SIMPLICITY AND TRUTHFULNESS IN ARITHMETIC

BY W. HOPE-JONES

Presidential Address to the Mathematical Association, January 1939

I am no artist, but I believe my artistic friends when they tell me that among the first requisites for good art are simplicity and truthfulness. Architecture is also outside my range; but I have architectural friends too, and they tell me of the need for simplicity and truthfulness in Architecture. After taking my degree in Mathematics I went for nearly a year to a Theological College; and I very soon found out for myself what were the qualities which a Theological College most urgently requires.

"And the moral of that", said the Duchess, "is 'Be what you would seem to be'—or if you'd like it put more simply, 'Never imagine yourself not to be otherwise than what it might appear to others that what you were or might have been was not otherwise than what you had been would have appeared to them to be otherwise'."

To those of you who deal in more advanced branches of Mathematics than come my way, I commend as a suggestion worth thinking over, "Could your teaching be improved by the use of simpler words without sacrifice of truthfulness?" But I have nothing to say to-day about advanced Mathematics: I believe that a child brought up to love simplicity and truthfulness in Arithmetic (assuming one to exist) will naturally import these same qualities into higher work when the time comes for that.

"CHECKABILITY"

Now, to begin with, a great deal of very obscure and disgusting work that is shown up to me is defended by its authors on the ground that "it gets the answer right". Pass over for the time the plain fact that excessively often it gets the answer wrong, and consider the question whether the whole object of Mathematics is to get the answer right. How many ways there are of getting the answer right! Looking it out in the crib is the quickest: another way is to sit next to a brainy boy and keep your dividers well sharpened. But for practical purposes, not only at school but in after-life, it is so often necessary to convince somebody else that your answer is right. If you are my fishmonger or greengrocer, you must obtain my agreement to your multiplication and addition before I will pay your bill. Even an Income-tax Collector presents his actual calculations in a form in which you can easily check them,
though he has you on the obscurity of his data, in which simplicity and truthfulness are not the most conspicuous features, and which are supported by an authority which has little in common with the "appeal to reason" which is the guiding star of all you good teachers of Mathematics. I take it then that in Arithmetic our pupils should learn to get answers right by working which will convince a reasonably intelligent reader that they are right. And as one of the first tests of a well-done piece of work I put forward this: "A well-done piece of work is easier to check by reading it than by doing it again from the beginning." I am sorry that there is no good English word to denote the facility with which written work can be checked by reading it: in the absence of that I shall have to call it "checkability"; and I put it to you that not only would our pupils receive a better education but also we ourselves would live more tolerable lives if we would insist on this "checkability" test: a well-done piece of work is easier to check by reading it than by doing it again from the beginning.

A book which came to my notice lately printed as the complete solution of an arithmetical problem:

7s.
£147 12s. 6d. + £13 12s. 0d. + 10s. 6d. - £17 10s. 0d.
-£101 3s. 6d. - £2 2s. 0d.,
in which it will be observed that the dead figures outnumber the living by twenty to eight. This was described by the author as "a reasoned form", but to me it seems to fall somewhat below the standard of checkability which I require in work that I have to look over.

Cancelling

It was the difficulty of checking work in which figures are crossed out that first led me to call in question the utility and necessity of the practice called "cancelling". The only defence of it which I hear when I discuss the practice with those who are addicted to it is that "it is impossible to get on without it". Now this is the same defence that you will get if you make a similar criticism of those who find it necessary to consume a large cocktail before, during, after, between, instead of and independently of every meal: "it is impossible to get on without it." But it is probably the experience of many of you that life without so much refreshment is not completely unliveable. And since I gave up cancelling nearly thirty years ago, I have never felt the need for it. My hair has not fallen out, nor my feet become a wilderness of bunions.

Boys insist that cancelling is necessary for a particular question; and when I show them how easy it is without it, they shift their ground and say, "Ah, but you might have another question just a
little different, in which you really might have to cancel.” Well, I suppose you might have a situation in which you might have to shoot your grandmother: yet many of us go on year after year without ever meeting it. So I go on year after year, hearing about the question in which I will have to cancel, yet never quite meeting it. Very occasionally I meet a question, generally one specially designed for the purpose, in which my preference for clean working costs me an extra five or ten seconds; but that is a price which I willingly pay for the many minutes which I gain every year by being able to check my own work—to which I would like to add the many hours which I would save if somebody in Europe would invent and send me a boy who believes in doing good, clean, checkable working. (And my reason for adding “in Europe” is that such a boy has actually been invented, and later I shall tell you something of him; but unfortunately he is a very long way off, and the cost of importing him in large quantities is prohibitive.)

In fractions like

\[
\frac{44C_6}{6!} = \frac{44 \cdot 43 \cdot 42 \cdot 41 \cdot 40 \cdot 39}{6!},
\]

a mass of cancelling involves great risk of error, as well as making the work uncheckable. A clear arrangement for this is to keep a score-sheet of the occurrence of the various prime factors involved, thus:

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>11</th>
<th>13</th>
<th>41</th>
<th>43</th>
</tr>
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<tbody>
<tr>
<td>44C_6</td>
<td>2^1 \cdot 7 \cdot 11 \cdot 13 \cdot 41 \cdot 43</td>
<td>\begin{align*} 44C_6 &amp;= 2^1 \cdot 7 \cdot 11 \cdot 13 \cdot 41 \cdot 43 \ &amp;= 4 \cdot 1001 \cdot 1763 \ &amp;= 7052 \ &amp;= 7059052 \end{align*}</td>
<td></td>
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These criticisms of “cancelling” have been based chiefly on the unpleasantness and obscurity of the working, even if the answer is right; but it ought also to be noticed that cancelling is the most frequent cause of wrong answers, first because figures are liable to be lost in the mud, and worse still, because the natural boy, who at eight years old understands the difference between subtracting and dividing, has so often had the distinction completely obliterated in his mind by the age of thirteen, when he has learnt to talk of both indiscriminately by the name of “cancelling”. (“Oh sir,
Mr. X taught me that when it cancels out I ought to put down 1, not 0; and now you say that it ought to be 0.

THE IMPROPER FRACTIONS SUPERSTITION

Now the main burden of all that I have to say to you is this—that boys have not enough ingenuity to invent their own bad methods of doing their working, but that they pick them up from those who have charge of the early stages of their education. By the time they reach a Public School they are mostly irretrievably ruined already, because their conservatism is such that to change a method taught them at their Preparatory Schools is abhorrent to their nature, especially if it has been taught them by a master known to be a good man because he can imitate a pig or has hit a ball through an unpopular neighbour's window. It is too much to ask that such men should learn or teach methods of making difficult questions easy; but could they somehow be restrained from concentrating the child's entire education upon dodges for making easy questions difficult? Unfortunately these good men are often not members of our Association; but you are, and if I can't appeal to the culprits themselves, you are the missionaries who must go among them with your hymn-books and harmoniums and convert them to the cause of clean and checkable working.

No boy, unaided by the perverted ingenuity of some older and more decayed brain, could ever have invented the Improper Fractions Superstition. The Improper Fractions Superstition is this: that whenever you see a number followed by a fraction, such as $4\frac{1}{3}$, you multiply the denominator by the whole number, add in the numerator, and "put it over" the denominator. "Put it over"—an intolerable form of words: there are few if any that do more harm. Boys arrive at Eton cram-full of rules about putting something over something else: you find the speed by putting the distance over the time, or the time over the distance—it is an even chance which you do, and by taking turn and turn about you can ensure a decent proportion of successes.

But to return to the Improper Fractions Superstition. If you look back into your own past, you will recognise that very early in life you understood "four"—unless you are the kind of philosopher who likes proving that nobody ever understands anything. Later you learnt to understand "an eighth", probably before you understood $3\frac{1}{8}$, and certainly long before you understood $33\frac{1}{8}$.

After that $4\frac{1}{3}$ is an easy idea: you can eat four biscuits and an eighth of another; you can walk four miles and an eighth of another; round a circle is three whole diameters and nearly a seventh of another, not a seventh of twenty-two diameters. The natural and untutored boy thinks in terms of wholes and bits in preference to
improper fractions. And there the natural and untutored boy is quite right, bless his soul; but this sort of thing can’t be allowed to last: so presently he is caught and put through the sacred mysteries of the Improper Fractions Superstition; and by the time he comes to Eton, and the first question I ask him is “Can you prove anything about the circumference of a circle?”, up goes his hand and he crows in triumph “22 over 7”, or some variation on it such as “22πr²”.

I am not talking now of stupid boys: for many years now I have had the top Division of Fourth Form, and this is how nearly all of them would multiply 2½ by 3½, correct to three figures:

\[
\begin{align*}
2.6 \times 3\frac{1}{2} &= \frac{13}{5} \times \frac{7}{2} = \frac{28}{5} \\
8.171 \quad (\text{the extra figure superstition}) &= 8.17 \\
* &= 35 \quad 286 \\
280 &= 286 \\
60 &= 26 \\
35 &= 26 \\
250 &= 26 \\
245 &= 26 \\
50 &= 26
\end{align*}
\]

I will not insult you by supposing that you don’t know a better way than this of doing it; but I do want to call your attention to this important fact: the right method is based on the conviction that 3 means three, \(\frac{1}{7}\) means a seventh, and \(3\frac{1}{7}\) means just what it says, three and then another seventh.

\[
\begin{align*}
2.6 &= 3.37\ldots (\text{because } 50 \text{ is nearer to } 49 \text{ than to } 56) \\
3\frac{1}{7} &= 2.6 \\
7.8 &= 3.37\ldots
\end{align*}
\]

Now there is no point in jeering at the natural mistakes that boys of thirteen and fourteen make; but that is just what I am not doing. These are not the natural mistakes of the young: they are absolutely unnatural, and no boy would ever think of such artificial methods by himself. There is somebody about who is teaching boys this stuff, and will continue to do so unless you can catch him and send him to a reformatory.

All boys know that a fraction remains unchanged by dividing top and bottom by the same number (though some of them won’t recognise this wording of the rule, but will ask if you mean “cancelling”);

*This “=” sign is a particular enemy of mine.*
but I find that most new boys are shocked and a little incredulous at the news that a fraction remains the same size if you multiply top and bottom by the same number. So if you ask them to simplify \( \frac{2\frac{1}{2}}{8\frac{1}{3}} \), instead of multiplying top and bottom by 6,

\[
\left( \frac{2\frac{1}{2}}{8\frac{1}{3}} = \frac{15}{50} = \frac{3}{10} \right),
\]

practically every boy falls back on the Improper Fractions Superstition; and probably most will cancel into the bargain.

\[
\frac{2\frac{1}{2}}{8\frac{1}{3}} = \frac{5}{2} \div \frac{25}{3} = \frac{5}{2} \times \frac{3}{25} = \frac{3}{10}.
\]

Even such an extreme instance as this is not uncommon:

\[
5\frac{1}{6} \times 7 = \frac{46}{9} \times \frac{7}{1} = \frac{322}{9} = 35\frac{7}{9}.
\]

In doing Compound Interest, it is rare to meet a boy, or a book, that will treat \( \frac{4}{3} \) as four and a quarter:

\[
\begin{align*}
\text{C.I. on £713} & \quad \text{at } 4\frac{1}{4}\% \\
+ 28.52 & \quad (1.7825) \quad I_1 \\
\frac{743.3025}{P_2}
\end{align*}
\]

Nearly all of them treat it as \( \frac{17}{4} \), doing a long-multiplication sum if they don’t know their 17 times table:

\[
\begin{array}{c|c|c}
\text{£713} & \text{P}_1 & 713 \\
121.21 & 30.3025 & I_1 \\
743.3025 & P_2 & 4991 \\
\hline & & 12121
\end{array}
\]

and some will cross out the 121.21 before they write the 30.3025, presumably on the theory that a number is easier to divide when it has first been slaughtered.

Before leaving the subjects of cancelling and the Improper Fractions Superstition, I would have you notice that they are accomplices and form what in modern politics we call “an axis”. It is the Improper Fractions Superstition that provides most of the cannon-fodder for the canceller: to get rid of war, get rid of the causes of war; and to get rid of the slaughter of innocent figures, as of our innocent fellow-creatures, the great requirement is that we should “live in the virtue of that life and power that takes away the occasion of all cancelling”: it was “wars” in the original,
but in Arithmetic we may well interpret that as cancelling, remembering that cancelling is the arithmetical equivalent of war.

If in $4\frac{1}{2}$ minutes a train goes $3\frac{1}{2}$ miles, how far will it go in $58\frac{1}{2}$ minutes?

In $4\frac{1}{2}$ minutes it goes $3\frac{1}{2}$ miles;

\[ \therefore \frac{3\frac{1}{2}}{4\frac{1}{2}} = \frac{7}{9} \text{ miles}; \]

\[ \therefore \frac{7}{9} \text{ of } 58\frac{1}{2} = \frac{7}{9} \times 6\frac{1}{2} = 45\frac{1}{2} \text{ miles}. \]

Here we are following the simple principle, "Take easy simplifications in your stride". But because this method gives nothing to cancel, many boys are taught to do this:

\[ \text{In } \frac{9}{2} \text{ minutes it goes } \frac{7}{2} \text{ miles; } \]

\[ \therefore \frac{7}{2} \times \frac{2}{9} = \frac{13}{9} \]

\[ \therefore \frac{13}{2} \text{ miles,} \]

which is absurdly indirect. The whole process suggests the mentality of those who, when urged for hygienic reasons to "kill that fly", proceed to breed innumerable bluebottles so as to have plenty to kill. But I would have you work for something better than a world full of dead bluebottles.

But there is one even worse vice at which I must have a tilt before I leave this question.

\[ \text{In } \frac{9}{2} \text{ minutes it goes } \frac{7}{2} \text{ miles; } \]

\[ \therefore \frac{7}{2} \times \frac{2}{9} \times \frac{117}{2} = \frac{91}{2} = 45\frac{1}{2} \text{ miles,} \]

again, you merely multiply what is written above).

The second line is now as direct a lie as it is possible for a child to tell; and anyone who can burden his soul with it is better fitted to be an election-agent than a teacher.

**The Unitary Method**

Of all forms of slavery, perhaps slavery to a name is the most contemptible, but I find it very prevalent where "The Unitary Method" is concerned. The point of the method is to use a con-
venient stepping-stone when required. *One* is often a convenient stepping-stone; therefore call it "The Unitary Method"; and when you have given it that name, you *must* use one as your stepping-stone, "because it is the Unitary Method".

By spending £160 he gained £72. What did he gain per cent.?

\[ \therefore \quad \frac{20}{1} \quad \frac{9}{1} \]

"No," says the boy; "you have to say 'By spending 1'." "Why should I, if I find 20 a more convenient stepping-stone between 160 and 100?" "It's the Unitary Method; and that means you use 1." In fact I even meet boys who object to the step from 50 to 100 on the ground that I ought to have used 1 as a stepping-stone and "put 100 over 50".

\[ \pi \]

I have expressed a preference for \( 3 \frac{1}{7} \) rather than \( \frac{22}{7} \) as the right name for a commonly-used approximation to \( \pi \), chiefly on the ground that it carries its meaning and an idea of its size more clearly stamped on it, and that multiplication by 3 is easier than multiplication by 22. As a small side-issue arising out of this, I would have you notice how easily multiplication by \( 3 \frac{1}{7} \) can be converted, if required, into multiplication by Adrian's much finer approximation \( 3 \frac{1}{128} \), or \( 3 \frac{1}{4} + \frac{1}{4} (1 - \frac{1}{128}) \).

Suppose that we want \( 7 \cdot 29 \pi \).

\[
\begin{array}{c}
7.29 \\
3 \frac{1}{7} \\
21.87 \\
+ 1.04 \\
22.9 \quad \text{... correct to 3 figures.}
\end{array}
\]

Now suppose that a better approximation to \( 7 \cdot 29 \pi \) is required.

\[
\begin{array}{c}
7.29 \\
3 \frac{1}{4} - \\
21.87 \\
+ 1.041429 - \\
- 0.009216 + \\
22.90221 \quad \text{... correct to 7 figures.}
\end{array}
\]

(Of course if we had started to use the finer approximation from the beginning without the rougher one first, it would have been better to subtract a 113th part from the 7.29 before the division by 7, so cutting out the need for the extra figure. If you put down the working as here, but without the 8th figure, there is a chance of nearly 1 in 4 of the 7th figure's being wrong through accumulation of errors.)
The diameter of the Earth multiplied by $3\frac{1}{7}$ differs from the circumference by ten miles. Multiplied by Adrian's approximation $3\frac{11}{3}$, it differs from the circumference by eleven feet. This king of approximations deserves to be more widely used than it is: I believe that people are afraid of it because they call it $\frac{22}{7}$ (the Improper Fractions Superstition), and never learn this way of making multiplication easy. Call it three and a tinkered seventh, the tinkering consisting of decreasing it by $\frac{X}{\frac{7}{3}}$ of itself, and you will make it much more readily available for general use.

Every boy knows that Hannibal lived a long time ago, though it is not so generally known that Archimedes fought against the Romans in the same war as Hannibal; and it is absurd that the world should be full of boys taught that "$\pi$ means 22 over 7" when Archimedes proved 22 centuries ago that $\pi$ is less than $3\frac{1}{7}$. Given the diameters of the Earth, if you calculate its area taking $\pi$ as $3\frac{1}{7}$, you get an answer differing from the true area by just about the area of England and Scotland, which I hope you will not class as negligible countries.

**The Extra Figure Superstition**

It is well known that if you want a sum, difference or product to be certainly correct to a required number of places of decimals, you must begin by having more places of decimals before the addition, subtraction or multiplication. "The Extra Figure Superstition" is that the same principle applies to division; and I have met examiners so convinced of this that they take off marks for a division in which an extra figure has not been used, even when the working and answer are completely right.

Now the fact is that in division the error is not cumulative; and in order to divide an approximate quantity by an integer, you never need to have it to more places of decimals than your final answer, nor yet to continue the division to an extra figure. I will not prove all cases of these, but only give instances and prove one or two cases. Answers are to the nearest integer: by merely shifting the decimal point, all these are applicable to any number of places of decimals.

(1) $857 \div 35.$

\[
\begin{array}{c|c}
5 & 857 \\
7 & 171 \\
\hline
24 & \\
\end{array}
\]

(2) $811 \div 108.$

\[
\begin{array}{c|c}
9 & 811 \\
12 & 90 + \\
\hline
8 & \\
\end{array}
\]

(3) $\sqrt{44091} \div 84.$

\[
\sqrt{44091} = 210 -
\]
2. HIGHLIGHTS

(Minus how much doesn’t matter: it is enough to know which side of 210 the exact square root is.)

\[
\begin{array}{c|c}
7 & 210 \\
12 & 30 \\
\hline
2 & \\
\end{array}
\]

(4) \(173 \times \frac{10}{11} \) [treated as \( \frac{1}{2} (1 - \frac{1}{113}) \)].

(In this case the 173 must be exact to make the method proof against cumulative error.)

\[
\begin{array}{c|c}
173 \\
-2 \\
\hline
7 & 171 \\
\hline
24 & \\
\end{array}
\]

Proof of (1). \( a \) and \( b \) are odd integers: \( x \) is not necessarily an integer.

\( y \) is the nearest integer to \( x/a \).

\( z \) is the nearest integer to \( y/b \).

Required to prove that \( z \) is the nearest integer to \( x/ab \).

\[
\frac{x}{a} \sim y < \frac{1}{2},
\]

\[
bz \sim y \leq \frac{b - 1}{2}.
\]

Thus

\[
\frac{x}{a} \sim bz < \frac{b}{2},
\]

and

\[
\frac{x}{ab} \sim z < \frac{1}{2}.
\]

Hence \( z \) is the nearest integer to \( x/ab \).

Proof of (4). (In this case \( x \) is an integer.)

\( y \) is the nearest integer to \( x/113 \).

\( z \) is the nearest integer to \( (x - y)/7 \).

Required to prove that \( z \) is the nearest integer to \( \frac{16x}{113} \).

\[
\frac{x}{113} \sim y < \frac{8}{113};
\]

\[
\therefore \left( x - \frac{x}{113} \right) \sim (x - y) < \frac{8}{113};
\]
But $z \sim \frac{x-y}{7} \leq \frac{3}{7} = \frac{30}{63}$.

\[ \therefore \frac{16x}{113} \sim \frac{z}{7} < \frac{30}{63} < \frac{1}{2}; \]

\[ \therefore z \text{ is the nearest integer to } \frac{16x}{113}. \]

The generalised form of this presents no difficulties, but is tedious to write and to read. In the case of an even divisor we sometimes need guidance, as in (2) and (3), whether to take the big or the little half.

**WHAT MAKES AN APPROXIMATION “CORRECT”?**

In all questions of an approximation correct to a certain degree of accuracy, the prominent idea should be “nearer to this than to that”. “$\sqrt{10} = 3.16 \ldots$” means that $\sqrt{10}$ is nearer to 3.16 than to 3.15 or 3.17. \[ \frac{1.7321}{3} = .5774 \ldots, \] the last 4 being got by the consideration that we are dividing 11 by 3, and 11 is nearer to 12 than to 9—a procedure which I consider hugely preferable to all this slushabout with an extra figure, with rules about 5 being more than five and therefore ten. (It is well known that $\log_{10} \pi = 0.4971499 \ldots$, which to 5 places is 0.49715; and therefore, according to somebody’s rule that 15 counts as 20, it is 0.4972, which is actually printed as the value in somebody’s tables which are still on the market.)

An approximation is correct to as far as it goes if the exact value is nearer to it than to any other expression of the same or less degree of complexity, which is sometimes capable of exact definition and sometimes a matter of opinion. For instance, each of these approximations to $\sqrt{\frac{1}{2}}$ is nearer to $\sqrt{\frac{1}{2}}$ than any other expression of the same sort: 1, .7, .71, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{6}$ (which is nearer to $\sqrt{\frac{1}{2}}$ than any other fraction made of smaller numbers).

Approximations of the type $\frac{a}{b}$ were more used by the ancients than by us, who have nearly killed them off with our severely tabulated decimal notation: they survive like fine but rarely worn hand-made fabrics into a pitiless machine-age; and for mental calculations especially they are much easier to use than decimals of the same degree of accuracy. For instance, find the side of a square whose diagonal is 61 metres. $61 \times 70$ is 4270 = 4300 - 30. Divide by 99: it goes 43 times: the remainder is the difference
between 43 and 30, that is 13. Divide 13 by 99 and get \( \frac{13}{99} \). 43.13 metres is the length, correct to four figures.

It is interesting to make a collection of expressions like \( 99 - 70 \sqrt{2} \), in which the squares of the two terms differ by 1, from which fractional approximations to square roots of integers may be obtained to any required degree of accuracy: I will say no more about that except that \( \sqrt{46} \) is a tough nut to crack.

**Remainders**

How many shares costing £6\( \frac{3}{8} \) each can be bought for £300? How much money is over?

The way in which most boys do this seems to me to offend against the principles both of simplicity and truthfulness.

Number of shares bought

\[
\begin{align*}
\text{Number of shares bought} & = \frac{300}{6\frac{3}{8}} \text{ (which is untrue, because this is not an integer)} \\
& = \frac{300}{1} \times \frac{8}{51} \text{ (the Improper Fractions Superstition and the cancelling disease)} \\
& = \frac{800}{17} \text{ (more cancelling)} = \frac{47}{1} \\
& = 47 \frac{1}{17} \\
\end{align*}
\]

Next we take the remainder, which is posing as a fraction of a share, and convert it back again into the money which it really was all the time.

\[
3 \frac{1}{17} \text{ share} = \frac{51}{8} \times \frac{1}{17} \text{ (more cancelling)} = \frac{3}{8} = 7s. 6d. \text{ remainder.}
\]

As a start towards doing this better, the first rule I would give is, "Call them pigs". Children have seen pigs and probably smelt them; but they are not familiar with the look or the smell of shares.

Now we will be a farmer starting out to buy pigs which cost £6\( \frac{3}{8} \) each—a simple common-sense man of little education, but knowing that eight half-crowns make a pound. He has £300 in his wheelbarrow; and because the price of pigs is in eighths of a pound, he has changed it all into 2400 half-crowns.

Arrived at the market, he chooses a pig and shovels out 51 half-crowns to pay for it, chooses another and does it again, and goes on till—till what? Till he has to stop because he hasn’t 51 half-crowns left in the barrow. Then what is left in it? A fraction of a pig? Bunkum: it is a few remaining half-crowns.
Now put down the working like this:

\[
\begin{array}{c}
\text{h.c.} & 47 \text{ pigs (or shares)} \\
51 & 2400 \text{ h.c.} \\
204 & \\
360 & \\
357 & 3 \text{ over. (3 what over? 3 of these 2400 things. What are they? Half-crowns.)} \\
7s. 6d. & \text{over.}
\end{array}
\]

Many boys will put down all this except the headings “half-crowns” or “£3’s”; and for want of that they don’t understand the remainder 3 when they get it, but embark on another calculation so as to find it over again. Stating your unit is the key to it, and behind that lies the principle, “Think about a real farmer buying real pigs with real coins in a real wheelbarrow.”

And closely allied with this principle is another: “Be a bit more generous in the use of plain English words, and avoid those with double meanings.” “On” is a dangerous word with a multiple meaning. “On 70 he gains £3” may mean that he spent £70, sold for £70, or bought 70 tons or ounces or tea-leaves. “By spending” can have only one meaning, and that is the clue to all questions in which you are asked for the gain per cent.

**Percentage**

We all know that Percentage is one of the most fruitful sources of confusion to the young. The veterans among you may remember “The Pinnacle Soap-dish.” Its floor was made of rubber spikes like the back of a batting-glove, which supported the soap above the slush-level and saved it from degenerating into that peculiarly unpleasant pale suety texture which is the fate of soap that welters in its own slime. “The Pinnacle Soap-dish”, according to its advertisements, “saved 50 per cent of your soap-bill”. But I have never met a parent who gave a really convincing answer when the young, as one man, lifted up their voice and demanded, “Why not buy two, and save the whole of the soap-bill?”

For percentage, as for so many other arithmetical diseases, my panacea is, “Be generous in the use of simple English words”. I think it is a great mistake to teach boys Sherlock Holmes questions, in which you are given what happened last in time and asked what happened first, before they are familiar with straightforward questions in which they are told what happened first. When this mistake is made, boys import the inverse methods of the backwards problem into straightforward questions, which is as logical as buying a stomach-pump to eat your breakfast with.
Here is an instance: "22 lb. of tea at 2s. per lb. are mixed with 27 lb. at 3s. per lb. The mixture is sold at 2s. 9d. per lb. Find the gain per cent".

\[
\begin{align*}
22 \text{ lb. of first tea} & \text{ cost } 44s. \\
27 \text{ lb. of second tea} & \text{ cost } 81s. \\
49 \text{ lb. of mixed tea} & \text{ cost } 125s.
\end{align*}
\]

After that most boys will proceed to find in 49ths of a penny what one pound or one ounce or one tea-leaf cost: but the few who avoid this trap and discover (probably by improper fractions or reduction to pence) that the selling price is \( \frac{134}{6} \) shillings will go on to write something like this:

"125 is 100 (or worse still, '125 = 100')

(a peculiarly unsatisfying abstraction, with neither simplicity nor truthfulness to commend it);

\[ \therefore 1 \text{ is } \frac{125}{100}; \]

\[ \therefore \frac{134}{6} \text{ is—a large mass of slaughtered figures, resembling a portrait of a hedgehog, and leading ultimately to 107.8;} \]

\[ \therefore \text{ the gain per cent is 7.8.} \]

My two chief objections to this are that it doesn’t mean anything obvious and that it uses unnecessarily big numbers. What I want is simple English words.

By spending 125s. he gained 9.8s.;

\[ \begin{align*}
'500 & ' 39; \\
'100 & ' 7.8;
\end{align*} \]

and this is the meaning of "gaining 7.8 per cent".

(At tea-time Mr. Boon will explain to you that the percentage ought really to be taken on the selling price; but I am following here the general practice of Arithmetic books and the principle that percentage is taken on the thing that happens first in time, as in interest and population questions. I am also assuming that time progresses in the same direction for all of us.)

**Young Methods**

And now if I inflict on you another scrap of autobiography, it is not that I expect you to be interested in my life for its own sake, but because I believe you will find it in some important respects the same as your own. When I had been teaching for some years, I began to notice that I often did questions better than the boys I taught. This may have been your experience too. But whenever you find this happening, I want you to look into the reason. You are older, probably cleverer, and have had a university education:
all these things would naturally enable you to use more advanced methods than your pupils. But is that how you excel them? In my own experience exactly the opposite thing has happened: in Arithmetic at any rate, when I do a question better than a boy, I nearly always find that I have done it by a more elementary method than the boy, by an appeal to the concrete and visible, by thinking what would interest a real grocer in a real apron picking real maggots out of a real cheese, and living on real shillings got by selling real coffee. In fact a strengthened and steadied version of the eight-year-old mind is more effective than the fourteen-year-old mind bunged up with the superstitions that somebody has choked it with at school. My own progress in Arithmetic has consisted largely in forgetting dodges for making easy things difficult, and reversion to the methods of my good Scottish governess who, before I ever went to school, taught me the Battle of Bannockburn for History and common sense for Arithmetic. Even if your own early education was conducted with less emphasis on the Battle of Bannockburn (thirteen-fourteen), I would have you look at every question in Arithmetic which you do better than your pupils, and notice how often you have done it by a younger method rather than an older one. Possibly something of the same sort was in the mind of the Psalmist who wrote, "Out of the mouths of very babes and sucklings hast thou ordained strength ". Another verse of his which I commend to your attention is, "Confounded be all they that worship carved images ". And that is just what they are, thoroughly confounded; and to straighten out their confusion you must turn them away from their elaborate idols.

**Addition and Subtraction**

\[
\begin{align*}
4 & \quad 8 & \quad 8 & \quad 3 & \quad 1 & \quad 2 \\
+ & \quad 3 & \quad 0 & \quad 3 & \quad 8 & \quad 6 & \quad 8 \\
\hline
7 & \quad 9 & \quad 2 & \quad 1 & \quad 8 & \quad 0 \\
- & \quad 1 & \quad 6 & \quad 0 & \quad 5 & \quad 9 & \quad 6 \\
\hline
6 & \quad 3 & \quad 1 & \quad 5 & \quad 8 & \quad 4
\end{align*}
\]

In this addition there has been one to carry three times in the six columns; and in the subtraction that follows one has been borrowed in three of the six columns. But if the two operations are combined together into one, four of the six columns are complete in themselves, in one we have borrowed, and in one we have carried.

\[
\begin{align*}
4 & \quad 8 & \quad 8 & \quad 3 & \quad 1 & \quad 2 \\
+ & \quad 3 & \quad 0 & \quad 3 & \quad 8 & \quad 6 & \quad 8 \\
- & \quad 1 & \quad 6 & \quad 0 & \quad 5 & \quad 9 & \quad 6 \\
\hline
6 & \quad 3 & \quad 1 & \quad 5 & \quad 8 & \quad 4
\end{align*}
\]
These are the actual expectations, and it can be proved that, whatever the scale of notation, the probability of a borrow in any column is $\frac{1}{3}$, of a carry $\frac{2}{3}$, and a $\frac{2}{3}$ chance of a column complete in itself. This makes mixed addition and subtraction of this sort, with two positive lines and one negative, easier than the pair of operations which it replaces, and in fact much easier even than the addition of three positive numbers, in which a column complete in itself occurs only once in six times.

The same method is possible when two of these three numbers are negative; but in that case I have not found it as easy as the method of making two bites at it, chiefly because only one column in six is complete in itself, and the borrowing becomes a strain on the memory.

**The Use of Tables**

When boys reach the stage of using tables, there is much to be said for the practice of trying to get right results from them rather than wrong ones. And for this the first requirement is to know the history of Jimmy and Sally.

Once upon a time there was a little boy and his name was Jimmy; and he lived in a house on a long, long road with milestones all along it, placed as accurately as a professional surveyor could place them. Buses ran along the road, but they wouldn't stop except at the milestones.

One day Jimmy's mother said to him: "Jimmy," she said, "go and see your grandmother: she lives 100-9 miles away along the road." So she did up two ham sandwiches and a piece of currant cake in a brown paper parcel for him, and put him into a bus, and paid his fare and asked the conductor to take great care of him; and the bus started.

When the bus got to milestone 100, Jimmy got out and began to try and pace another $\frac{3}{5}$ of a mile. But he wasn't very good at pacing, and his little legs were rather podgy: so by the time he had paced $\frac{3}{5}$ of a mile he was a good deal out of his reckoning. He went in through the wrong gap in the hedge; and instead of reaching his grandmother's house he got to a house that was full of big, black, bearded, Bulgarian brigands. When they saw little Jimmy coming up the garden path, they laughed a hoarse laugh, "Ha-ha"; and one of them took a long knife out of his boot, and they took his little collar off him and cut him up into vermicelli and stewed him with onions for their supper. And that was the end of Jimmy who tried to pace $\frac{3}{5}$ of a mile.

About a week later, Jimmy's mother said to his little sister Sally: "Sally," she said, "go and visit your grandmother: she lives in a house 100-9 miles away along the road". So she did up one ham sandwich and a piece of the best currant cake in brown paper, and
put Sally in the bus, and paid only half-fare for her because she was under 12, and was so flustered that she nearly kissed the conductor instead of Sally; and the bus started.

Now Sally was a thoughtful child; and when she got to milestone 100 she didn’t get out, “because”, she said, “my granny lives nearer to milestone 101”. So she got out at 101 and began to pace back one-tenth of a mile towards home. Now she wasn’t really any better at pacing than Jimmy; but because she had only one-tenth of a mile to pace, she didn’t go anything like as wrong as Jimmy: so she got to her grandmother’s house, and her grandmother was awfully pleased to see her; and they had prawns and lemonade for supper, and a lovely tall white creamy pudding with almonds stuck into it all over to make it look like a porcupine.

<table>
<thead>
<tr>
<th>Mean Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>10</td>
</tr>
</tbody>
</table>

Now if you look out the Jimmylog of 100-9 in the four-figure tables which you have so thoughtfully brought with you, you will see that it comes to 2.0037 (or 8 if you have a split-line table), but the Sallylog, obtained by getting out at the nearest milestone, 101, and then pacing back \(\frac{1}{10}\), is 2.0039. As these are different, they can’t all be right; and as the seven-figure table gives 2.0038912, there is no doubt which of them is the truth.

“"And the moral of that”, as the Duchess would have said, “is ‘Always get out at the nearest stop’.” In fact, completely abolish all mean differences for more than half the interval tabulated, and use the small ones only, getting out at the nearest carefully-surveyed milestone and doing the smallest possible quantity of the most inaccurate work, this interpolation which I have compared to pacing.

Here is a line out of a table of logarithms of sines of angles, arranged by six-minute intervals, as is usual, and giving mean differences up to 5 minutes.

A line of the log sin table, with 6' intervals

<table>
<thead>
<tr>
<th>00'</th>
<th>06'</th>
<th>12'</th>
<th>18'</th>
<th>24'</th>
<th>30'</th>
<th>36'</th>
<th>42'</th>
<th>48'</th>
<th>54'</th>
<th>Mean diff.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.436</td>
<td>1.489</td>
<td>1.542</td>
<td>1.594</td>
<td>1.646</td>
<td>1.697</td>
<td>1.747</td>
<td>1.797</td>
<td>1.847</td>
<td>1.895</td>
<td>8</td>
</tr>
</tbody>
</table>

Here, for comparison, is the corresponding line of a table in which the interval is five minutes, and mean differences are given for 1' and 2' only. It is written on the same scale as the other, and you will notice that it is exactly the same length.
A LINE OF THE LOG SIN TABLE, WITH 5' INTERVALS

<p>| | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>00'</td>
<td>05'</td>
<td>10'</td>
<td>15'</td>
<td>20'</td>
<td>25'</td>
<td>30'</td>
<td>35'</td>
<td>40'</td>
<td>45'</td>
</tr>
<tr>
<td>1.1436</td>
<td>1480</td>
<td>1525</td>
<td>1568</td>
<td>1612</td>
<td>1655</td>
<td>1697</td>
<td>1739</td>
<td>1781</td>
<td>1822</td>
</tr>
<tr>
<td>50'</td>
<td>55'</td>
<td>1'</td>
<td>2'</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1863</td>
<td>1903</td>
<td>8</td>
<td>17</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

I have looked out the log sin of every minute from 8° 01’ to 8° 59’ in three ways: (A) by using the 6’ table and always adding the mean differences; (B) by using the 6’ table and small mean differences only, subtracting when necessary; (C) by using the 5’ table and small mean differences for 1’ and 2’ only. The next graph shows the comparative accuracy of the three methods by giving you the frequency-distributions of the errors in the 4th figure. (No distinction is made here between positive and negative errors: they are grouped by absolute magnitude only.) The sizes of the errors are measured from left to right; and up the page is the number of times that errors of each size occur.

The Jimmy method A, in which big mean differences are used, is shown to be very inaccurate by the relative frequency of the big errors, some even more than two. B, in which Sally’s principle is followed, and the big mean differences for 4’ and 5’ are never used, is much more accurate, as is shown by the rarity of the bigger errors.
Frequency-distribution for those cases only in which the number of minutes is $6n + (4$ or $5)$. Interpret $A$ and $B$ as before.

Frequency-distributions of errors in the 4th figure of log sin $(8^\circ + n')$, obtained

$D$ by 10′ intervals tables, adding mean differences up to 9′ (— — —).

$E$ by 15′ interval tables, using mean differences up to 7′ (— — —).
In four cases out of six, methods $A$ and $B$ are identical, and therefore their disparity is here shown greatly diluted, and should be multiplied by three to show what really happens when the distinctively Sally principle is in action.

The graph $C$ is got from the table with 5' intervals, using small mean differences only. System $C$ gives no errors more than $1\frac{1}{2}$; it is incomparably better than $A$, and is got from a table with lines of the same length. Table-maker, please note. Notice also the great advantage of an odd number of minutes in the interval: with an even number you are always being confronted with two equally likely values for things like log sin $8^\circ 27'$.

If you look at these two frequency-distributions of errors, it is not very obvious which is the better. I think I prefer $E$ to $D$ on the whole, as giving higher frequency to the smaller errors and lower to the bigger ones than $D$. But $D$ is got from the Cambridge University Table, with 10' intervals and using mean differences up to 9', and $E$ from a regular cad table, with 15' intervals, but using no mean differences above 7'. The odd number of minutes and the expurgation of the foulest mean differences are enough to compensate the greater length of the interval.

By choosing a line of the table in which the mean differences behave even worse, I could have made the disparity between the good method and the bad much greater; but here I have taken a line in which the mean differences are good enough to use for most purposes. For very small angles they are not, and that is why the four-figure tables which I use most give the log sin of every single minute, and not only every 6th minute, up to $8^\circ$.

**Where shall Wisdom be found?**

It only remains to tell you where to find the boys who write clean, sensible, checkable working. Nearly twenty years ago I examined about my own weight of Arithmetic papers for the Cambridge Local. They came from every latitude and longitude, and were done by children of every colour of the rainbow and the coal-mine. Good papers came from West Africa, done by Macaulay Babington Gladstone and O. O. Onabanjo, and delightful papers from Mauritius in broken English that was half French. But the gem of the collection was a batch of twenty supremely beautiful papers from Penang. Most of them were up to the Distinction standard; and every one of them was a work of art, a joy to the eye and a rest to the brain. Every question was easier to check by reading it than by doing it again from the beginning. The boys who did them had names that would sound queer to you, Chng Kah Sim, Ng Ching Choon, and (particularly endeared to me by his second name) Chew Boon Huck. But to my mind the important point is this: there is already in
the world at least one place where Arithmetic is a thing of beauty
and a joy to its doer and its corrector; and when once a man has
seen the kind of work they do in Penang, it is impossible to persuade
him again that Arithmetic need be the depressing slush that we make
of it in this country. Therefore let us either teach our children to
write clean sense in simple English words, or let us charter a large
ship and invite the entire population of Penang to get aboard of
her, come to this dark island of superstitions, slaughter and lies, and
undertake the gigantic task of our education.

[Vol. XXIII, No. 253, 1939.]

3. HELPING THE TEACHER

THE DALTON PLAN AND THE TEACHING OF
MATHEMATICS

BY MISS F. A. YELDHAM, B.Sc.

The Dalton Plan, which has been discussed so much recently, is a
change of organisation within school which affects a girl's whole
school life and her general development more than her work in
individual subjects. I am obliged in describing it to devote a few
minutes to speaking of general school life, and when I turn to the
teaching of mathematics there seems so little to say, that could not
be said as appropriately of teaching in ordinary circumstances that I
do not know if it will justify the consideration of this meeting.

There has been a certain organisation in schools since girls’
secondary schools opened, and the women from them who have had
the enterprise to give public service in so many and new directions
are evidence of its success in the main. There were a few points in this
organisation which appeared as defects, and the aim of the Dalton
Plan is to remedy these.

The chief failing perhaps was due to the desire on the part of
teachers to have a well-ordered school with every movement well
thought out and arranged beforehand for the benefit of the girls,
and the forgetfulness that the autocracy which succeeded best in this,
did so at the expense of the full and free development of the indivi­
dual. In the extreme case we arranged every minute of a girl's
life for her from ten minutes before prayers in the morning, until she
was out of the school gates in the afternoon. Whatever the import­
ance of the subject under attention, at each bell, several in the course
of the day, she must stop doing it and do something else. There was
a consensus of opinion that a girl's interest in any subject lasted
exactly forty minutes, and that the periods of forty minutes must be