = \{ m_3 \} \). \( M \) is the full collineation group of \( \text{PG}(3, 9) \) which fixes \( U \) linewise.

Lemmas 6.2 and 6.3 follow from Lemma 6.1.

**LEMMA 6.2.** \( M \) is the subgroup of \( H \) fixing each regulus \( R_i \).
Therefore \( H/M \) acts on \( \{ R_1, R_2, ..., R_6 \} \) faithfully.

**Proof.** Let \( \phi \) be the correspondence defined in Section 3. Then \( \Phi(H) = H \) and \( \Phi(H) \) acts on \( \{ R_1, R_2, ..., R_6 \} \) as \( H \) on \( \{ P_1, P_2, ..., P_6 \} \). By Lemma 6.6, \( H \) is faithful on \( \{ P_1, P_2, ..., P_6 \} \).
Thus \( \text{Ker} \phi \) is the subgroup of \( H \) fixing each \( R_i \). As we have shown in Section 3, \( \text{Ker} \phi = M \). This proves Lemma 6.2.

Since \( |H| = 48 \) and \( |M| = 10 \), from Lemma 6.2 it follows that

**LEMMA 6.3.** \( |H| = 480 \).

Let us consider the collineations

(6.2) \( h : px' = 5x, \ py' = 5y, \ pz' = z, \ pu' = u \),
(6.3) \( s : px' = z, \ py' = 6u, \ pz' = 6x, \ pu' = y \),
(6.4) \( c : px' = 5y + 7z, \ py' = 8x + 5u, \ pz' = 3x + 3u, \ pu' = 2y + 2z \).

It is easy to check that \( h, s, c \) have order 4, 2, 6 respectively and

\[
\begin{align*}
\{ h(r_\infty) &= r_\infty , \\
 h(r(a, b)) &= r(5a, 5b) ,
\}
\end{align*}
\]
\[
\begin{cases}
    s(r_\infty) = r(0, 0), \\
    s(r(0, 0)) = r_\infty, \\
    s(r(a, b)) = r(a(a^2 + 6b^2)^{-1}, b(a^2 + 6b^2)^{-1}) \text{ for } (a, b) \neq (0, 0), \\
    c(r_\infty) = r(7, 0), \\
    c(r(2, 0)) = r_\infty, \\
    c(r(a, b)) = r((7a^2 + 4b^2 + 5)((a + 1)^2 + 6b^2)^{-1}, 7b((a + 1)^2 + 6b^2)^{-1}) \text{ for } (a, b) \neq (2, 0).
\end{cases}
\]

Therefore

**Lemma 6.4.** \(\langle h, s, c, M \rangle\) is a subgroup of \(H\).

From (6.5), (6.6) and (6.7) we infer that \(h, s, c\) act on \(\{R_1, R_2, \ldots, R_6\}\) as follows:

\[(6.9)\quad h : (R_1 R_3 R_2 R_4)[R_5, R_6], \quad s : (R_1 R_2)[R_3, R_4, R_5, R_6], \quad c : (R_1 R_3 R_2 R_4).\]

Thus \(h, s, c\) act on \(\{R_1, R_2, \ldots, R_6\}\) as \(\bar{h}, \bar{s}, \bar{c}\) on \(\{P_1, P_2, \ldots, P_6\}\). As the subgroup \(\langle \bar{h}, \bar{s}, \bar{c} \rangle\) of \(S_6\) gives a faithful representation of \(\bar{H}\) on \(\{P_1, P_2, \ldots, P_6\}\), we get

\(\langle h, s, c, M \rangle / M = \langle \bar{h}, \bar{s}, \bar{c} \rangle\). Therefore from Lemma 6.3 it follows that

**Lemma 6.5.** \(H = \langle h, s, c, M \rangle\).

Next we prove

**Lemma 6.6.** \(\langle h, s, c \rangle \cap M = \{1\}\).

Proof. The orbit of \(X_\infty\) in \(\langle h, s, c \rangle\) consists of the points \(X_\infty, Y_\infty, Z_\infty, 0, (1, 0, 0, 1), (0, 1, 1, 0), (2, 0, 0, 1), (0, 2, 1, 0), (5, 0, 0, 1), (0, 5, 1, 0), (7, 0, 0, 1), (0, 7, 1, 0)\). The collineation \(m_\alpha\) maps \(X_\infty\) into \((\alpha, 3, 0, 0)\), so \(m_\alpha \notin \langle h, s, c \rangle\) if \(\alpha \neq 0\). Now let us consider \(m_0\): it maps \((1, 1, 0, 0)\) into \((4, 1, 0, 0)\); the orbit of \((1, 1, 0, 0)\) in \(\langle h, s, c \rangle\) consists of the points \((1, 1, 0, 0), (0, 0, 1, 1), (1, 1, 1, 1), (8, 1, 0, 0), (0, 0, 6, 1), (2, 2, 1, 1), (5, 5, 1, 1), (7, 7, 1, 1), (1, 6, 6, 1), (2, 3, 6, 1), (7, 4, 6, 1), (5, 8, 6, 1)\). Therefore \(m_0 \notin \langle h, s, c \rangle\).
From the preceding lemmas we obtain

**THEOREM 6.7.** The collineation group $H$ of $\text{PG}(3, 9)$ which maps $F$ into itself and leaves $U$ invariant is $\langle h, s, c \rangle \cdot M$. Moreover $\langle h, s, c \rangle \simeq \overline{H}$ (cf. Theorem 5.11) and $\langle h, s, c \rangle$ acts on $\{R_1, R_2, \ldots, R_6\}$ as $\overline{H}$ on $\{P_1, P_2, \ldots, P_6\}$.

7. The inherited group $K$

Bruen [4] proved that if $q \geq 7$ the collineation group $K$ preserving the partial spread $G = (F-U) \cup V$ is the inherited group, that is, the subgroup of $H$ fixing $G$. In our case $K$ is a proper subgroup of $H$. In fact one can check that $m_3$ does not preserve $V$. On the other hand

$h, s, c, m_3^2 \in K$, as

$h : \langle r'(1, 1)r'(3, 3)r'(2, 2)r'(4, 5) \rangle \langle r'(1, 2)r'(3, 4)r'(2, 1)r'(4, 4) \rangle \langle r'(1, 3)r'(3, 5)r'(2, 3)r'(4, 3) \rangle \langle r'(1, 4)r'(3, 2)r'(2, 5)r'(4, 1) \rangle \langle r'(1, 5)r'(3, 1)r'(2, 4)r'(4, 2) \rangle \langle r'(5, 1) \rangle \langle r'(5, 2) \rangle \langle r'(5, 3) \rangle

\langle r'(5, 4) \rangle \langle r'(5, 5) \rangle \langle r'(6, 1) \rangle \langle r'(6, 2) \rangle \langle r'(6, 3) \rangle \langle r'(6, 4) \rangle \langle r'(6, 5) \rangle$,

$s : \langle r'(1, 1)r'(2, 1) \rangle \langle r'(1, 2)r'(2, 2) \rangle \langle r'(1, 3)r'(2, 3) \rangle \langle r'(1, 4)r'(2, 4) \rangle \langle r'(1, 5)r'(2, 5) \rangle \langle r'(3, 1)r'(3, 2) \rangle \langle r'(3, 3)r'(3, 4) \rangle \langle r'(3, 5) \rangle \langle r'(4, 1)r'(4, 2) \rangle \langle r'(4, 3) \rangle

\langle r'(4, 4)r'(4, 5) \rangle \langle r'(5, 1)r'(5, 3) \rangle \langle r'(5, 2)r'(5, 4) \rangle \langle r'(5, 5) \rangle \langle r'(6, 1)r'(6, 5) \rangle \langle r'(6, 2) \rangle \langle r'(6, 3)r'(6, 4) \rangle$,

$c : \langle r'(1, 1)r'(6, 5)r'(3, 3)r'(2, 1)r'(5, 1)r'(4, 4) \rangle \langle r'(1, 2)r'(6, 1)r'(3, 4)r'(2, 2)r'(5, 3)r'(4, 5) \rangle \langle r'(1, 3)r'(6, 2)r'(3, 5)r'(2, 3)r'(5, 5)r'(4, 3) \rangle \langle r'(1, 4)r'(6, 4)r'(3, 2)r'(2, 4)r'(5, 2)r'(4, 2) \rangle

\langle r'(1, 5)r'(6, 3)r'(3, 1)r'(2, 5)r'(5, 4)r'(4, 1) \rangle$,

$m_3^2 : \langle r'(1, 1)r'(1, 3)r'(1, 2)r'(1, 4)r'(1, 5) \rangle \langle r'(2, 1)r'(2, 5)r'(2, 4)r'(2, 2)r'(2, 3) \rangle \langle r'(3, 1)r'(3, 3)r'(3, 5)r'(3, 4)r'(3, 2) \rangle \langle r'(4, 1)r'(4, 2)r'(4, 5)r'(4, 3)r'(4, 4) \rangle \langle r'(5, 1)r'(5, 5)r'(5, 3)r'(5, 2)r'(5, 4) \rangle

\langle r'(6, 1)r'(6, 2)r'(6, 5)r'(6, 3)r'(6, 4) \rangle$.
Therefore we can state the following:

**LEMMA 7.1.** \( \langle h, s, c \rangle \leq K \), \( m_3 \notin K \), \( m_3^2 \in K \).

Thus \( \langle h, s, c, m_3^2 \rangle \) is a subgroup of \( K \). Since \( K \cap M = \langle m_3^2 \rangle \), \( |\langle m_3^2 \rangle| = |M|/2 \), it follows that \( [H : K] = 2 \). By \( K \neq H \), \( K = \langle h, s, c, m_3^2 \rangle \). Moreover \( \langle m_3^2 \rangle \) is a normal subgroup of \( K \). So we have the following:

**THEOREM 7.2.** The collineation group \( K \) preserving the spread \( G = (F-U) \cup V \) is the semidirect product of a group of order 5 lying in the center of \( K \) with a group isomorphic to \( H \). Therefore \( K \) is a solvable group of order 24. Moreover the orbits of \( K \) are

\[
[r(3, 0)r(4, 0)r(6, 0)r(8, 0)],
\]
\[
[r(0, 1)r(0, 2)r(0, 3)r(0, 4)r(0, 5)r(0, 6)r(0, 7)r(0, 8)r(3, 3)
r(3, 5)r(3, 6)r(3, 7)r(4, 1)r(4, 2)r(4, 4)r(4, 8)r(6, 3)r(6, 5)
r(6, 6)r(6, 7)r(8, 1)r(8, 2)r(8, 4)r(8, 8)],
\]
\[
[r(1, 3)r(1, 5)r(1, 6)r(1, 7)r(2, 3)r(2, 5)r(2, 6)r(2, 7)r(3, 4)
r(3, 8)r(4, 3)r(4, 6)r(5, 1)r(5, 2)r(5, 4)r(5, 8)r(6, 4)r(6, 8)
r(7, 1)r(7, 2)r(7, 4)r(7, 8)r(8, 3)r(8, 6)],
\]
\[
[r'(1, 1)r'(1, 2)r'(2, 1)r'(2, 2)r'(3, 3)r'(3, 4)r'(4, 4)r'(4, 5)
r'(5, 1)r'(5, 3)r'(6, 6)r'(6, 5)],
\]
\[
[r'(1, 3)r'(2, 3)r'(3, 5)r'(4, 3)r'(5, 5)r'(6, 2)],
\]
\[
[r'(1, 4)r'(1, 5)r'(2, 4)r'(2, 5)r'(3, 1)r'(3, 2)r'(4, 1)r'(4, 2)
r'(5, 2)r'(5, 4)r'(6, 3)r'(6, 4)].
\]

8. The translation complement

As it is well known a translation plane \( \tau \) of order \( q^2 \) arises from any spread of \( PG(3, q) \) (cf. [6], p. 220 and [11] and [2]). Here we give an outline for the case \( q = 9 \).

The points of \( \tau \) are those of \( PG(4, 9) - PG(3, 9) \), the lines of \( \tau \) are the planes of \( PG(4, 9) \) which meet \( PG(3, 9) \) in a line of the spread and do not belong to \( PG(3, 9) \). If we consider \( PG(4, 9) - PG(3, 9) \) as a
vector space $V_h(9)$ of dimension 4 over $GF(9)$, the points of $\tau$ can be identified with the vectors of $V_h(9)$, the lines through the zero vector 0 are some 2-dimensional subspaces (called also components), the other lines of $\tau$ are translates of the components. The group of collineations of $\tau$ fixing 0 is called the "translation complement" and consists of some linear or semilinear transformations of $V_h(9)$. In particular such a transformation $\alpha$ of $V_h(9)$ belongs to the translation complement of $\tau$ if and only if the collineation of PG(3, 9) defined by the matrix of $\alpha$ leaves the spread invariant. Moreover if we denote the translation complement of $\tau$ by $\Lambda_0$, the cyclic group of order 8 of all dilations with center 0 by $\Delta_0$ and the collineation group of PG(3, 9) which leaves the spread invariant by $K$, then $K \cong \Lambda_0/\Delta_0$.

As $\tau$ is the translation plane arising from the spread $G = (F-U) \cup V$, by Theorem 7.2 we get

**THEOREM 8.1.** The translation complement $\Lambda_0$ of $\tau$ is $\langle \lambda, \sigma, \gamma, \mu \rangle$, where 
\[
\begin{align*}
\lambda : x' &= 5x, \quad y' = 5y, \quad z' = z, \quad u' = u, \\
\sigma : x' &= z, \quad y' = 6u, \quad z' = 6x, \quad u' = y, \\
\gamma : x' &= 5y + 7z, \quad y' = 8x + 4z, \quad z' = 3x + 3z, \quad u' = 2y + 2z, \\
\mu : x' &= 8x + y, \quad y' = 3x + 8y, \quad z' = 8a + 3u, \quad u' = z + 8u.
\end{align*}
\]

As $\langle \mu \rangle$ is a normal subgroup of $\Lambda_0$ and $\langle \lambda, \sigma, \gamma \rangle \cap \langle \mu \rangle = \{1\}$, $\Lambda_0$ is semidirect product of $\langle \lambda, \sigma, \gamma \rangle$ and $\langle \mu \rangle$. Therefore $\Lambda_0$ is a solvable group of order $240 \cdot 8 = 1920$.

**References**

Translational plane of order 81


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