

## ON THE DYNAMICS OF A FAMILY OF THIRD-ORDER ITERATIVE FUNCTIONS

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### Abstract

We study the dynamics of a family of third-order iterative methods that are used to find roots of nonlinear equations applied to complex polynomials of degrees three and four. This family includes, as particular cases, the Chebyshev, the Halley and the super-Halley root-finding algorithms, as well as the so-called c-methods. The conjugacy classes of these iterative methods are found explicitly.

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### 1. Introduction

One of the most important applications of an iterative method is the search for roots of a nonlinear equation  $f(z) = 0$ . We begin with an initial guess  $z_0$ , which is then improved by means of an iterative function  $\Phi$ , that is,  $z_{n+1} = \Phi(z_n)$ . Conditions are imposed on  $z_0$  (and, eventually, on  $f$  or  $\Phi$ ) to ensure the convergence of the sequence of iterates  $(z_n)_{n \geq 0}$  to a solution  $z^*$  of the equation  $f(z) = 0$ , as well as to establish the order of convergence of the iterative method defined by  $\Phi$ .

For example, Newton's root-finding algorithm

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} = N_f(z_n)$$

and other second-order root-finding algorithms have been extensively used and studied, see [30] and the references therein. For the polynomial equations  $p(z) = 0$ , the

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iterative function  $N_p$  defines a rational mapping on the Riemann sphere  $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ . The simple roots of  $p$ , that is, the roots that are not critical points are superattracting fixed points of  $N_p$ , which means that, if  $z^*$  is a simple root of  $p$ , then  $N_p(z^*) = z^*$  and  $N'_p(z^*) = 0$ . This implies that if  $z_0$  is chosen conveniently near  $z^*$ , then the sequence  $z_{n+1} = N_p(z_n)$  converges quadratically to  $z^*$ . For the dynamics of Newton's method, see [30].

Although third-order root-finding algorithms require more computational cost than simpler algorithms, in some cases they are preferred on account of greater speed of convergence of the sequence iterates when the initial guess is chosen conveniently. For instance, these algorithms have been successfully used in the search for solutions of nonlinear integral equations [11, 12]. They have also been used to solve quadratic equations in a very fast way [3]. On the other hand, third-order root-finding algorithms are also interesting from a theoretical point of view because they provide a variety of results on existence and uniqueness of solutions that improve, by and large, the results obtained by using Newton-type algorithms, see [25] or [6].

In this paper, we are interested in understanding a family of classical third-order root-finding algorithms from the point of view of their dynamics.

The paper is organized as follows. In Section 2, we review the basic notions on complex dynamics to be used. For a comprehensive study of the theory of iteration of rational maps, see [9], [28] and [29]. In Section 3, we introduce a general family of third-order root-finding algorithms whose dynamics we are interested in understanding. These root-finding algorithms are induced by iterative methods. The conjugacy classes of the family of iterative methods presented in Section 3 are constructed in Section 4. Finally, in Section 5, the equations of the critical points and additional fixed points of the iterative methods under study are obtained.

## 2. Basic notions

Before presenting the iterative methods we are interested in, we shall recall some basic notions of complex dynamics. Let

$$R(z) = \frac{P(z)}{Q(z)}$$

be a rational map on the Riemann sphere, where  $P$  and  $Q$  are complex polynomials with no common factors. We say that  $z_0$  is a *fixed point* of  $R$  if  $R(z_0) = z_0$ . For  $z \in \overline{\mathbb{C}}$ , we define its *forward orbit* as the set  $\text{orb}(z) = \{z, R(z), R^2(z), \dots, R^k(z), \dots\}$  where  $R^k$ , for  $k \in \mathbb{N}$ , means the  $k$ -fold iterate of  $R$ . A *periodic point* of period  $n$  is a point  $z_0$  such that  $R^n(z_0) = z_0$  and  $R^j(z_0) \neq z_0$  for  $0 < j < n$ . Observe that if  $z_0 \in \overline{\mathbb{C}}$  is a periodic point of period  $n \geq 1$ , then  $z_0$  is a fixed point of  $R^n$ . Also, recall that

a fixed point  $z_0$  is, respectively, *attracting*, *repelling* or *indifferent* in the case where  $|R'(z_0)|$  is less than, greater than, or equal to 1. A periodic point of period  $n$  is said to be attracting, repelling or indifferent if, as a fixed point of  $R^n$ , it is respectively attracting, repelling or indifferent. A *superattracting fixed point* of  $R$  is a fixed point which is also a zero of the derivative  $R'$ . A periodic point of period  $n$  is said to be a *superattracting periodic point* of  $R$  if, as a fixed point of  $R^n$ , it is superattracting.

Let  $\zeta$  be an attracting fixed point of  $R$ . The *basin of attraction* of  $\zeta$  is the set  $B(\zeta) = \{z \in \mathbb{C} : R^n(z) \rightarrow \zeta \text{ as } n \rightarrow \infty\}$ . The *immediate basin of attraction* of an attracting fixed point  $\zeta$  of  $R$ , denoted  $B^*(\zeta)$ , is the connected component of  $B(\zeta)$  containing  $\zeta$ . Finally, if  $z_0$  is an attracting periodic point of period  $n$  of  $R$ , the *basin of attraction of the orbit*  $\text{orb}(z_0)$  is the set  $B(\text{orb}(z_0)) = \bigcup_{j=0}^{n-1} R^j(B(z_0))$ , where  $B(z_0)$  is the attraction basin of  $z_0$  as a fixed point of  $R^n$ . The *Julia set* of a rational map  $R$ , denoted  $\mathcal{J}(R)$ , is the closure of the set of repelling periodic points. Its complement is the *Fatou set*  $\mathcal{F}(R)$ . If  $R$  has an attracting fixed point  $z_0$ , then the basin of attraction  $B(z_0)$  is contained in the Fatou set and  $\mathcal{J}(R) = \partial B(z_0)$ . Therefore, the chaotic dynamics of  $R$  are contained in its Julia set (see [9]).

In what follows, we will assume that  $f : U \rightarrow \mathbb{C}$  is an analytic function where  $U \subset \mathbb{C}$  is an open set. Our main interest is the case where  $U = \mathbb{C}$  and  $f$  is a polynomial function.

**DEFINITION 1.** We say that a map  $f \rightarrow T_f$  carrying a complex-valued function  $f$  to a function  $T_f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  is a *one-point iterative root-finding algorithm* if every root of  $f$  is an attracting fixed point of  $T_f$ . It is *convergent* if, given an initial guess  $z_0$ , the sequence of iterates  $(z_k)_{k \geq 0}$ , where  $z_{k+1} = T_f(z_k)$ , converges to a root  $z^* \in \overline{\mathbb{C}}$  of  $f$  whenever  $z_0$  is sufficiently close to  $z^*$ .

### 3. The third-order family

We consider the family of iterative root-finding algorithms given for  $n \geq 0$  by

$$z_{n+1} = M_{f,\theta,c}(z_n) = z_n - \left( 1 + \frac{L_f(z_n)}{2(1 - \theta L_f(z_n))} + c[L_f(z_n)]^2 \right) u_f(z_n)$$

where  $z_0$  is an initial guess,  $L_f(z) = f(z)f''(z)/f'(z)^2$ ,  $u_f(z) = f(z)/f'(z)$ , if  $f'(z) \neq 0$ , and  $\theta, c$  are complex parameters, both to be chosen conveniently in each case. This iterative root-finding algorithm is induced by the family of iterative functions

$$M_{f,\theta,c}(z) = z - \left( 1 + \frac{L_f(z)}{2(1 - \theta L_f(z))} + c[L_f(z)]^2 \right) u_f(z). \tag{3.1}$$

Observe that when we apply any of these iterative functions (3.1) to a polynomial, we obtain a rational map on the Riemann sphere.

This family of third-order iterative algorithms includes, as particular cases, the following (see [2]).

(1) When  $c = 0$  and the parameter  $\theta$  is real and nonnegative, we obtain the family of third-order iterative functions  $M_{f,\theta}(z) = M_{f,\theta,0}(z)$  studied in [23]

$$M_{f,\theta}(z) = z - \left( 1 + \frac{L_f(z)}{2(1 - \theta L_f(z))} \right) u_f(z). \tag{3.2}$$

In particular, we have that

(a) The well-known Chebyshev iteration function

$$C_f(z) = z - \left( 1 + \frac{1}{2} L_f(z) \right) u_f(z)$$

is obtained from family (3.2) when  $\theta = 0$ .

(b) The well-known Halley iterative method

$$H_f(z) = z - \left( \frac{2}{2 - L_f(z)} \right) u_f(z) = z - \frac{2f(z)f'(z)}{2f'(z)^2 - f(z)f''(z)}$$

is obtained from family (3.2) when  $\theta = 1/2$ . The dynamics of Halley’s iterative method have been studied in [10] and [31].

(c) Another third-order iterative method perhaps not as well known as the previous two is the so-called *super-Halley iterative method* (see [24]), which is given by

$$SH_f(z) = z - \left( 1 + \frac{L_f(z)}{2(1 - L_f(z))} \right) u_f(z)$$

and which is obtained from family (3.2) when  $\theta = 1$ .

(2) When  $c$  is a nonzero parameter and  $\theta = 0$ , we obtain the so-called *c-iterative methods*

$$M_{f,c}(z) = z - \left( 1 + \frac{1}{2} L_f(z) + cL_f(z)^2 \right) u_f(z).$$

This class of iterative methods has been introduced in [1] and [22], where we can find a study of their convergence for  $0 \leq c \leq 2$ .

Many papers have been published on the convergence of third-order iterative root-finding algorithms. Thus, for Chebyshev’s method we have [7, 12, 26]; for Halley’s method [11, 15]; for the super-Halley method [16, 24], for the *c-methods* [1]; as well as for Laguerre’s iterative root-finding algorithm, [20] and the references therein.

#### 4. Conjugacy classes

We recall the definition of conjugacy.

**DEFINITION 2.** Let  $R_1, R_2 : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$  be two rational maps. We say that  $R_1$  and  $R_2$  are conjugated if there exists a Möbius transformation (an analytic homeomorphism of the Riemann sphere)  $\phi : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$  such that  $R_2 \circ \phi(z) = \phi \circ R_1(z)$ , for all  $z \in \bar{\mathbb{C}}$ .

The maps  $R_1$  and  $R_2$  can be regarded as the same map viewed in different coordinate systems.

An important feature of conjugation of rational maps is given by the following classical result.

**THEOREM 4.1.** *Let  $R_1$  and  $R_2$  be two rational maps, and let  $\phi$  be a Möbius transformation conjugating  $R_1$  and  $R_2$ , that is,  $R_2 = \phi \circ R_1 \circ \phi^{-1}$ . Then  $\mathcal{F}(R_2) = \phi(\mathcal{F}(R_1))$  and  $\mathcal{J}(R_2) = \phi(\mathcal{J}(R_1))$ .*

Conjugacy plays a central role in understanding the behaviour of classes of maps from the point of view of dynamical systems in the following sense. Suppose that one wishes to describe both the quantitative and the qualitative behaviour of a map  $z \mapsto \Phi_f(z)$ , where  $\Phi_f$  is some iteration function as in Section 3. Since conjugacy preserves fixed and periodic points and their types as well as basins of attraction, the dynamical features concerning  $f$  are carried by the fixed points of  $\Phi_f$  as well as by the nature of such fixed points which may be (super)attracting, repelling or indifferent. Therefore, it is worthwhile to build up, for polynomials of degree two and three, a parameterized family consisting of polynomials which are *as simple as possible* such that a conjugacy exists between the corresponding iterative methods.

For the family of iterative methods (3.1), we begin with a useful result.

**THEOREM 4.2 (Scaling).** *Let  $f(z)$  be an analytic map on the Riemann sphere, and let  $T(z) = \alpha z + \beta$ , with  $\alpha \neq 0$ , be an affine map. If  $g(z) = \lambda(f \circ T(z))$ , where  $\lambda \in \mathbb{C}$  is a nonzero constant, then  $T \circ M_{g,\theta,c} \circ T^{-1}(z) = M_{f,\theta,c}(z)$ . That is,  $M_{f,\theta,c}$  and  $M_{g,\theta,c}$  are conjugated by  $T$ .*

**PROOF.** See [4]. □

The scaling theorem allows us, via suitable changes of coordinates, to reduce the study of the dynamics of iterative methods  $M_{f,\theta,c}$  to the study of specific families of iteration functions of simpler maps.

We recall that a polynomial is said to be *generic* when all of its roots are simple.

**4.1. Quadratic polynomials** It is well known that every quadratic polynomial  $p(z) = \alpha z^2 + 2\beta z + \gamma$  ( $\alpha \neq 0$ ), reduces, via the affine change of coordinates  $L(z) = (z - \beta)/\alpha$ , to the polynomial  $p_\mu(z) = z^2 - \mu$ , where  $\mu = \beta^2 - \alpha\gamma - \beta$ , that is, any quadratic polynomial, via an affine change of coordinates, reduces to a polynomial belonging to the one-parameter family of quadratic polynomials  $\{p_\mu\}_\mu$ . Note that  $p_\mu$  is generic if  $\mu \neq 0$ . Now we have the following result.

**THEOREM 4.3.** *Let  $p(z) = \alpha z^2 + 2\beta z + \gamma$ , with  $\alpha \neq 0$ , be a generic quadratic polynomial, and let  $p_\mu(z) = z^2 - \mu$ , where  $\mu = \beta^2 - \alpha\gamma - \beta$ . Then the affine scaling of coordinates  $L(z) = (z - \beta)/\alpha$  provides a conjugacy between  $M_{p,\theta,c}$  and  $M_{p_\mu,\theta,c}$ .*

**PROOF.** It is straightforward. □

**REMARK.** For the non-generic polynomial  $p(z) = z^2$ , the iterative method  $M_{p,\theta,c}(z)$  reduces to the simpler linear iterative method

$$G(z) = \frac{z(6 - 4\theta - 2c + c\theta)}{8(\theta - 2)},$$

which has  $z_0 = 0$  as its unique attracting fixed point.

**DEFINITION 3 (Universal Julia set).** We say that an iterative root-finding algorithm  $f \rightarrow T_f$  has a universal Julia set for polynomials of degree  $d$  if there exists a rational map  $R$  such that for every polynomial  $f$  of degree  $d$ ,  $\mathcal{J}(T_f)$  is conjugated by a Möbius transformation  $\phi$  to  $\mathcal{J}(R)$ , that is,  $\mathcal{J}(T_f) = \phi(\mathcal{J}(R))$ .

The result that follows, due to Cayley [13, 14] and to Schröder [32], has great historical importance. In an attempt to understand the dynamics of Newton’s method in the complex plane, Cayley investigated the dynamics of Newton’s method applied to polynomials of a particularly simple form. He realized that major difficulties would arise when attempting to extend the following result for quadratics to cubics and beyond. It is believed that this circumstance motivated further work of P. Fatou and G. Julia along these lines.

**THEOREM 4.4.** *Let*

$$N_p(z) = \frac{z^2 - ab}{2z - (b + a)}$$

*be the rational map obtained from Newton’s method applied to a generic quadratic polynomial  $p(z) = (z - a)(z - b)$ . Then  $N_p$  is conjugated to the map  $R(z) = z^2$  by the Möbius transformation  $M(z) = (z - a)/(z - b)$ , that is,  $R(z) = M \circ N_p \circ M^{-1}(z)$ . Moreover,  $\mathcal{J}(N_p)$  is the straight line in the complex plane corresponding to the locus of points equidistant from  $a$  and  $b$ .*

It follows from Theorem 4.4 that the Julia set of Newton's algorithm applied to a quadratic polynomial, up to a change of coordinates, is the unit circle in the complex plane. That is, Newton's iteration function has a universal Julia set when it is applied to quadratic polynomials.

We also have extensions of this theorem due to K. Kneisl.

**THEOREM 4.5** (See [27]). *Let*

$$H_p(z) = \frac{z^3 - 3abz + ab(a + b)}{3z^2 - 3(a + b)z + a^2 + ab + b^2}$$

*be the rational map obtained from Halley's iterative root-finding algorithm applied to a generic quadratic polynomial  $p(z) = (z - a)(z - b)$ . Then  $H_p$  is conjugated to the map  $Q(z) = z^3$  by the Möbius transformation  $M(z) = (z - a)/(z - b)$ , that is,  $Q(z) = M \circ H_p \circ M^{-1}(z)$ . Moreover,  $\mathcal{J}(H_p)$  is the straight line in the complex plane corresponding to the locus of points equidistant from  $a$  and  $b$ .*

Theorem 4.5 shows that the Julia set of Halley's iterative root-finding algorithm applied to a quadratic polynomial, up to a change of coordinates, is the unit circle in the complex plane. That is, Halley's iterative method has a universal Julia set when it is applied to quadratic polynomials.

For Chebyshev's iterative root-finding algorithm, also known as the super-Newton algorithm, the following result holds.

**THEOREM 4.6** (See [27]). *Let*

$$C_p(z) = \frac{3z^4 - 2(a + b)z^3 - 6abz^2 + 6ab(a + b)z - ab(a^2 + 3ab + b^2)}{(2z - a - b)^3}$$

*be the rational map obtained from the super-Newton algorithm applied to a generic quadratic polynomial  $p(z) = (z - a)(z - b)$ . Then  $C_p$  is conjugated to the map  $S_3(z) = (z^4 + 2z^3)/(2z + 1)$  via the Möbius transformation  $M(z) = (z - a)/(z - b)$ . That is,  $S_3(z) = M \circ C_p \circ M^{-1}(z)$ .*

Theorem 4.6 shows that the Julia set of Chebyshev's iterative root-finding algorithm applied to a quadratic polynomial, up to a change of coordinates, is the same as that of the rational map  $S_3(z) = (z^4 + 2z^3)/(2z + 1)$ . This means that Chebyshev's iterative method has a universal Julia set when it is applied to quadratic polynomials.

Now we have the following extension of the above theorems.

**THEOREM 4.7.** *The rational map obtained from the  $M_{f,\theta,c}(z)$  algorithm applied to a generic quadratic polynomial  $p(z) = (z - a)(z - b)$  is conjugated to the rational*

map

$$\begin{aligned}
 R(z) = & z^3(z^5 + (6 - 2\theta)z^4 + (14 - 8\theta)z^3 + (16 - 4c - 12\theta)z^2 \\
 & + (9 - 8c - 8\theta + 8c\theta)z + 2 - 4c - 2\theta)/((2 - 4c - 2\theta)z^5 \\
 & + (9 - 8c - 8\theta + 8c\theta)z^4 + (16 - 4c - 12\theta)z^3 \\
 & + (14 - 8\theta)z^2 + (6 - 2\theta)z + 1)
 \end{aligned}$$

via the Möbius transformation  $M(z) = (z - a)/(z - b)$ . In particular, for the super-Halley algorithm we obtain the simplest rational map  $T(z) = z^4$ , and for the  $c$ -algorithms we obtain the following rational map:

$$R(z) = \frac{z^3(z^3 + 4z^2 + 5z + 2 - 4c)}{(2 - 4c)z^3 + 5z^2 + 4z + 1}.$$

The proof of Theorem 4.7 is straightforward.

Theorem 4.7 shows that the Julia set of the super-Halley iterative root-finding algorithm applied to a quadratic polynomial, up to a change of coordinates, is the unit circle in the complex plane. This means that the super-Halley iteration function has a universal Julia set when it is applied to quadratic polynomials. The  $c$ -algorithms also have the properties of a universal Julia set when they are applied to quadratic polynomials.

**4.2. Cubic polynomials** For cubic polynomials we have the following result.

**THEOREM 4.8.** *Let  $p(z) = (z - z_0)(z - z_1)(z - z_2)$  be a generic cubic polynomial with roots ordered as follows:  $0 \leq |z_0| \leq |z_1| \leq |z_2|$ . Let  $T(z) = (z_2 - z_0)z + z_0$  and  $\lambda = (z_1 - z_0)/(z_2 - z_0)$ . Using the affine change of coordinates  $T$ , the polynomial  $p$  reduces to an element belonging to the parameterized family  $\{q_\rho\}_{\rho \in \mathbb{C}}$ , where  $q_\rho(z) = z(z - 1)(z - \rho)$ . Moreover,  $T$  is a conjugacy between  $M_{p,\theta,c}$  and  $M_{q_\lambda,\theta,c}$ , or in other words,  $T^{-1} \circ M_{q_\lambda,\theta,c} \circ T = M_{p,\theta,c}$ .*

The proof of Theorem 4.8 is straightforward.

It is easy to show that the one-parameter family  $q_\rho(z) = z(z - 1)(z - \rho)$  reduces to the well-known one-parameter family  $p_A(z) = z^3 + (A - 1)z - A$  (see [17]) by an affine change of coordinates. Therefore, we have reduced the study of the dynamics of  $M_{q_\rho,\theta,c}$  to the study of the dynamics of  $M_{p_A,\theta,c}$ , and since every cubic polynomial  $p(z) = az^3 + bz^2 + cz + d$  reduces, via an affine change of coordinates, to a polynomial belonging to either the one-parameter family  $p_A$  for a suitable choice of the parameter  $A$  or to  $z^3$ , it follows that an appropriate scaling puts  $M_{p,\theta,c}$  inside the conjugacy class of  $M_{Q,\theta,c}$ , where  $Q$  stands for either  $p_A$  or  $z^3$  in each case. Now for the polynomial  $p(z) = z^3$ , we have

$$M_{p,\theta,c}(z) = \frac{z(45 - 12c - 36\theta + 8c\theta)}{27(3 - 2\theta)}.$$

The relationship between the parameter  $A$  and the parameter  $\rho$  is given above and the relationship between the parameter  $\rho$  and the roots of the original polynomial is given in the statement of the theorem. A study of the  $A$ -parameter space for Newton's iterative method associated to the one-parameter family  $p_A$  can be found in [17]. Recall that König's iterative method applied to a complex function  $f$  is given by

$$K_{f,\sigma}(z) = z + (\sigma - 1) \frac{(1/f(z))^{\sigma-2}}{(1/f(z))^{\sigma-1}}, \quad (\sigma = 2, 3, \dots).$$

This is a family of iterative methods depending on the parameter  $\sigma$ . It includes Halley's iterative method as a member. A fair amount of progress in the study of the  $A$ -parameter space when this family of root-finding algorithms is applied to the cubic polynomials  $p_A$  has been achieved in [5, 33, 34].

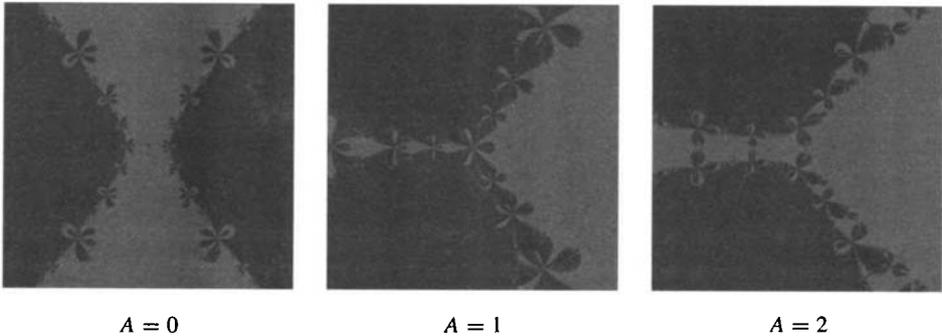


FIGURE 1. Attraction basins of the roots of  $p_A$  and the Julia set of the rational obtained when Chebyshev's iterative method is applied to  $p_A$ .

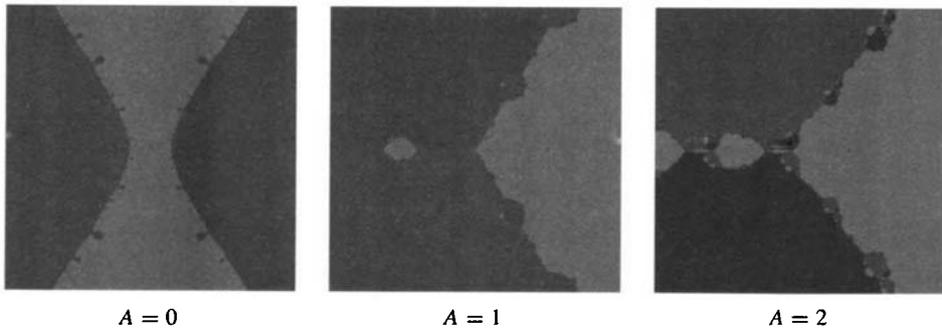


FIGURE 2. Attraction basins of the roots of  $p_A$  and the Julia set of the rational obtained when Halley's iterative method is applied to  $p_A$ .

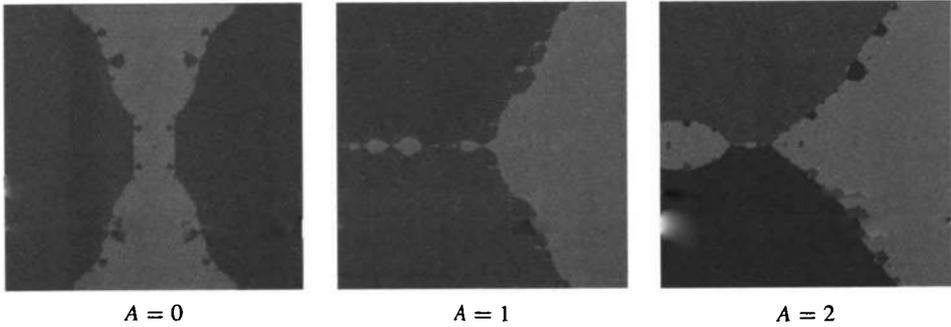


FIGURE 3. Attraction basins of the roots of  $p_A$  and the Julia set of the rational obtained when the super-Halley iterative method is applied to  $p_A$ .

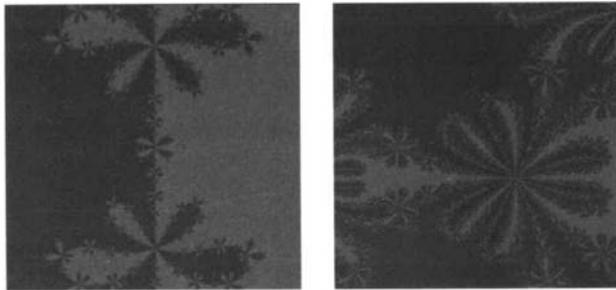


FIGURE 4. Attraction basins of the roots of  $p_\mu$  and of  $p_A$ , and the corresponding Julia set when the  $c$ -method, for  $c = 2$ , is applied to  $p_\mu$ , with  $\mu = 2$ ; and to  $p_A$ , with  $A = 1$ , respectively.

**4.3. Quartic polynomials** In the case of quartic polynomials, the following result holds.

**THEOREM 4.9.** *Let  $p(z) = (z - z_0)(z - z_1)(z - z_2)(z - z_4)$  be a degree four generic polynomial having its roots ordered as follows:  $0 \leq |z_0| \leq |z_1| \leq |z_2| \leq |z_3|$ . Let  $\eta = (z_1 - z_0)/(z_3 - z_0)$  and  $\zeta = (z_2 - z_0)/(z_3 - z_0)$ . Using the affine change of coordinates  $T$ , the polynomial  $p$  reduces to an element belonging to the parameterized family of quartic polynomials  $\{q_{\rho,\mu}\}_{\rho,\mu \in \mathbb{C}}$  where  $q_{\rho,\mu}(z) = z(z - 1)(z - \rho)(z - \mu)$ . Moreover,  $T$  is a conjugacy between  $M_{p,\theta,c}$  and  $M_{q_{\eta,\zeta},\theta,c}$ .*

The proof of Theorem 4.9 is straightforward.

The degree four case is somewhat less well understood. However, some computational progress has been made concerning the description of the parameter space for the family of polynomials  $p_\lambda(z) = z^4 + (\lambda - 1)z^2 - \lambda = (z - 1)(z + 1)(z^2 + \lambda)$ , where  $\lambda$  is a complex parameter. This is achieved in [8] for Newton’s method.

### 5. Equation of critical and additional fixed points for the third-order family of iterative methods

In this section, we give the equations for critical points and for additional fixed points, if any, for the iterative methods  $M_{p,\theta,c}$ , when  $p$  is a generic polynomial.

Let  $p$  be a generic polynomial. The derivative of the iterative method  $M_{p,\theta,c}$  is given by

$$M'_{p,\theta,c}(z) = 1 - \left( \frac{L'_p(z)}{[2(1 - \theta L_p(z))]^2} + 2cL_p(z)L'_p(z) \right) u_p(z) - \left( 1 + \frac{L_p(z)}{2(1 - \theta L_p(z))} + c[L_p(z)]^2 \right) u'_p(z).$$

Therefore, if  $p(z) = 0$  and  $p'(z) \neq 0$ , then  $M'_{p,\theta,c}(z) = 0$ . Hence, the roots of  $p$  are superattracting fixed points of  $M_{p,\theta,c}$ . The solutions of the equation  $M'_{p,\theta,c}(z) = 0$ , other than the roots of  $p$ , are called *free critical points*.

(1) For Chebyshev’s iterative function  $C_p$ , the derivative is given by

$$C'_p(z) = \frac{1}{2} \left( \frac{p(z)}{p'(z)} \right)^2 \left( 3 \left( \frac{p''(z)}{p'(z)} \right)^2 - \frac{p'''(z)}{p'(z)} \right).$$

Therefore, if  $C'_p(z) = 0$  then either  $p(z) = 0$  (roots of  $p$ ) or  $3p''(z)^2 - p(z)p'''(z) = 0$ . The solutions of  $3p''(z)^2 - p(z)p'''(z) = 0$  that are not solutions of  $p(z) = 0$ , if any, are the free critical points of Chebyshev’s iterative method. For example, Chebyshev’s iterative method has no free critical points when it is applied to the quadratic polynomial  $p_\mu$  and has  $c_1, c_2 = \pm\sqrt{(A - 1)/15}$  as its free critical points when it is applied to the cubic polynomial  $p_A$  (see [33]).

(2) For Halley’s iterative method  $H_p(z)$ , the derivative is given by

$$H'_p(z) = [p(z)]^2 \frac{3[p''(z)]^2 - 2p'(z)p'''(z)}{(2[p'(z)]^2 - p(z)p''(z))^2} = -2[p(z)]^2 [p'(z)]^2 \frac{S(p)(z)}{(2[p'(z)]^2 - p(z)p''(z))^2},$$

where

$$S(p)(z) = \frac{p'''(z)}{p'(z)} - \frac{3}{2} \left( \frac{p''(z)}{p'(z)} \right)^2$$

is the *Schwarzian derivative* of  $p(z)$ . Therefore, if  $H'_p(z) = 0$ , then either  $p(z) = 0$  (roots of  $p$ ) or  $3p''(z)^2 - 2p'(z)p'''(z) = 0$ . Solutions of  $3p''(z)^2 - 2p'(z)p'''(z) = 0$  that are not roots of  $p(z)$ , if any, are the free critical points of Halley’s iteration

function. For example, Halley’s iterative method has no free critical points when it is applied to the quadratic polynomial  $p_\mu$  and has  $c_1, c_2 = \pm\sqrt{(A - 1)/6}$  as free critical points when it is applied to the cubic polynomial  $p_A$ . A study of the parameter space for Halley’s iterative method associated to the polynomial  $p_A$  was done in [31].

(3) For the super-Halley iterative method, the derivative is given by

$$SH'_p(z) = \frac{1}{2} \left( \frac{p(z)}{p'(z)} \right)^2 \frac{p(z)p''(z)^3 - p'(z)^3 p'''(z)}{(p'(z)^2 - p(z)p''(z))^2}.$$

Therefore, if  $SH'_p(z) = 0$ , then either  $p(z) = 0$  (roots of  $p$ ) or  $p(z)p''(z)^3 - p'(z)^3 p'''(z) = 0$ . The solutions of  $p(z)p''(z)^3 - p'(z)^3 p'''(z) = 0$  that are not roots of  $p(z)$ , if any, are the free critical points of the super-Halley iterative method. The super-Halley iterative method applied to the quadratic polynomial  $p_\mu$  has no free critical points and when it is applied to the cubic polynomial  $p_A$  the free critical points are the solutions of the equation

$$54z^6 + 54(A - 1)z^4 - 216z^3 - 54(A - 1)^2z^2 - 6(A - 1)^3 = 0.$$

(4) For the  $c$ -iterative methods, the derivative is given by

$$M'_{p,c}(z) = -\frac{p(z)^2}{2p'(z)^6} [3(2c - 1)p'(z)^2 p''(z)^2 - 10cp(z)p''(z)^3 + p'(z)^3 p'''(z) + 4cp(z)p'(z)p''(z)p'''(z)].$$

Therefore, for a  $c$ -iterative method the solutions of the equation

$$3(2c - 1)p'(z)^2 p''(z)^2 - 10cp(z)p''(z)^3 + p'(z)^3 p'''(z) + 4cp(z)p'(z)p''(z)p'''(z) = 0$$

that are not roots of  $p$ , if any, are the free critical points of the  $c$ -iterative method. For the  $c$ -iterative function applied to the quadratic polynomial  $p_\mu$ , the free critical points are  $c_1, c_2 = \pm\sqrt{(15 - 5c)c\mu}/(3 - c)$  and when it is applied to the cubic polynomial  $p_A$  the free critical points are the solutions of the equation

$$2(405 - 108c)z^6 + 2(243 + 144c)(A - 1)z^4 + 18(3(A^2 + 1) + 20(2 - A)Ac - 2(3A + 10c))z^2 + 144Ac(A - 1)z - 6(A - 1)^3 = 0.$$

The reason why free critical points are important is due to the following classical result.

**THEOREM 5.1 (Fatou-Julia).** *Let  $R$  be a rational map. If  $z_0$  is an attracting periodic point, then the immediate basin of attraction  $B^*(z_0)$  contains at least one critical point.*

Thus, as a consequence of this theorem, the existence of attracting periodic orbits interferes with our search for roots of the equation  $p(z) = 0$  because if this is the case, the orbit of each free critical point must be computed and its set of limit points determined. If the set of limit points of the orbit of some free critical point is not a root of  $p$ , then it must be an attracting periodic orbit of  $M_{p,\theta,c}$ .

We notice that the existence of an attracting periodic orbit is not all that interferes with the basins of attraction of the roots of  $p(z)$ . In some cases, there may exist additional fixed points, that is, fixed points other than the roots of  $p$ . It is clear that the roots of  $p$  are fixed points of  $M_{p,\theta,c}$ . In fact, as we saw earlier, they are superattracting fixed points.

To find the additional fixed points of the iterative methods  $M_{p,\theta,c}$ , we must solve the equation  $M_{p,\theta,c}(z) = z$  subject to the condition  $p(z) \neq 0$ .

We have  $M_{p,\theta,c}(z) = z$  if and only if

$$\left(1 + \frac{L_p(z)}{2(1 - \theta L_p(z))} + cL_p(z)^2\right) u_p(z) = 0.$$

Since the roots of  $p$  are fixed points of  $M_{p,\theta,c}$ , we may assume that  $p(z) \neq 0$ . Thus the additional fixed points are the solutions, other than the roots of  $p$ , if any, of the equation

$$2p'(z)^6 + (1 - 2\theta)p(z)p'(z)^4p''(z) + 2cp(z)^2p'(z)^2p''(z)^2 - 2c\theta p(z)^3p''(z)^3 = 0.$$

(a) For Chebyshev's iterative method, the additional fixed points are the solutions, if any, of the equation  $2p'(z)^2 + p(z)p''(z) = 0$ .

For example, Chebyshev's method applied to the quadratic polynomial  $p_\mu$  has two additional fixed points given by  $z_1, z_2 = \pm\sqrt{\mu/5}$  and when it is applied to the cubic polynomial  $p_A$  the additional fixed points are the solutions of the equation  $12z^4 + 9(A - 1)z^2 - 3Az + (A - 1)^2 = 0$ .

(b) For Halley's iterative method, the additional fixed points are given by the solutions of the equation  $p'(z) = 0$ , that is, the additional fixed points are the critical points of  $p$ . Since

$$H'_p(z) = \frac{p(z)^2(3p''(z)^2 - 2p'(z)p'''(z))}{(-2p'(z)^2 - p(z)p''(z))^2},$$

if  $p(z_0) \neq 0$  and  $p'(z_0) \neq 0$ , then  $H'_f(z_0) = 3$ , or in other words any additional fixed points of Halley's method are repelling fixed points (see [19, Proposition 1]).

(c) For the super-Halley iterative method, the additional fixed points are the solutions of the polynomial equation  $2p'(z)^2 - p(z)p''(z) = 0$ .

The super-Halley method applied to the quadratic polynomial  $p_\mu$  has two additional fixed points  $z_1, z_2 = \pm\sqrt{-\mu/3}$  and when it is applied to the cubic polynomial  $p_A$ , the additional fixed points are the solutions of the equation

$$9z^6 + 9(A - 1)z^4 - 36Az^3 - 9(A - 1)^2z^2 - (A - 1)^3 = 0.$$

(d) For the  $c$ -methods, the additional fixed points are the solutions of another polynomial equation, namely  $2p'(z)^4 + p(z)p'(z)^2p''(z) + 2cp(z)^2p''(z)^2 = 0$ .

When the  $c$ -methods are applied to the quadratic polynomial  $p_\mu$ , we have four free fixed points

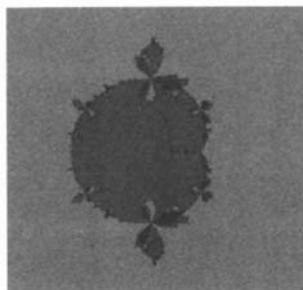
$$z_1, z_2 = \pm \frac{\sqrt{2}\sqrt{(c + 5)(1 + 2c + \sqrt{1 - 16c})\mu}}{2c + 10}$$

and

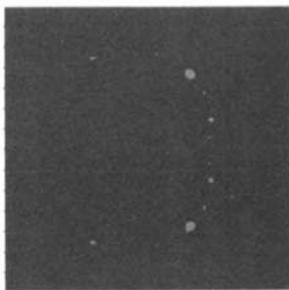
$$z_3, z_4 = \pm \frac{\sqrt{2}\sqrt{(c + 5)(1 + 2c - \sqrt{1 - 16c})\mu}}{2c + 10},$$

and when they are applied to the cubic polynomial  $p_A$ , the additional fixed points are the solutions of the equation

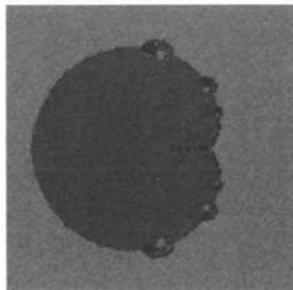
$$\begin{aligned} &36(3 + c)z^8 + (153 + 72c)(A - 1)z^6 - (27 + 72c)Az^5 \\ &+ (75 + 36c)(A - 1)^2z^4 - (18 - 72c)A(A - 1)z^3 \\ &+ (15(A - 1)^3 + 36A^2c)z^2 - 3A(A - 1)^2z - (A - 1)^4 = 0. \end{aligned}$$



Chebyshev's method



Halley's method



Newton's method

FIGURE 5. The regions in parameter space  $A \in [-2, 2] \times [-2, 2]$  in which the corresponding critical points are attracted to some of the roots of the polynomial  $p_A$  when the indicated iterative methods are applied to it.

## 6. Conclusions

To summarize, in this paper we study a large class of third-order methods from the point of view of their dynamics. For each method, we establish its conjugacy classes and find explicit formulations for both its additional fixed points and its free critical points.

TABLE 1.  $x_0 = 1.5$ ,  $x^* = 2$ ,  $l_\infty$  - error,  $n = 100$ .

Iterations	Newton	Chebyshev	Halley	$c = 2$
1	0.1826	0.1474	0.0408	0.16206
2	0.0146	0.0036	$2.59 \times 10^{-5}$	$3.86 \times 10^{-4}$
3	$1.06 \times 10^{-4}$	$2.31 \times 10^{-7}$	$4.86 \times 10^{-13}$	$1.23 \times 10^{-12}$
4	$1.46 \times 10^{-8}$	$8.41 \times 10^{-13}$	0	0
5	$1.24 \times 10^{-12}$	0		
6	0			

We finalize with an application where the studied methods are applied to quadratic equations of the type

$$F(x) = x^T Ax + Bx + C = 0, \quad (6.1)$$

where  $\dim(A) = (n \times n) \times n$ ,  $\dim(B) = n \times n$  and  $\dim(C) = \dim(x) = n$ .

In this case, the second Fréchet derivative is constant,  $F''(x) = A + A^T$ .

The above kind of equations may come from the discretization of equilibrium problems, where interacting forces between particles determine the output. The discretization of the Laplacian using divided differences also gives quadratic equations [21]. Other interesting examples may be found in Fredholm equations [18] as

$$x(s) = f(s) + \lambda \int_0^1 e^{t+s} x(t)^2 dt.$$

However, the actual case we are going to analyze will have an exact solution in order to make the evaluation of errors easier. We randomly generate  $A$  and  $B$ , and then we determine  $C$  such that  $X^*(i) = 2$ ,  $i = 1, \dots, n$ , is a solution of (6.1).

In Table 1 the dimension is  $n = 100$ .

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