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On lattices acting on boundaries of semi-simple groups

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Abstract. It is shown that for a lattice Γ in a semi-simple group of real rank 1 the action on the boundary always admits an equivariant topological Γ-factor. We also show that there are no such factors for SL $(n, \mathbb{Z})$ acting on $\mathbb{P}^{n-1}$, $n \geq 3$.

Introduction

In his proof of the finiteness theorem in [**3**] and [**4**] Margulis classifies all measurable Γ-quotients of the maximal boundary G/P for an irreducible lattice Γ in a semi-simple Lie group G (finite centre, no compact factors) of real rank at least two. In fact, he shows that they are all of the form G/P' for some parabolic subgroup P'. At the same time his method allows one to 'construct' non-trivial SL $(2, \mathbb{Z})$-quotients of S^1 in the measurable sense. In fact, his method generalizes to some other rank 1 lattices.

At the end of his paper Margulis asks whether one could have topological (Hausdorff) quotients for SL $(n, \mathbb{Z})$ acting on $\mathbb{P}^{n-1}$, $n \geq 2$. R. Zimmer proved in [**7**] that, for $n > 2$, any such quotient is trivial. Here we first propose a geometrical method to construct factors of the boundary for any lattice in a rank 1 group. Measure-theoretically though, these quotients will be trivial. Then we present another argument for the triviality for $n \geq 2$. For general Γ and G the question is still open. Our argument might essentially carry over to the case of split lattices.

I. *The rank* 1 *case*

To fix notation, let G be a connected simple Lie group without compact factors of real rank 1. Let Γ be a lattice in G. H will denote the globally symmetric space G/K for K a maximal compact subgroup of G. B will be the boundary of H. We shall use the geometric interpretation of [**2**] for B, i.e. B is the set of equivalence classes of asymptotic geodesics. Finally, M will be the locally symmetric space $\Gamma\backslash G/K$.

We start with a closed geodesic $\bar{\alpha}$ in M. Pick a covering geodesic α in H and an axial isometry γ for α, i.e. γ translates α into itself (cf. [**2**, § 6]). We identify a geodesic with its pair of endpoints in B. Let (x, y) be the endpoints of α. Suppose that $\gamma_n(x, y)$ converges to a pair of distinct points (s, t) for some γ_n in Γ (in the cone topology) (cf. [**2**, § 2]) and let β be the geodesic joining s to t. Then any point on β is a limit point of points on the $\gamma_n\alpha$'s. Since $\bar{\alpha}$ is closed, its pre-image in H

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is closed and hence $\beta \in \Gamma\alpha$. Since Γ is countable, there are at least two points on β that lie on the same $\delta\alpha$ for some δ in Γ. By uniqueness of geodesics in negative curvature,

$$\beta = \delta\alpha.$$

We summarize our discussion in:

LEMMA 1. *The orbit* $\Gamma(x, y)$ *is closed in* $B \times B$*-diagonal.*

Consider the equivalence relation $\sim$ generated by $A = \Gamma(x, y)$: i.e. if (s, t) is in A then so is (t, s), and so on. Let Γ_t denote the isotropy subgroup of t in Γ.

LEMMA 2. *If* $y \notin \Gamma x$ *and* $\Gamma_x = \Gamma_y$ *then* $a \sim b$ *iff either* $a = b$ *or* $(a, b) \in A$ *or* $(b, a) \in A$.

Proof. If $a \neq b$ and $a \sim b$ then a and b lie in the Γ-orbit of either x or y. Hence we assume that $a = x$. Now $x \sim b$ iff there exists a chain $z_1, \ldots, z_n$ such that

$$(x, z_1) \in A \quad \text{or} \quad (z_1, x) \in A \quad \text{and} \quad z_i \sim z_{i+1}.$$

If $(z_1, x) \in A$ then $x \in \Gamma y$, in contradiction to the assumptions. If $(x, z_1) \in A$ then there is a $\delta \in \Gamma$ s.t.

$$\delta(x, y) = (x, z_1).$$

By the assumption on the isotropy groups

$$y = z_1.$$

The same reasoning applies to y, and the lemma is clear. □

Suppose for the moment that α satisfies the assumptions of lemma 2. Then the quotient space of B under $\sim$ is Hausdorff by the two lemmas, and clearly the action of Γ on B factors through $\sim$.

We have to see that the new action is not equivalent to the old one. The axial isometry γ fixes two points on B which are identified under $\sim$. Moreover, there are no new fixed points for γ, since otherwise there is an $s \in B$ such that

$$(s, \gamma s) = \delta(x, y) \quad \text{or} \quad (\gamma s, s) = \delta(x, y)$$

for some $\delta \in \Gamma$. In either case this implies that $y \in \Gamma x$.

To verify the assumptions in lemma 2 first assume that Γ does not have torsion. [**2**, prop. 6.8] shows the equality of the isotropy groups (even if there is torsion). If $y = \delta x$ then $\delta\gamma\delta^{-1}$ has the geodesic through $(y, \delta y)$ as axis. On the other hand, γ fixes y and hence δy, by [**2**, prop. 6.8]. Since every non-elliptic isometry of H has at most two fixed points in B [**2**, prop. 6.5], δ permutes x and y. If δ is non-elliptic, it has fixed points on the boundary, by the Brouwer fixed point theorem. So δ^2 has at least three fixed points in B. In any case, δ^2 is elliptic and hence is a torsion element, since an elliptic isometry in a lattice has finite order.

If Γ has torsion not every axis α will satisfy the first assumption of lemma 2. But most α will do.

Consider a fundamental region F for Γ in H. The previous argument shows that, if $\delta x = y$, then δ^2 is elliptic. Hence δ is a torsion element. In particular, δ^2 will fix α pointwise. Since δ is orientation preserving, δ fixes α pointwise. By a conjugation we may assume that α goes through F^-. Since F is a fundamental domain, α cannot pass through the interior of F. Therefore, an axis α that passes through the interior of F satisfies the assumptions of lemma 2. Since the axes are dense in the geodesics [**6**, lemma 8.3′], such an axis will always exist. We obtain the following:

PROPOSITION 1. *Any lattice Γ in a non-compact real rank 1 connected semi-simple Lie group G with finite centre has a non-trivial (Hausdorff) quotient of the action on the boundary.*

Proof. Consider the projection $p: G \to G'$, where G' does not have compact factors. Then $p(\Gamma)$ is a lattice in G'. So we can apply the construction above. □

PROPOSITION 1′. *Let M be a manifold of non-positive curvature whose universal cover H satisfies: any two boundary points are joined by a unique geodesic. Then there always exists a non-trivial (Hausdorff) quotient of the action of $\pi_1(M)$ on the boundary.*

Proof. In the construction above we only used [**2**, props. 6.5, 6.8] which hold for $\pi_1(M)$ for M satisfying our conditions (cf. [**2**]). Since $\pi_1(M)$ does not have torsion (cf. [**5**, cor. 19.3]), any closed geodesic gives rise to a quotient. □

II. SL $(n, \mathbb{Z})$ *acting on* $\mathbb{P}^{n-1}$

Let $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ for short.

PROPOSITION 2. *All (Hausdorff) quotients of Γ acting on $\mathbb{P}^{n-1}$, $n > 2$, are trivial.*

We first observe:

LEMMA 3. *Let a group Γ act on a compact Hausdorff space M. If the diagonal action of Γ on $M \times M$-diagonal is minimal, then all equivariant Γ-quotients of M are trivial. (Recall that an action of a group is called minimal if every orbit is dense.)*

Proof. Let Y be such a quotient and let $C \subset C(M)$ = continuous functions on M be the pullback of $C(Y)$ to M. Then C is Γ-invariant. If $C \neq$ constants, pick $f \in C$ such that

$$f(x_0) \neq f(y_0)$$

for some x_0, y_0 in M. Pick neighbourhoods U, V of x_0, y_0 such that $f(U)$ is disjoint from $f(V)$. If $x_1 \neq y_1$ is any other pair of points in M, there exists a $\gamma \in \Gamma$ such that

$$\gamma x_1 \in U \quad \text{and} \quad \gamma y_1 \in V.$$

Hence

$$f(x_1) \neq f(y_1)$$

and C separates points. By Stone–Weierstrass

$$C = C(M).$$

Gelfand duality yields

$$Y = M.$$ □

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Proposition 2 follows from the stronger:

PROPOSITION 3. Γ *acts minimally on* $\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}$*-diagonal.*

Proof. First, on the level of $\mathbb{P}^{n-1}$ itself, we have:

LEMMA 4. Γ *acts minimally on* $\mathbb{P}^{n-1}$, $n>1$.

Note. This is completely general: i.e. any lattice Γ in a semi-simple Lie group G without compact factors acts minimally on G/P, P any parabolic (cf. [**6**, lemma 8.5]). Of course, the case at hand is standard and follows from elementary arguments.

Now the proof of proposition 3 develops in two stages. For notation let $\bar{x}$ be the line through x for any $x\in\mathbb{R}^n$.

(1) Let e_i be the standard basis of $\mathbb{R}^n$. Let $x\in\mathbb{R}^n$, $\bar{x}\neq\bar{e}_1$. We claim that $\Gamma(\bar{e}_1,\bar{x})$ is dense in $\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}$-diagonal. The stabilizer subgroup Γ_0 of Γ at $\bar{e}_1$ looks like

$$\begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \cdot & & \cdots & \cdot \\ 0 & * & \cdots & * \end{pmatrix}.$$

In particular, embed SL $(n-1,\mathbb{Z})$ into Γ_0 in the obvious way. Clearly, it suffices to prove that $\Gamma(\bar{e}_1,\bar{y})$ is dense in $\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}$-diagonal for any $\bar{y}$ in the closure of $\Gamma_0(\bar{x})$. By lemma 4 and the above, we may assume that the coordinates $x_2,\dots,x_n$ of x are linearly independent over $\mathbb{Q}$. Let $\bar{y}\neq\bar{z}$ be two lines and V, W neighbourhoods of them. By lemma 4 there is a $\gamma\in\Gamma$ such that $\gamma(\bar{e}_1)$ is in W. Hence it suffices to find γ_0 such that $\gamma_0(\bar{x})\in\gamma^{-1}(V)$. Let $\bar{t}=\gamma^{-1}(\bar{y})$. We can find $\gamma_1\in\mathrm{SL}\,(n-1,\mathbb{Z})$ such that

$$\gamma_1\overline{(0,x_2,\dots,x_n)}$$

is close to

$$\overline{(0,t_2,\dots,t_n)}$$

by lemma 4. Let $x_2',\dots,x_n'$ be coordinates of

$$\gamma_1\overline{(0,x_2,\dots,x_n)}$$

and pick coordinates t_i for $\bar{t}$ such that t_i is close to x_i' for $i>1$. The $x_2',\dots,x_n'$ are clearly linearly independent over $\mathbb{Q}$. The group generated by them is dense in $\mathbb{R}$ and we can find

$$\gamma_2=\begin{pmatrix} 1 & m_2 & \cdots & m_n \\ 0 & & & \\ \vdots & & \mathrm{id} & \\ 0 & & & \end{pmatrix}\in\Gamma_0$$

such that $x_1'+m_2x_2'+\cdots m_nx_n'$ is close to t_1. Since γ_2 leaves the other coordinates alone we have finished.

(2) Consider any two lines $\bar{y}\neq\bar{z}$. We claim that the closure of their Γ-orbit contains $(\bar{x},\bar{e}_1)$ or $(\bar{e}_1,\bar{x})$. We consider two cases:

(a) $\bar{z}$ *is rational.* Then we have the well-known result:

LEMMA 5. $\Gamma\bar{e}_1$ = *rational lines* (*i.e. all coprime n-tuples of integers lie in* $\Gamma(1, 0, \ldots)$).

Proof. For $n = 2$ this is clear. For $n > 2$ let $(m_1, \ldots, m_n)$ be a point on a given line l with integer entries. Then l lies in the plane spanned by

$$(m_1, \ldots, m_{n-1}, 0) \quad \text{and} \quad \bar{e}_n.$$

By induction, pick $\gamma \in \mathrm{SL}(n-1, \mathbb{Z})$ such that

$$\gamma\bar{e}_1 = (m_1, \ldots, m_{n-1}, 0).$$

Then $\gamma^{-1}(l)$ lies in the plane spanned by $\bar{e}_1$ and $\bar{e}_n$ and we can use the case $n = 2$. □

(*b*) $\bar{z}$ *is irrational.* Then there are i, j such that z_i and z_j are rationally independent, say $i = 2$, $j = 3$. In particular, $\mathbb{Z}z_2 + \mathbb{Z}z_3$ is dense in $\mathbb{R}$. Hence there are matrices

$$\gamma_n = \begin{pmatrix} 1\, m_2^n m_3^n 0 & \cdots & 0 \\ 0 & & \\ \vdots & \mathrm{id} & \\ 0 & & \end{pmatrix} \in \Gamma$$

such that

$$\gamma_n z \to (0, z_2, \ldots, z_n)$$

as $n \to \infty$ while

$$\gamma_n y = (1 + m_2^n y_2 + m_3^n y_3, y_2, y_3, \ldots).$$

If $\overline{(z_2, z_3)} \neq \overline{(y_2, y_3)}$, then

$$\frac{m_2^n y_2 + m_3^n y_3}{m_2^n z_2 + m_3^n z_3} = \frac{y_2}{z_2} + m_3^n \frac{y_3 - z_3 y_2 z_2^{-1}}{m_2^n z_2 + m_3^n z_3} \to \pm\infty,$$

since the denominator stays bounded and w.l.o.g.

$$|m_3^n| \to +\infty,$$

unless the slopes and so the lines are the same. We find that

$$\gamma_n \bar{y} \to \overline{(1, 0, \ldots)}$$

and we have finished.

If $\overline{(z_2, z_3)} = \overline{(y_2, y_3)}$, we can still pick γ_n as above. Let

$$a(z_2, z_3) = (y_2, y_3).$$

Then

$$\gamma_n y \to (y_1 - az_1, y_2, \ldots).$$

If

$$\overline{(z_1, z_2, z_3)} \neq \overline{(y_1, y_2, y_3)}$$

then

$$y_1' \overset{\text{def}}{=} y_1 - az_1 \neq 0.$$

Notice that y_2 and y_3 are rationally independent, so one of (y_1', y_2) or (y_3', y_3) is rationally independent, say the first. Since

$$\overline{(y_1', y_2)} \neq \overline{(0, z_2)}$$

we can apply the previous argument to (y_1', y_2). Instead of z_1 we could have used any z_i, $i > 3$. We are left with the case

$$\overline{(z_2, z_3, z_i)} = \overline{(y_2, y_3, y_i)}$$

for all i, i.e.

$$\bar{z} = \bar{y}.$$

This is the final contradiction. □

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