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ON DIHEDRAL GALOIS COVERINGS

HIRO-O TOKUNAGA

ABSTRACT. In this paper, we shall give a method in constructing dihedral Galois covering with prescribed branch locus. As an application, we shall look into dihedral Galois covering of $\mathbf{P}^2$, where torsion elements of the Mordell-Weil group of an elliptic surface play key roles in constructing coverings.

Introduction. The main purpose of this article is to give an explicit method for constructing finite dihedral Galois coverings. As an application, we shall look into dihedral Galois coverings of $\mathbf{P}^2$.

We shall start with the definition of dihedral Galois coverings of a smooth projective variety.

DEFINITION 0.1. Let Y be a smooth projective variety, and let X be a normal variety with a finite morphism $\pi: X \longrightarrow Y$. The fields of rational functions of X and Y are denoted by $\mathbf{C}(X)$ and $\mathbf{C}(Y)$, respectively. Then π induces the inclusion map $\pi^*: \mathbf{C}(Y) \hookrightarrow \mathbf{C}(X)$, and $\mathbf{C}(Y)$ is identified with a subfield of $\mathbf{C}(X)$ by this map. We call X a dihedral $\mathcal{D}_{2n}$ covering of Y if $\mathbf{C}(X)$ is a Galois extension of $\mathbf{C}(Y)$ having dihedral group, $\mathcal{D}_{2n}$, of the order $2n$ as its Galois group.

Our study on dihedral $\mathcal{D}_{2n}$ coverings is motivated by the following problem, which was one of the main subjects discussed in Namba [11].

PROBLEM 0.2. Let Y be a smooth projective variety and let B be a reduced divisor on Y. Give a necessary and sufficient condition and (Y, B) for the existence of a finite Galois covering, X, of Y with the covering morphism $\pi: X \longrightarrow Y$ such that B is the branch locus $\Delta(X/Y) := \left\{y \in Y \mid \sharp\left(\pi^{-1}(y)\right) < \deg \pi\right\}$.

Answers to this problem may be divided into two steps:

(I) to give a general existence theorem on coverings,

(II) to construct Galois coverings having the prescribed Galois groups as well as the prescribed branch locus.

For (I) one investigates an open variety $Y \setminus B$. In fact, many answers to (I) have been obtained as applications of the study of the fundamental group $\pi_1(Y \setminus B)$. They are given in terms of topology and do not seem to give constructive answers to Problem 0.2. On the other hand, (II) takes care of this missing part. The step (II) resembles the situation of constructive aspects of the inverse problem of Galois theory: to construct a field extension of $\mathbf{Q}$ having a prescribed group as its Galois group over $\mathbf{Q}$.

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For abelian coverings, there are satisfactory answers in both (I) and (II) by several authors, *e.g.*, Namba [11] for (I), and Pardini [12] for (II). In his book [11], Namba also discusses general Galois coverings, and gives an answer for (I). However, he does not seem to give any explicit method for constructing Galois coverings. At the moment, (II) for non-abelian Galois coverings seems to be missing. Therefore, it is worthwhile to consider a constructive step for the simplest non-abelian Galois coverings such as dihedral $\mathcal{D}_{2n}$ coverings.

Now we shall explain our strategy. Let Y be a smooth variety, X a normal variety with a finite morphism $\pi: X \longrightarrow Y$. The field of rational functions $\mathbf{C}(X)$ gives a finite field extension of $\mathbf{C}(Y)$. Conversely, given a finite extension, K, of $\mathbf{C}(Y)$, the K-normalization, X_1, of Y (see Iitaka [7], §2.14 for normalization of varieties) is a normal variety satisfying (i) $\mathbf{C}(X_1)$ is K, and (ii) there is a finite morphism $\pi_1: X_1 \longrightarrow Y$ determined by the inclusion map $\mathbf{C}(Y) \hookrightarrow K$. In this way, we have a correspondence between finite coverings of Y and finite field extensions of $\mathbf{C}(Y)$. Using this correspondence, we shall translate elementary Galois theory of function fields into geometry of varieties. Let X be a dihedral $\mathcal{D}_{2n}$ covering of Y with the covering morphism $\pi: X \longrightarrow Y$. Then, $\mathbf{C}(X)$ is a Galois extension of $\mathbf{C}(Y)$ with the Galois groups $\mathcal{D}_{2n}$. We choose generators σ, τ of $\mathcal{D}_{2n}$ as follows: $\mathcal{D}_{2n} = \langle \sigma^2 = \tau^n = (\sigma\tau)^2 = e \rangle$. The invariant subfield, $\mathbf{C}(X)^\tau$, of $\mathbf{C}(X)$ by τ is a quadratic extension of $\mathbf{C}(Y)$. Let $D(X/Y)$ be the $\mathbf{C}(X)^\tau$-normalization of Y. Then $D(X/Y)$ is a finite double covering of Y canonically determined by X and we denote the covering morphism by β_1. Also, as $\mathbf{C}(X)$ is a cyclic extension of $\mathbf{C}\big(D(X/Y)\big)$ of degree n, X is an n-fold cyclic covering of $D(X/Y)$. We denote the covering morphism from X to $D(X/Y)$ by β_2. These varieties satisfy the following commutative diagram

$$\begin{tikzcd} X \arrow[d, "\pi"] \arrow[dr, "\beta_2"] & \\ Y & D(X/Y) \arrow[l, "\beta_1"] \end{tikzcd}$$

In this way, we reduce the study of the dihedral $\mathcal{D}_{2n}$ covering $\pi: X \longrightarrow Y$ to that of the two cyclic coverings $\beta_1: D(X/Y) \longrightarrow Y$ and $\beta_2: X \longrightarrow D(X/Y)$. We now formulate our problem in terms of these coverings.

PROBLEM 0.3. Let $f: Z \longrightarrow Y$ be a finite smooth double covering of Y and let D be a divisor on Z. Give a necessary and sufficient condition on D and (Y, Z, f) for the existence of a finite dihedral $\mathcal{D}_{2n}$ covering X such that

(i) $D(X/Y) = Z$, and

(ii) the branch locus of $\beta_2: X \longrightarrow Z$ is $\operatorname{Supp} D$.

Once this problem is settled, we shall then consider Problem 0.2 with dihedral $\mathcal{D}_{2n}$ coverings. Now we shall state our results. The first results are the following two propositions which we will prove in §2. (For the notation, see Notation and Conventions.)

PROPOSITION 0.4. *Let $f: Z \longrightarrow Y$ be a smooth finite double covering of a smooth projective variety Y. Let σ be the involution on Z determined by the covering transformation of f. Let D_1, D_2 and D_3 be effective divisors on Z and let n be an odd integer with $n \geq 3$. Suppose that D_i $(i = 1, 2, 3)$ satisfy the following properties:*

(a) D_1 and $\sigma^* D_1$ have no common component.

(b) If $D_1 = \sum_i a_i D_i^{(1)}$ denotes the decomposition into irreducible components, then $0 < a_i \leq \frac{n-1}{2}$ for every i, and the greatest common divisor of the a_i's and n is 1.

(c) $D_1 + nD_2 \sim \sigma^* D_1 + nD_3$.

Then, there exists a dihedral $\mathcal{D}_{2n}$ covering, X, of Y such that (i) the variety $D(X/Y)$ is Z and (ii) $\Delta(X/Y) = \Delta(Z/Y) \cup f\big(\mathrm{Supp}(D_1)\big)$. ■

Note that the condition (b) in Proposition 0.4 is automatically satisfied if there is an irreducible component of D_1 whose coefficient is 1.

Now we see that this construction is universal in the following sense.

PROPOSITION 0.5. *Let $\pi: X \longrightarrow Y$ be a dihedral $\mathcal{D}_{2n}$ ($n \geq 3$: odd) covering such that $D(X/Y)$ is smooth, and let σ be the involution on $D(X/Y)$ determined by the covering transformation of β_1. Then, there exist three effective divisors D_1, D_2 and D_3 on $D(X/Y)$ satisfying the following three conditions:*

(i) D_1 and $\sigma^* D_1$ have no common component.

(ii) If $D_1 = \sum_i a_i D_i^{(1)}$ denotes the decomposition into irreducible components, then $0 \leq a_i \leq \frac{n-1}{2}$ for every i.

(iii) $D_1 + nD_2 \sim \sigma^* D_1 + nD_3$, and $D_2 + \sigma^* D_2 \sim D_3 + \sigma^* D_3$.

(iv) $\mathrm{Supp}(D_1 + \sigma^* D_1)$ is the branch locus of β_2. ■

Propositions 0.4 and 0.5 deal only with the case of odd n. In §3, we shall also consider the case of even n. In the even cases, we need four divisors on $D(X/Y)$ to describe dihedral $\mathcal{D}_{2n}$ coverings as well as more complicated conditions on these divisors.

PROPOSITION 0.6. *Let $f: Z \longrightarrow Y$ be a smooth finite double covering of a smooth projective variety Y. Let σ be the involution on Z determined by the covering transformation of f. Let D_1, D_2, D_3, and D_4 be effective divisors on Z and let n be an even integer with $n \geq 4$. Suppose that D_i $(i = 1, 2, 3, 4)$ satisfy the following properties:*

(a) D_1 and $\sigma^* D_1$ have no common component.

(b) If $D_1 = \sum_i a_i D_i^{(1)}$ denotes the decomposition into irreducible components, then $0 < a_i \leq \frac{n-1}{2}$ for every i, and the greatest common divisor of the a_i's and n is 1.

(c) D_2 is either a reduced positive divisor, or $D_2 = 0$. In the former case, if $D_2 = \sum_j D_j^{(2)}$ denotes the decomposition into irreducible components, then for every $D_j^{(2)}$, there exists divisor $B_j^{(2)}$ on Y such that $f^* B_j^{(2)} = D_j^{(2)}$.

(d) $D_1 + \frac{n}{2} D_2 + nD_3 \sim \sigma^* D_1 + nD_4$.

(e) There exist an odd integer r_0 dividing n and a rational function $b \in \mathbf{C}(Y)$ such that

$$(f^* b) = r_0(D_2 + D_3 + \sigma^* D_3) - r_0(D_4 + \sigma^* D_4)$$

Then, there exists a dihedral $\mathcal{D}_{2n}$ covering, X, of Y such that (i) the variety $D(X/Y)$ is Z and (ii) $\Delta(X/Y) = \Delta(Z/Y) \cup f\big(\mathrm{Supp}(D_1 + D_2)\big)$. ∎

The "converse" of the assertion in Proposition 0.6 also holds and is formulated in the following proposition.

PROPOSITION 0.7. *Let $\pi: X \longrightarrow Y$ be a dihedral $\mathcal{D}_{2n}$ ($n \geq 4$: even) covering of which $D(X/Y)$ is smooth, and let σ be the involution on $D(X/Y)$ determined by the covering transformation of β_1. Then there exist four effective divisors D_1, D_2, D_3 and D_4 on $D(X/Y)$ satisfying the following four conditions:*

(i) D_1 and $\sigma^* D_1$ have no common component.

(ii) If $D_1 = \sum_i a_i D_i^{(1)}$ denotes the decomposition into irreducible components, then $0 \leq a_i \leq \frac{n-1}{2}$ for every i.

(iii) D_2 is either a reduced positive divisor, or $D_2 = 0$. In the former case, if $D_2 = \sum_j D_j^{(2)}$ denotes the decomposition into irreducible components, then for every $D_j^{(2)}$, there exists divisor $B_j^{(2)}$ on Y such that $f^* B_j^{(2)} = D_j^{(2)}$.

(iv) $D_1 + \frac{n}{2} D_2 + nD_3 \sim \sigma^* D_1 + nD_4$, and $D_2 + D_3 + \sigma^* D_3 \sim D_4 + \sigma^* D_4$.

(v) $\mathrm{Supp}(D_1 + \sigma^* D_1 + D_2)$ is the branch locus of β_2. ∎

Note that Proposition 0.6 is still unsatisfactory, as it does not cover the case that r_0 is even.

The conditions on the divisors in Propositions 0.4, 0.5, 0.6 and 0.7 are so complicated that it seems intractable to find divisors satisfying those conditions. In other words, dihedral $\mathcal{D}_{2n}$ coverings seem to exist rather rarely. Indeed, this is the case for dihedral $\mathcal{D}_{2n}$ covering of $\mathbf{P}^2$. Because, if a plane curve C is the branch locus of a dihedral $\mathcal{D}_{2n}$ covering of $\mathbf{P}^2$, then the fundamental group $\pi_1(\mathbf{P}^2 \setminus C)$ is non-abelian. But, we have the following theorem.

THEOREM 0.8 (DELIGNE [3], FULTON [4]). *Let $C \subset \mathbf{P}^2$ be a plane curve, which has only nodes as its singularities. Then $\pi_1(\mathbf{P}^2 \setminus C)$ is abelian.*

Therefore, if a plane curve C is the branch locus of the dihedral $\mathcal{D}_{2n}$ covering of $\mathbf{P}^2$, it must have singularities other than node (*e.g.*, cusps and triple points.) In particular, if n is an odd prime a curve C has degree ≤ 4, we can characterize C in terms of its singularities. Now we state our result which we will prove in Section 4 and Section 5.

THEOREM 0.9. *Let $\pi: S \longrightarrow \mathbf{P}^2$ be a dihedral $\mathcal{D}_{2p}$ (p : odd prime) covering of $\mathbf{P}^2$, and let $\Delta(S/\mathbf{P}^2)$ be the branch locus of π. Then, $\deg \Delta(S/\mathbf{P}^2) \geq 3$. Furthermore if $\deg \Delta(S/\mathbf{P}^2) \leq 4$, then possibilities for $\Delta(S/\mathbf{P}^2)$ are listed as follows:*

(a) If $\deg \Delta(S/\mathbf{P}^2) = 3$, then, for arbitrary p, $\Delta(S/\mathbf{P}^2)$ is three distinct lines intersecting at one point.

(b) If $\deg \Delta(S/\mathbf{P}^2) = 4$ and $\Delta\big(D(S/\mathbf{P}^2)/\mathbf{P}\big)$ is a conic, then, for arbitrary p,

(i) $\Delta(S/\mathbf{P}^2)$ is two distinct smooth conics tangent at two points,

(ii) $\Delta(S/\mathbf{P}^2)$ is two distinct smooth conics tangent at one point,

(iii) $\Delta(S/\mathbf{P}^2)$ is a smooth conic and two distinct lines tangent to the conic, or

(iv) $\Delta(S/\mathbf{P}^2)$ *is four distinct lines intersecting at one point.*

(c) if $\deg \Delta(S/\mathbf{P}^2) = 4$ *and* $\Delta\big(D(S/\mathbf{P}^2)/\mathbf{P}^2\big) = \Delta(S/\mathbf{P}^2)$, *then*

(i) $p = 3$, *and* $\Delta(S/\mathbf{P}^2)$ *is an irreducible quartic curve with three cusps,*

(ii) $p = 3$, *and* $\Delta(S/\mathbf{P}^2)$ *is a cubic curve with a cusp and a line; here the line is tangent to the cubic curve at an inflection point, or*

(iii) p *is arbitrary, and* $\Delta(S/\mathbf{P}^2)$ *is four distinct lines intersecting at one point.*

Conversely, for each curve described above, there exists a dihedral $\mathcal{D}_{2p}$ *covering of* $\mathbf{P}^2$ *branched along it.* ■

REMARK. The case (i) in (c) in Theorem 0.9 is consistent with Zariski's result [15]. For irreducible plane quartic curves, a quartic curve Q with three cusps is the only one having non-abelian $\pi_1(\mathbf{P}^2 \setminus Q)$. Furthermore, the group $\pi_1(\mathbf{P}^2 \setminus Q)$ is a non-abelian group of order 12.

As a corollary of Theorem 0.9, we have

COROLLARY 0.10. *Let* C *be a quartic curve described in Theorem 0.9. Then,* $\pi_1(\mathbf{P}^2 \setminus C)$ *is non-abelian.* ■

Now we shall sketch a proof of Theorem 0.9. All the cases except for cases (c)(i) and (ii) are straightforward from Propositions 0.4 and 0.5. To prove the remaining two cases, we shall make use of the following fact (*cf.* Miranda and Persson [9]): *For any double covering, W, of* $\mathbf{P}^2$ *branched along a quartic curve, there exist a rational elliptic surface,* $\mathcal{E}$, *and a birational morphism from* $\mathcal{E}$ *to W.*

The composite morphism from the elliptic surface $\mathcal{E}$ to $\mathbf{P}^2$ is degree 2. But it is not finite. So we can not apply our results on dihedral coverings to this composite morphism of degree 2. However, blowing up $\mathbf{P}^2$ several times, we get a surface, Σ, which has the following properties: (i) Σ is birational to $\mathbf{P}^2$ and (ii) the elliptic surface $\mathcal{E}$ is a finite double covering of Σ. Let $\tilde{S}$ be the $\mathbf{C}(S)$-normalization of Σ. We denote the covering map by $\tilde{\pi}$. In this way, we get a dihedral $\mathcal{D}_{2p}$ covering of Σ with $D(\tilde{S}/\Sigma) = \mathcal{E}$. Conversely, once we construct a dihedral $\mathcal{D}_{2p}$ covering, S_1, of Σ which is birational to S, then the Stein factorization of the composite morphism $S_1 \longrightarrow \Sigma \longrightarrow \mathbf{P}^2$ is nothing but S. Therefore, it is enough to consider the dihedral $\mathcal{D}_{2p}$ covering $\tilde{S}$ in order to investigate the original covering S.

In order to show that only the cases (c)(i) and (ii) can occur, we shall apply Proposition 0.5 to the covering $\tilde{\pi}: \tilde{S} \longrightarrow \Sigma$. Then we have the three divisors on $\mathcal{E}$ described in Proposition 0.4. We shall translate the conditions on these divisors into an arithmetic property of the Mordell-Weil group, $\mathrm{MW}(\mathcal{E})$, of $\mathcal{E}$ and show that $\mathrm{MW}(\mathcal{E})$ has a torsion of order three. This determines the configuration of the singular fibers. Singular fibers of $\mathcal{E}$ come from singularities of $D(S/\mathbf{P}^2)$. Hence, once we get the configuration of the singular fibers of $\mathcal{E}$, we can determine the singularities of $\Delta\big(D(S/\mathbf{P}^2)/\mathbf{P}\big)$, and ultimately prove Theorem 0.9.

Our proof of Theorem 0.9 gives a relation between dihedral $\mathcal{D}_{2p}$ coverings of $\mathbf{P}^2$ branched along quartic curves and the Mordell-Weil groups of rational elliptic surfaces.

While the former is a purely geometric object, the latter is an arithmetic object which has been studied by many mathematicians ([1], [2], [10]).

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Notation and conventions. Throughout the paper, the ground field will always be the complex number field $\mathbf{C}$. However, with great care, $\mathbf{C}$ may be replaced by the algebraic closure of $\mathbf{Q}$.

$\mathbf{C}(X) :=$ the rational function field of X.

Let φ be an element of $\mathbf{C}(X)$. We denote the zero divisor and the polar divisor of φ by $(\varphi)_0$ and $(\varphi)_\infty$, respectively. Also, (φ) means a divisor defined by $(\varphi) = (\varphi)_0 - (\varphi)_\infty$.

Let X be a normal variety, Y a smooth variety and let $\pi: X \longrightarrow Y$ be a finite morphism from X to Y. We define the branch locus of f, which we denote by $\Delta(X/Y)$, as follows:

$$\Delta(X/Y) = \left\{ y \in Y \mid \sharp\left(\pi^{-1}(y)\right) < \deg \pi \right\}.$$

For a divisor D on Y, $\pi^{-1}(D)$ denotes the set-theoretic inverse image of D, while $\pi^*(D)$ denotes the ordinary pullback. Also, $\operatorname{Supp} D$ means the supporting set of D.

For a divisor R on X, $\pi_* R$ denotes the push-forward of R defined in Fulton [5], 1.4.

Let $\pi: X \longrightarrow Y$ be a dihedral Galois covering of Y. Morphisms β_1, β_2 and the variety $D(X/Y)$ always mean those defined in the introduction.

Let S be a finite double covering of a smooth projective surface Σ. The "*canonical resolution*" of S always means the resolution given by Horikawa in [5].

Let S be an elliptic surface over C. We call S *minimal* if the fibration is relatively minimal. In this paper, we always assume that an elliptic surface is minimal. For singular fibers of an elliptic surface, we use the notation of Kodaira [7].

Let D_1, D_2 be divisors.

$D_1 \sim D_2$: linear equivalence of divisors.

$D_1 \approx D_2$: algebraic equivalence of divisors.

$D_1 \approx_{\mathbf{Q}} D_2$: $\mathbf{Q}$-algebraic equivalence of divisors.

By a (p,q) cusp, we mean a curve singularity which is isomorphic to one defined by a local equation $x^p + y^q = 0$. For simplicity, a $(2,3)$ cusp is called a *cusp*.

1. Preliminaries.

In this section, we shall review some elementary results from Galois theory.

LEMMA 1.1. *Let F be a field of characteristic zero containing all the n-th roots of unity ($n \geq 3$), and let E be a quadratic extension of F. Let σ denote the non-trivial element of* $\operatorname{Gal}(E/F)$. *Let K be a cyclic extension of degree n of E: $K = E(\xi)$, $\xi^n = \varphi$*

where φ is an element in E satisfying (i) $\varphi^\sigma \neq \varphi$, and (ii) there exists an element b in F such that $\varphi\varphi^\sigma = b^n$. Then K is a dihedral Galois extension of F having Galois group $\mathcal{D}_{2n}$, and E is the invariant subfield of the cyclic subgroup of order n of $\mathcal{D}_{2n}$.

PROOF. Put $\varphi + \varphi^\sigma = a$. Then ξ satisfies an equation $x^{2n} - ax^n + b^n = 0$ over F. Since $E = F(\varphi)$ and $K = F(\xi)$, this polynomial is the minimal polynomial of ξ over F. As all conjugate elements of ξ, which are $\epsilon^i\xi$, $\epsilon^j b/\xi$ $(i,j = 1,\ldots,n-1)$ where ϵ is a primitive n-th root of unity, are in K, K is a Galois extension of F. Let $\mathcal{D}_{2n} = \langle \sigma_1, \tau \mid \sigma_1^2 = \tau^n = (\sigma_1\tau)^2 = e\rangle$. We define an action of $\mathcal{D}_{2n}$ on K over F as follows: $\sigma_1 \colon \xi \longmapsto b/\xi, \tau \colon \xi \longmapsto \epsilon\xi$. Since σ_1 induces σ on E and τ is trivial on E, we have $K^\tau = E$. ∎

LEMMA 1.2. *Let F be as above and let K be a dihedral Galois extension of F with* $\mathrm{Gal}(K/F) = \mathcal{D}_{2n} = \langle \sigma, \tau \mid \sigma^2 = \tau^n = (\sigma\tau)^2 = e\rangle$ $(n \geq 3)$. *Then there exist $\theta \in K$ and $b \in F$ such that (i) $K = F(\theta)$ and (ii) the action of $\mathcal{D}_{2n}$ on K is given by $\sigma \colon \theta \longmapsto b/\theta$, $\tau \colon \theta \longmapsto \epsilon\theta$.*

PROOF. Let K^τ be the invariant subfield of K by τ. Then K is a cyclic extension of degree n of K^τ. Since F contains all the n-th roots of unity, there exists an element $\theta \in K$ such that the minimal polynomial of θ is of form $x^n - a$ for some $a \in K^\tau$. For this θ, the action of τ is given by $\tau \colon \theta \longmapsto \epsilon\theta$ where ϵ is a primitive n-th root of unity. Put $b = \theta\theta^\sigma$. Then as b is an invariant element of $\mathcal{D}_{2n}$, $b \in F$. Now we shall show that $F(\theta) = K$. To see this, it is enough to show that $a \notin F$. Assume that $a \in F$. Then $F(\theta) \subset K$, $[F(\theta) : F] = n$ and $F(\theta)/F$ is a cyclic extension of degree n. This shows that $\mathcal{D}_{2n}$ has a normal subgroup N of order 2 with $\mathcal{D}_{2n}/N \cong \mathbf{Z}/n\mathbf{Z}$; but this impossible. ∎

2. **Construction of dihedral Galois coverings for $\mathcal{D}_{2n}, n$: odd.** In this section, we shall prove Propositions 0.4 and 0.5.

PROOF OF PROPOSITION 0.4. By the assumption (c), there exists a rational function $\varphi \in \mathbf{C}(Z)$ satisfying $(\varphi)_0 = D_1 + nD_2$ and $(\varphi)_\infty = \sigma^*D_1 + nD_3$. Then we have $(\varphi\sigma^*\varphi) = n(D_2 + \sigma^*D_2) - n(D_3 + \sigma^*D_3)$. Put $\tilde{\varphi} = (\varphi\sigma^*\varphi)^{\frac{n-1}{2}}\varphi$. Then we have

CLAIM. *The polynomial $x^n - \tilde{\varphi}$ is irreducible over* $\mathbf{C}(Z)$.

PROOF OF CLAIM. Suppose that $x^n - \tilde{\varphi} = h_1(x)h_2(x)$, $\deg h_i(x) > 0$ $(i = 1,2)$ over $\mathbf{C}(Z)$. As roots of the equation $x^n - \tilde{\varphi} = 0$ are $\epsilon^i(\sqrt[n]{\varphi})$ $(i = 0,\ldots,n-1)$ where ϵ is a primitive n-th root of unity, the constant term of $h_1(x)$ is of form $\epsilon^m(\sqrt[n]{\varphi})^k$, $(0 \leq m \leq n-1$, $\deg h_1(x) = k)$. Let d be the greatest common divisor of k and n, and let s, t be integers such that $sn + tk = d$. Then, we have

$$(\sqrt[n]{\varphi})^d = \big((\sqrt[n]{\varphi})^k\big)^t\varphi^s \in \mathbf{C}(Z).$$

Putting $(\sqrt[n]{\varphi})^d = \eta$, we get $\varphi = \eta^{\frac{n}{d}}$. From this equality, we have

$$D_1 - \sigma^*D_1 = \frac{n}{d}\big(dD_3 - dD_2 + (\eta)_0 - (\eta)_\infty\big).$$

Therefore $\frac{n}{d}|a_i$ for every i. But this contradicts to the assumption (b) as $\frac{n}{d} > 1$.

Let $K = \mathbf{C}(Z)(\xi)$, where ξ is an n-th root of $\tilde{\varphi}$. By the Claim, K is a cyclic extension of degree n, and we have $\sigma^*\tilde{\varphi} = (\varphi\sigma^*\varphi)^n/\tilde{\varphi}$. Therefore, by Lemma 1.1, K is a dihedral Galois extension of $\mathbf{C}(Y)$ with $\mathrm{Gal}\big(K/\mathbf{C}(Y)\big) = \mathcal{D}_{2n}$.

Let X be the K-normalization of Y. Then, it is a Galois covering of Y with the Galois group $\mathcal{D}_{2n}$ and, by Lemma 1.1, $D(X/Y) = Z$. The statement $\Delta(X/Y) = \Delta(Z/Y) \cup f(D_1)$ easily follows from the assumption (a) and

$$(\tilde{\varphi}) = (D_1 - \sigma^*D_1) + n(D_2 - D_3) + \frac{n(n-1)}{2}(D_2 + \sigma^*D_2 - D_3 - \sigma^*D_3). \qquad \blacksquare$$

PROOF OF PROPOSITION 0.5. Applying Lemma 1.2 to the case $F = \mathbf{C}(Y)$, $E = \mathbf{C}\big(D(X/Y)\big)$ and $K = \mathbf{C}(X)$, we have $\theta \in \mathbf{C}(X)$ and $b \in \mathbf{C}(Y)$ such that

$$\theta^n = \varphi \in \mathbf{C}\big(D(X/Y)\big) \text{ and } \varphi\sigma^*\varphi = \beta_1^*b^n, \quad b \in \mathbf{C}(Y).$$

In order to find the three divisors stated in Proposition 0.5, we shall look into the divisors $(\varphi)_0$ and $(\varphi)_\infty$. Let $(\varphi)_0 = \sum_i \mu_i D_i^{(0)}$ and $(\varphi)_\infty = \sum_j \nu_j D_j^{(\infty)}$ denote the decomposition into irreducible components of $(\varphi)_0$ and $(\varphi)_\infty$, respectively. Put $\mu_i = \mu_i' + n[\frac{\mu_i}{n}]$ and $\nu_j = \nu_j' + n[\frac{\nu_j}{n}]$ where $0 \le \mu_i', \nu_j' \le n-1$. (Here [ ] means the greatest integer function.) We shall first investigate the divisor $(\varphi)_0$.

LEMMA 2.1. *Let $D_{i_1}^{(0)}$ be an irreducible component of $(\varphi)_0$ such that $\mu_i' \neq 0$. Then $\sigma^*D_{i_1}^{(0)} \neq D_{i_1}^{(0)}$.*

PROOF. Suppose $\sigma^*D_{i_1}^{(0)} = D_{i_1}^{(0)}$. Then, as $(\varphi\sigma^*\varphi)_0 = \sum_i \mu_i(D_i^{(0)} + \sigma^*D_i^{(0)}) = n(\beta_1^*b)_0$, we have $2\mu_{i_1} \equiv 0 \pmod{n}$. Hence we have $2\mu_{i_1}' = n$; but this is impossible because n is odd. $\blacksquare$

LEMMA 2.2. *Let $D_{i_1}^{(0)}$ be the same as above. Then $\sigma^*D_{i_1}^{(0)}$ is either an irreducible component of $(\varphi)_0$, or that of $(\varphi)_\infty$. Moreover, (i) if $\sigma^*D_{i_1}^{(0)}$ is an irreducible component of $(\varphi)_0$, then its coefficient, μ_{i_2}, satisfies $\mu_{i_1}' + \mu_{i_2}' \equiv 0 \pmod{n}$, and (ii) if $\sigma^*D_{i_1}^{(0)}$ is an irreducible component of $(\varphi)_\infty$, then its coefficient, ν_{i_1}, satisfies $\mu_{i_1} \equiv \nu_{i_1} \pmod{n}$, that is, $\mu_{i_1}' = \nu_{i_1}'$.*

PROOF. Assume that $\sigma^*D_{i_1}^{(0)}$ is neither an irreducible component of $(\varphi)_0$ nor of $(\varphi)_\infty$. Then the coefficients of the divisors $D_{i_1}^{(0)}$ and $\sigma^*D_{i_1}^{(0)}$ in $(\varphi\sigma^*\varphi)_0$ are both equal to μ_i. Since $(\varphi\sigma^*\varphi) = n(\beta_1^*b)$, we have $\mu_{i_1} \equiv 0 \pmod{n}$. But this contradicts our choice of $D_i^{(0)}$. Therefore $\sigma^*D_{i_1}^{(0)}$ is either an irreducible component of $(\varphi)_0$, or of $(\varphi)_\infty$. The statement on the coefficients easily follows from the identity $(\varphi\sigma^*\varphi) = n(\beta_1^*b)$. $\blacksquare$

For any irreducible component $D_j^{(\infty)}$ of $(\varphi)_\infty$ with non-zero ν_j', similar results to those in Lemmas 2.1 and 2.2 hold. Furthermore, since n is odd and $\sigma^2 = \mathrm{id}$, we may assume $\mu_{i_1}' \le \frac{n-1}{2}$. (Just replace $D_{i_1}^{(0)}$ by $\sigma^*D_{i_1}^{(0)}$, if necessary.) Also, if $D_j^{(\infty)}$ with non-zero ν_j' is the image of some $D_i^{(0)}$ by σ^*, we rewrite $D_j^{(\infty)}$ by $\sigma^*D_i^{(0)}$.

Combining all results so far, we rewrite $(\varphi)_0$ and $(\varphi)_\infty$ as follows:

$$(\varphi)_0 = \sum_{\substack{1\le\mu_i'\le n-1\\ \sigma^*D_i^{(0)}\not\subset(\varphi)_0}} \mu_i' D_i^{(0)} + \sum_{\substack{1\le\mu_i'\le\frac{n-1}{2}\\ \sigma^*D_i^{(0)}\subset(\varphi)_0}} \left(\mu_i' D_i^{(0)} + (n-\mu_i')\sigma^* D_i^{(0)}\right) + n\sum_{\mu_i''\neq 0} \mu_i'' D_i^{(0)},$$

$$(\varphi)_\infty = \sigma^*\Big(\sum_{\substack{1\le\mu_i'\le n-1\\ \sigma^*D_i^{(0)}\not\subset(\varphi)_0}} \mu_i' D_i^{(0)}\Big) + \sum_{\substack{1\le\nu_j'\le\frac{n-1}{2}\\ \sigma^*D_j^{(\infty)}\subset(\varphi)_\infty}} \left(\nu_j' D_j^{(\infty)} + (n-\nu_j')\sigma^* D_j^{(\infty)}\right) + n\sum_{\nu_j''\neq 0} \nu_j'' D_j^{(\infty)}.$$

Now, we define three effective divisors D_1, D_2, and D_3 as follows:

$$D_1 = \sum_{1\le\mu_i'\le\frac{n-1}{2}} \mu_i' D_i^{(0)} + \sum_{\substack{\frac{n-1}{2}<\mu_i'\le n-1\\ \sigma^*D_i^{(0)}\not\subset(\varphi)_0}} (n-\mu_i')\sigma^* D_i^{(0)} + \sum_{\substack{1\le\nu_j'\le\frac{n-1}{2}\\ \sigma^*D_j^{(0)}\subset(\varphi)_\infty}} \nu_j'\sigma^* D_j^{(\infty)},$$

$$D_2 = \sum_{\substack{\frac{n-1}{2}<\mu_i'\le n-1\\ \sigma^*D_i^{(0)}\not\subset(\varphi)_0}} D_i^{(0)} + \sum_{\substack{1\le\mu_i'<\frac{n-1}{2}\\ \sigma^*D_i^{(0)}\subset(\varphi)_0}} \sigma^* D_i^{(0)} + \sum_{\mu_i''\neq 0} \mu'' D_i^{(0)},$$

$$D_3 = \sum_{\substack{1\le\nu_j'\le\frac{n-1}{2}\\ \sigma^*D_j^{(\infty)}\subset(\varphi)_\infty}} \sigma^* D_j^{(\infty)} + \sum_{\substack{\frac{n-1}{2}<\mu_i'\le n-1\\ \sigma^*D_i^{(0)}\not\subset(\varphi)_0}} \sigma^* D_i^{(0)} + \sum_{\nu_j''\neq 0} \nu_j'' D_j^{(\infty)}.$$

Since $(\varphi) = (D_1 + nD_2) - (\sigma^* D_1 + nD_3)$, it is clear that these three divisors satisfy the conditions (i), (ii), (iii) and (iv). ∎

COROLLARY 2.3. *Let X be a dihedral $\mathcal{D}_{2p}$ (p : odd prime) covering of Y with the covering morphism $\pi: X \longrightarrow Y$. Suppose that variety $D(X/Y)$ is smooth. Then we can choose the divisor D_1 with $a_1 = 1$.*

PROOF. Let $D_1 = \sum_i a_i D_i^{(1)}$. As p is prime, there exists m with $1 \le m \le p-1$ such that $ma_1 \equiv 1 \pmod p$. Since $\mathbf{C}\big(D(X/Y)\big)(\theta) = \mathbf{C}\big(D(X/Y)\big)(\theta^m)$, we can replace θ and φ by θ^m and φ^m, respectively. Then we get new three effective divisors D_1', D_2' and D_3' satisfying the three properties in Proposition 0.5. This D_1' is the desired divisor. ∎

COROLLARY 2.4. *Let Σ be a smooth projective surface. Let S be a dihedral $\mathcal{D}_{2n}$ (n : odd) covering of Σ with $\pi: S \longrightarrow \Sigma$. Let D be an irreducible component of $\Delta(S/\Sigma)$ such that $D \not\subset \Delta\big(D(S/\Sigma)/\Sigma\big)$. Then there exists an irreducible divisor D' on $D(S/\Sigma)$ such that (i) D' and σ^*D' have no common component, and (ii) $\beta_1^* D = D' + \sigma^* D'$.*

This corollary is straightforward from Proposition 0.5.

COROLLARY 2.5. *Using the same notation as in Corollary 2.4, any intersection point between D and $\Delta\big(D(S/\Sigma)/\Sigma\big)$ has the multiplicity ≥ 2.*

PROOF. Assume that D meets $\Delta\big(D(S/\Sigma)/\Sigma\big)$ at a point P transversely. Then $\beta_1^* D$ is smooth around $\beta^{-1}(P)$; but this contradicts Corollary 2.4. ∎

3. **Construction of dihedral Galois coverings for $\mathcal{D}_{2n}$, n : even.**

PROOF OF PROPOSITION 0.6. By the assumption (d), there exists a rational function $\varphi \in \mathbf{C}(Z)$ which satisfies $(\varphi)_0 = D_1 + \frac{n}{2}D_2 + nD_3$ and $(\varphi)_\infty = \sigma^* D_1 + nD_4$. Let b be the rational function in the assumption (e), and put $\tilde{\varphi} = f^* b^{\frac{n(r_0-1)}{2r_0}} \varphi$. Then we have

CLAIM. *The polynomial $x^n - \tilde{\varphi}$ is irreducible over* $\mathbf{C}(Z)$.

PROOF OF CLAIM. Suppose that $x^n - \tilde{\varphi} = h_1(x)h_2(x)$, $\deg h_i(x) \geq 1$. By the similar argument to that of the proof of Claim in Proposition 0.4, there exists a rational function $\eta \in \mathbf{C}(Z)$ such that $\tilde{\varphi} = \eta^{\frac{n}{d}}$, $d =$ the greatest common divisor of $\deg h_1(x)$ and n. From this equality, we have

$$D_1 - \sigma^* D_1 + \frac{n}{2}D_2 = \frac{n}{d}\left\{(\eta) + d(D_4 - D_3) + \frac{d(r_0-1)}{2}\{(D_2 + D_3 + \sigma^* D_3) - (D_4 + \sigma^* D_4)\}\right\}.$$

Hence, $\frac{n}{d} \mid \frac{n}{2}$ and $\frac{n}{d} \mid a_i$ for every i, but this contradicts to the assumption (b) as $\frac{n}{d} > 1$.

Let $K = \mathbf{C}(Z)(\xi)$ where ξ is an n-th root of $\tilde{\varphi}$. Since $\sigma^* \tilde{\varphi} = f^* b^n / \tilde{\varphi}$, by Lemma 1.1, K is a dihedral Galois extension of $\mathbf{C}(Y)$ with the Galois group $\mathcal{D}_{2n}$.

Let X be the K-normalization of Y. It is a finite Galois covering of Y with the Galois group $\mathcal{D}_{2n}$ and by Lemma 1.1, $D(X/Y) = Z$. The statement on the branch locus easily follows from the assumptions (a), (c) and

$$(\tilde{\varphi}) = (D_1 - \sigma^* D_1) + \frac{n}{2}D_2 + n(D_3 - D_4) + \frac{n(r_0-1)}{2}\{(D_2 + D_3 + \sigma^* D_3) - (D_4 + \sigma^* D_4)\}. \blacksquare$$

REMARK 3.1. If Y is simply connected, the condition (e) of Proposition 0.6 is automatically satisfied for $r_0 = 1$.

PROOF. Let B_1 and B_2 be divisors on Y such that $f^* B_1 = D_2 + D_3 + \sigma^* D_3$ and $f^* B_2 = D_4 + \sigma^* D_4$, respectively. Then $nf^* B_1 - nf^* B_2 \sim 0$. Taking the push-forward, this implies

$$f_*(nf^* B_1 - nf^* B_2) = 2n(B_1 - B_2) \sim 0.$$

As Y is simply connected, the Picard group, Pic(Y), has no torsion. Hence $B_1 - B_2 \sim 0$, and there exists a rational function $b \in \mathbf{C}(Y)$ such that $(f^* b) = f^* B_1 - f^* B_2 = (D_2 + D_3 + \sigma^* D_3) - (D_4 + \sigma^* D_4)$. $\blacksquare$

Next we shall prove Proposition 0.7.

PROOF OF PROPOSITION 0.7. By Lemma 1.2, there exists an element $\theta \in \mathbf{C}(X)$ such that

$$\theta^n = \varphi \in \mathbf{C}\big(D(X/Y)\big) \text{ and } \theta^n(\sigma^*\theta)^n = \beta_1^* b^n, \quad b \in \mathbf{C}(Y).$$

Let $(\varphi)_0 = \sum_i \mu_i D_i^{(0)}$ and $(\varphi)_\infty = \sum_j \nu_j D_j^{(\infty)}$ denote the decomposition into irreducible components, respectively. We rewrite these decompositions in the following way. The notations μ_i' and ν_i' are the same as those in the proof of Proposition 0.5. Let $D_i^{(0)}$ be an irreducible component of $(\varphi)_0$ such that $\sigma^* D_i^{(0)} = D_i^{(0)}$. If $\operatorname{Supp} D_i^{(0)} \not\subset \operatorname{Supp}\Big(\beta_1^* \Delta\big(D(X/Y)/Y\big)\Big)$, then there exists an irreducible divisor $B_i^{(0)}$ on Y such that

$\beta_1^* B_i^{(0)} = D_i^{(0)}$, while if $\operatorname{Supp} D_i^{(0)} \subset \operatorname{Supp}\Big(\beta_1^*\Delta\big(D(X/Y)/Y\big)\Big)$, then there exists an irreducible divisor $B_i'^{(0)}$ on Y such that $\beta_1^* B_i'^{(0)} = 2D_i^{(0)}$. Since both of $B_i^{(0)}$ and $B_i'^{(0)}$ are irreducible components of $(b)_0$, comparing the coefficients of both sides of $(\varphi\sigma^*\varphi) = n(\beta_1^* b)$, we have $\mu_i' = \frac{n}{2}$ or 0 for the former case, and $\mu_i' = 0$ for the latter case. For an irreducible component $D_j^{(\infty)}$ with $\sigma^* D_j^{(\infty)} = D_j^{(\infty)}$, there are also two cases: (i) $\operatorname{Supp} D_j^{(\infty)} \not\subset \operatorname{Supp} \beta_1^*\Delta\big(D(X/Y)/Y$ or (ii) $\operatorname{Supp} D_j^{(\infty)} \subset \operatorname{Supp} \beta_1^*\Delta\big(D(X/Y)/Y\big)$. By using the same equality $(\varphi\sigma^*\varphi) = n(\beta_1^* b)$, we also get $\nu_j' = \frac{n}{2}$ or 0 and if $\nu_j' = \frac{n}{2}$, there exists a divisor $B_j^{(\infty)}$ such that $\beta_1^* B_j^{(\infty)} = D_j^{(\infty)}$.

For any irreducible component $D_i^{(0)}$ (resp. $D_j^{(\infty)}$) of $(\varphi)_0$ (resp. φ_∞) with $\sigma^* D_i^{(0)} \neq D_i^{(0)}$ (resp. $\sigma^* D_j^{(\infty)} \neq D_j^{(\infty)}$), the similar results to those in Lemma 2.2 hold. Therefore we can rewrite $(\varphi)_0$ and $(\varphi)_\infty$ as follows:

$$
\begin{aligned}
(\varphi)_0 = &\sum_{\substack{1\le\mu_i'\le n-1\\ \sigma^* D_i^{(0)}\not\subset(\varphi)_0}} \mu_i' D_i^{(0)} + \sum_{\substack{1\le\mu_i'\le\frac{n}{2}\\ \sigma^* D_i^{(0)}\subset(\varphi)_0\\ \sigma^* D^{(0)}\neq D^{(0)}}} \big(\mu_i' D_i^{(0)} + (n-\mu_i')\sigma^* D_i^{(0)}\big)\\
&+\frac{n}{2}\sum_{\substack{\mu_i'=\frac{n}{2}\\ \sigma^* D_i^{(0)}=D_i^{(0)}}} D_i^{(0)} + n\sum_{\mu_i''\neq 0}\mu_i'' D_i^{(0)},\\
(\varphi)_\infty = &\,\sigma^*\Big(\sum_{\substack{1\le\mu_i'\le n-1\\ \sigma^* D_i^{(0)}\not\subset(\varphi)_0}} \mu_i' D_i^{(0)}\Big) + \sum_{\substack{1\le\nu_i'\le\frac{n}{2}\\ \sigma^* D_j^{(\infty)}\subset(\varphi)_\infty\\ \sigma^* D^{(\infty)}\neq D^{(\infty)}}} \big(\nu_j' D_j^{(\infty)} + (n-\nu_j')\sigma^* D_j^{\infty}\big)\\
&+\frac{n}{2}\sum_{\substack{\nu_j'=\frac{n}{2}\\ \sigma^* D_j^{(\infty)}=D_j^{(\infty)}}} D_j^{(\infty)} + n\sum_{\nu_j''\neq 0}\nu_j'' D_j^{(\infty)}.
\end{aligned}
$$

Now we define effective divisors as follows:

CASE (A). There exists no irreducible component $D_i^{(0)}$ (resp. $D_j^{(\infty)}$) satisfying $\sigma^* D_i^{(0)} = D_i^{(0)}$ (resp. $\sigma^* D_j^{(\infty)} = D_j^{(\infty)}$) with $\mu_i' = n/2$ (resp. $\nu_j' = n/2$). In this case, we set $D_2 = 0$, and let D_3 (resp. D_4) be the divisor as D_2 (resp. D_3) in Proposition 0.5.

CASE (B). There exist some irreducible components $D_i^{(0)}$ (resp. $D_j^{(\infty)}$) satisfying $\sigma^* D_i^{(0)} = D_i^{(0)}$ (resp. $\sigma^* D_j^{(\infty)} = D_j^{(\infty)}$) with $\mu_i' = n/2$ (resp. $\nu_j' = n/2$). In this case, we define four divisors D_1, D_2, D_3, and D_4 as follows:

$$
\begin{aligned}
D_1 &= \sum_{1\le\mu_i'\le\frac{n}{2}} \mu_i' D_i^{(0)} + \sum_{\substack{\frac{n}{2}<\mu_i'\le n-1\\ \sigma^* D_i^{(0)}\not\subset(\varphi)_0}} (n-\mu_i')\sigma^* D_i^{(0)} + \sum_{\substack{1\le\nu_j'<\frac{n}{2}\\ \sigma^* D_j^{(\infty)}\subset(\varphi)_\infty}} \nu_j'\sigma^* D_j^{(\infty)},\\
D_2 &= \sum_{\substack{\mu_i'=\frac{n}{2}\\ \sigma^* D_i^{(0)}=D_i^{(0)}}} D_i^{(0)} + \sum_{\substack{\nu_j'=\frac{n}{2}\\ \sigma^* D_j^{(\infty)}=D_j^{(\infty)}}} D_j^{(\infty)},
\end{aligned}
$$

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$$D_3 = \sum_{\substack{\frac{n}{2}<\mu_i'\le n-1 \\ \sigma^* D_i^{(0)} \not\subset (\varphi)_0}} D_i^{(0)} + \sum_{\substack{1\le\mu_i'<\frac{n}{2} \\ \sigma^* D_i^{(0)} \subset (\varphi)_0}} \sigma^* D_i^{(0)} + \sum_{\mu_i''\neq 0} \mu_i'' D_i^{(0)},$$

$$D_4 = \sum_{\substack{1\le\nu_j'\le\frac{n}{2} \\ \sigma^* D_j^{(\infty)} \subset (\varphi)_\infty}} \sigma^* D_j^{(\infty)} + \sum_{\substack{\frac{n}{2}<\mu_i'\le n-1 \\ \sigma^* D_i^{(0)} \not\subset (\varphi)_0}} \sigma^* D_i^{(0)} + \sum_{\substack{\nu_j'=\frac{n}{2} \\ \sigma^* D_j^{(\infty)} = D_j^{(\infty)}}} \sigma^* D_j^{(\infty)} + \sum_{\nu_j''\neq 0} \nu_j'' D_j^{(\infty)}.$$

Since $(\varphi) = (D_1 + nD_2) - (\sigma^* D_1 + nD_3)$ (resp. $(D_1 + \frac{n}{2}D_2 + nD_3) - (\sigma^* D_1 + nD_4)$) for Case (A) (resp. Case (B)), it is clear that these divisors satisfy the required conditions. ∎

4. **Dihedral $\mathcal{D}_{2p}$ (p : odd prime) coverings of $\mathbf{P}^2$ branched along curves of degree ≤ 4.** The purpose of this section is to study dihedral $\mathcal{D}_{2p}$ coverings of $\mathbf{P}^2$. Let $\pi: S \longrightarrow \mathbf{P}^2$ be a dihedral $\mathcal{D}_{2p}$ covering and let $\Delta(S/\mathbf{P}^2)$ be the branch locus. We shall consider the cases $\deg \Delta(S/\mathbf{P}^2) \le 4$. As the fundamental group $\pi_1(\mathbf{P}^2 \setminus C)$ is an abelian group for a plane curve of degree ≤ 2, our problem is the cases $\deg \Delta(S/\mathbf{P}^2) = 3, 4$.

4.1. We shall first consider the case $\deg \Delta(S/\mathbf{P}^2) = 3$. As the branch locus $\Delta\big(D(S/\mathbf{P}^2)/\mathbf{P}^2\big)$ of β_1 is a curve of even degree, $\Delta\big(D(S/\mathbf{P}^2)/\mathbf{P}^2\big)$ is a smooth conic or two distinct lines. Hence, for the former case, $\Delta(S/\mathbf{P}^2)$ is a smooth conic and a tangent line to it by Corollary 2.5. For the latter case, by considering the canonical resolution of $D(S/\mathbf{P}^2)$, we can apply Corollary 2.5 to this case. Looking into the inverse process of the canonical resolution, we know that $\Delta(S/\mathbf{P}^2)$ is three distinct lines intersecting at one point.

In case that $\Delta\big(D(S/\mathbf{P}^2)/\mathbf{P}^2\big)$ is a smooth conic, $D(S/\mathbf{P}^2)$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. Let l be the line component of $\Delta(S/\mathbf{P}^2)$. Then $\beta_1^* l$ has two irreducible components l_1 and $\sigma^* l_1$ generating $\mathrm{NS}\big(D(S/\mathbf{P}^2)\big)$. By Proposition 0.5(iii), $l_1 - \sigma^* l_1$ is p-divisible in $\mathrm{NS}\big(D(S/\mathbf{P}^2)\big)$; but this is impossible. Therefore, this case does not occur.

In case that $\Delta(S/\mathbf{P}^2)$ is distinct three lines intersecting at one point, we can replace our problem by that of dihedral $\mathcal{D}_{2p}$ covering of $\mathbf{P}^1$ branched at three points by blowing up the intersecting point. Using well-known results on coverings of $\mathbf{P}^1$ (*cf.* Namba [11] pp. 29–31), we can easily show that there exists the desired covering.

4.2. Now we go on to the problem on dihedral $\mathcal{D}_{2p}$ coverings of $\mathbf{P}^2$ branched along quartic curves. Our problem is divided in two cases as follows:

(A) $\Delta\big(D(S/\mathbf{P}^2)/\mathbf{P}^2\big)$ is a conic,

(B) $\Delta\big(D(S/\mathbf{P}^2)/\mathbf{P}^2\big)$ is a quartic.

For the Case (A), $\Delta\big(D(S/\mathbf{P}^2)/\mathbf{P}^2\big)$ is either a smooth conic or two distinct lines. In either case, using Proposition 0.5 and Corollary 2.5, it is straightforward to show that $\Delta(S/\mathbf{P})^2$ is one of the following:

(i) $\Delta(S/\mathbf{P}^2)$ is two smooth conics tangent at two points,

(ii) $\Delta(S/\mathbf{P}^2)$ is two smooth conics tangent at one point,

(iii) $\Delta(S/\mathbf{P}^2)$ is a smooth conic and two distinct lines tangent to the conic; $\Delta\big(D(S/\mathbf{P}^2)/\mathbf{P}^2\big)$ is the smooth conic,

(iv) $\Delta(S/\mathbf{P}^2)$ is a smooth conic and two distinct lines tangent to the conic; $\Delta\big(D(S/\mathbf{P}^2)/\mathbf{P}^2\big)$ is the two lines, or

(v) $\Delta(S/\mathbf{P}^2)$ is four distinct lines; $\Delta\big(D(S/\mathbf{P}^2)/\mathbf{P}^2\big)$ is two of the four lines.

The Case (B) is rather complicated. In case that $\Delta(S/\mathbf{P}^2)$ is four distinct lines intersecting at one point, we can reduce our problem to that of $\mathbf{P}^1$ by blowing up at the intersection point, and we can easily show the existence of the covering. Therefore, in what follows, we always assume that

(*) $\Delta(S/\mathbf{P}^2)$ is not four distinct lines intersecting at one point.

We choose a smooth point x on $\Delta(S/\mathbf{P}^2)$ such that

(a) if $\Delta(S/\mathbf{P}^2)$ is not distinct four lines, then the tangent line, l_x, at x intersects $\Delta(S/\mathbf{P}^2)$ at two other distinct points,

(b) if $\Delta(S/\mathbf{P}^2)$ is four distinct lines, then the line component passing through x, which we denote by l_x, intersects other components at three distinct points.

Let $\mu_1\colon \mathbf{P}^2_x \longrightarrow \mathbf{P}^2$ be a blowing-up at x, and let B and E denote the proper transform of $\Delta(S/\mathbf{P}^2)$ and the exceptional divisor, respectively. Next let $\mu_2\colon \hat{\mathbf{P}}^2 \longrightarrow \mathbf{P}^2_x$ be a blowing-up at $B\cap E$, and let B_1, E_1 and E_2 denote the strict transform of B, E and the exceptional divisor of μ_2, respectively. Let $D(\widetilde{S/\mathbf{P}^2})$ be the $\mathbf{C}\big(D(S/\mathbf{P}^2)\big)$-normalization of $\hat{\mathbf{P}}^2$. $D(\widetilde{S/\mathbf{P}^2})$ is a finite normal double covering of $\hat{\mathbf{P}}^2$ branched along B_1+E_1. Let $\mathcal{E}$ be the canonical resolution of $D(\widetilde{S/\mathbf{P}^2})$. Then $\mathcal{E}$ satisfies the following:

(i) $\mathcal{E}$ is a finite double covering of a smooth surface Σ obtained from $\hat{\mathbf{P}}^2$ by a succession of blowing-ups. We denote the covering morphism and one from Σ to $\hat{\mathbf{P}}^2$ by $\tilde{\beta}_1$ and g, respectively.

(ii) Let $\tilde{\sigma}$ be the involution coming from the covering transformation of β_1 on $D(S/\mathbf{P}^2)$. Then $\tilde{\beta}_1$ is the quotient morphism by $\tilde{\sigma}$.

By Miranda and Persson [10], §6, $\mathcal{E}$ is a rational elliptic surface with a section s_0 coming from E_1. Its elliptic fibration comes from the family of lines passing through the point x. For this reason, x is called the *distinguished point*. Note that $\mathcal{E}$ has a singular fiber of type I_2 if x satisfies the condition (a), while $\mathcal{E}$ has a singular fiber of type I_0^* if x satisfies the condition (b). In either case, the singular fiber is determined by $g^*\Big(\big((\mu_1\circ\mu_2)^*l_x\big)\backslash E_1\Big)$. Other singular fibers of $\mathcal{E}$ arise from the singularities of $D(\widetilde{S/\mathbf{P}^2})$. They are determined by the singularities of B_1. (For explicit relations between singular fibers and singularities of B_1, see Miranda and Persson [10], §6).

Let $\tilde{S}$ be the $\mathbf{C}(S)$-normalization of Σ. Then $\tilde{S}$ is a dihedral $\mathcal{D}_{2p}$ covering with $D(\tilde{S}/\Sigma)=\mathcal{E}$. We denote the covering morphism from $\tilde{S}$ to Σ and one from $\tilde{S}$ to $\mathcal{E}$ by $\tilde{\pi}$ and $\tilde{\beta}_2$, respectively. Now the branch locus of $\tilde{\pi}$ consists of B_1, E_1 and some of exceptional divisors of g. Thus, S is the Stein factorization of the composite morphism from $\tilde{S}$ to $\mathbf{P}^2$. Therefore, it suffices to study the covering $\tilde{\pi}\colon\tilde{S}\longrightarrow\Sigma$ to investigate $\pi\colon S\longrightarrow\mathbf{P}^2$.

Under these circumstances, using Proposition 0.5 and the fact that a rational elliptic surface is simply connected, we find three effective divisors D_1, D_2 and D_3 on $\mathcal{E}$ such that

(i) all coefficients of the irreducible components of D_1 are positive and $\leq \frac{p-1}{2}$,

(ii) $D_1 - \tilde{\sigma}^* D_1 \sim p(D_2 - D_3)$.

(iii) $\mathrm{Supp}(D_1 + \tilde{\sigma}^* D_1)$ is the branch locus of $\tilde{\beta}_2$.

The branch locus of $\tilde{\pi}$ consists of B_1, E_1 and some of the exceptional divisors of g, and $\mathrm{Supp}(B_1 + E_1)$ is the branch locus of $\tilde{\beta}_1$. Hence, the branch locus of $\tilde{\beta}_2$ consists of the inverse images of exceptional divisors of g, which are irreducible components of singular fibers not intersecting s_0. Therefore, the condition (iii) means that all irreducible components of D_1 and $\tilde{\sigma}^* D_1$ are irreducible components of singular fibers not intersecting the section s_0. Under these circumstances, we have

CLAIM. *Let T be a subgroup of the Néron-Severi group,* $\mathrm{NS}(\mathcal{E})$, *generated by s_0 and all irreducible components of fibers. Then, $D_2 - D_3$ is a divisor satisfying that $D_2 - D_3 \notin T$; $p(D_2 - D_3) \in T$.*

PROOF OF CLAIM. By Shioda [14], Proposition 2.3, T is a torsion free group generated by s_0, the class of a fiber, F, of the elliptic fibration, and all irreducible components of the singular fibers not intersecting s_0, which we denote by Θ_i, $1 \leq i \leq rkT - 2$. Suppose that $D_2 - D_3 \in T$. Then we have

$$D_2 - D_3 \approx m s_0 + nF + \sum_i c_i \Theta_i.$$

where m, n and c_i's are integers.

On the other hand, as every irreducible component of D_1 and $\tilde{\sigma}^* D_1$ is one of Θ_i's, we can rewrite $D_1 - \tilde{\sigma}^* D_1$ as follows:

$$D_1 - \tilde{\sigma}^* D_1 = \sum_i a_i' \Theta_i$$

where a_i' are integers with $1 \leq |a_i'| \leq \frac{p-1}{2}$ by the condition (i). Using these expressions, by the condition (ii), we have

$$pms_0 + pnF + \sum_i (pc_i - a_i')\Theta_i \approx 0.$$

Since $pc_i - a_i' \neq 0$, this gives a non-trivial relation for divisors, s_0, F and Θ_i's. But this is a contradiction as these divisors form a basis of T.

By Shioda [14], Theorem 1.3, the Claim means that the rational elliptic surface $\mathcal{E}$ has a p-torsion elements in its Mordell-Weill group, $\mathrm{MW}(\mathcal{E})$. In our case, p is an odd prime, and $\mathcal{E}$ is a rational elliptic surface with at least one singular fiber of type I_2 or I_0^*. Therefore, by Persson [13], it follows that $p = 3$ and the configuration of the singular fibers is one of the following:

$$IV, I_3, I_3, I_2, \quad I_3, I_3, I_3, I_2, I_1, \quad \text{or} \quad I_6, I_3, I_2, I_1.$$

Looking into the inverse process of the canonical resolution, we have

$\Delta(S/\mathbf{P}^2)$ is irreducible and has three $(2, 3)$ cusps for the first two cases of $p = 3$,

$\Delta(S/\mathbf{P}^2)$ is a cubic with one cusp and a line, and the line is the tangent at an inflection point of the cubic for the last case of $p = 3$.

We summarize the above discussion with the following theorem.

THEOREM 4.3. *Let $\pi: S \longrightarrow \mathbf{P}^2$ be a dihedral $\mathcal{D}_{2p}$ (p : odd prime) covering of $\mathbf{P}^2$, and let $\Delta(S/\mathbf{P}^2)$ be the branch locus of π. Then, $\deg \Delta(S/\mathbf{P}^2) \geq 3$. Furthermore, if $\deg \Delta(S/\mathbf{P}^2) \leq 4$, $\Delta(S/\mathbf{P}^2)$ is one of the following:*

If $\deg \Delta(S/\mathbf{P}^2) = 3$, $\Delta(S/\mathbf{P}^2)$ is three distinct lines intersecting at one point.

If $\deg \Delta(S/\mathbf{P}^2) = 4$ and $\Delta\big(D(S/\mathbf{P}^2)/\mathbf{P}^2\big)$ is a conic,

(i) $\Delta(S/\mathbf{P}^2)$ is two distinct smooth conics tangent at two points,

(ii) $\Delta(S/\mathbf{P}^2)$ is two distinct smooth conics tangent at one point,

(iii) $\Delta(S/\mathbf{P}^2)$ is a distinct smooth conic and two distinct lines tangent to the conic, or

(iv) $\Delta(S/\mathbf{P}^2)$ is four distinct lines intersecting at one point.

If $\deg \Delta(S/\mathbf{P}^2) = 4$ and $\Delta\big(D(S/\mathbf{P}^2)/\mathbf{P}^2\big) = \Delta(S/\mathbf{P}^2)$

(v) $p = 3$ and $\Delta(S/\mathbf{P}^2)$ is an irreducible quartic curve with three cusps,

(vi) $p = 3$ and $\Delta(S/\mathbf{P}^2)$ is a cubic curve with a cusp and a line; the line is the tangent line at an inflection point of the cubic curve, or

(vii) p is arbitrary and $\Delta(S/\mathbf{P}^2)$ is four distinct lines intersecting at one point.

5. **Existence of coverings.** The notation of §4 is utilized in the following. In Theorem 4.3, we have characterized all possible curves of degree 3 and 4 that can be the branch loci of dihedral $\mathcal{D}_{2p}$ coverings of $\mathbf{P}^2$. For any curve described in Theorem 4.3 except for the cases (v) and (vi), we have already seen that there exists a dihedral $\mathcal{D}_{2p}$ covering branched along it. The goal of this section is to show that, using the results in §2, there exists a dihedral $\mathcal{D}_6$ covering for the remaining cases. Here we recall that, in order to get the desired covering S, it suffices to show that there exists $\tilde{S}$.

We first consider the case (v). Choosing the distinguished point x such that (i) no line that meets $\Delta(S/\mathbf{P}^2)$ at a cusp with multiplicity 3 passes through x, and (ii) the tangent line at x intersects $\Delta(S/\mathbf{P}^2)$ at two other distinct points, we may assume that the configuration of the singular fibers of the elliptic surface $\mathcal{E}$ is I_3, I_3, I_3, I_2, I_1. We label the irreducible components of each singular fiber as in Figure 1.

Consider two sections l_1, l_2 as above. To get these sections, choose two lines each of which pass through two of the three cusps of $\Delta(S/\mathbf{P}^2)$ and take two suitable sections determined by these two lines. Let $\psi: \mathrm{NS}(\mathcal{E}) \longrightarrow \mathrm{MW}(\mathcal{E})$ be the group homomorphism from the Néron-Severi group to the Mordell-Weil group introduced by Shioda [14] and let $\langle\, ,\, \rangle$ be Shioda's height pairing on $\mathrm{MW}(\mathcal{E})$. (For the definition of the pairing, see Shioda [13].) For the sections l_1, l_2, we have $\langle (l_1 - l_2), \psi(l_1 - l_2)\rangle = 0$. Therefore, by Shioda [14] Lemma 8.1 and Theorem 8.4, we have

$$l_1 - l_2 \approx_{\mathbf{Q}} \frac{1}{3}(\tilde{\sigma}^*\Theta_1 - \Theta_1) - \frac{1}{3}(2\Theta_2 + \tilde{\sigma}^*\Theta_2) + \frac{1}{3}(\Theta_3 + 2\tilde{\sigma}^*\Theta_3).$$

This means

$$(\Theta_1 + \tilde{\sigma}^*\Theta_2 + \tilde{\sigma}^*\Theta_3) - \tilde{\sigma}^*(\Theta_1 + \tilde{\sigma}^*\Theta_2 + \tilde{\sigma}^*\Theta_3) \approx 3(l_2 + \tilde{\sigma}^*\Theta_3) - 3(l_1 + \Theta_2).$$

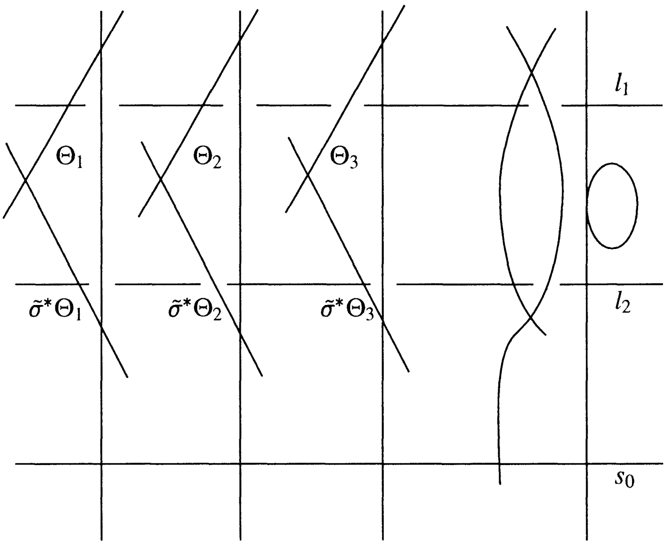

FIGURE 1

As a rational elliptic surface is simply connected, we can replace algebraic equivalence by linear equivalence. Now we put

$$D_1 = \Theta_1 + \tilde{\sigma}^*\Theta_2 + \tilde{\sigma}^*\Theta_3,$$
$$D_2 = l_1 + \Theta_2,$$
$$D_3 = l_2 + \tilde{\sigma}^*\Theta_3,$$

and apply Proposition 0.4 in case of $n = 3$ to these three divisors on $\mathcal{E}$. In this way, we obtain the desired covering.

Next we consider the case (vi). Choosing a general point on the cubic curve as the distinguished point x, we may assume that the configuration of the singular fibers of the elliptic surface $\mathcal{E}$ is I_6, I_3, I_2, I_1. We label the irreducible components of each singular fiber as in Figure 2.

Consider a section s as above. This section comes from the line passing through the cusp and the intersection point of the cubic and the line component of the branch locus. Using the same notation as in the case (v), we have $\langle\psi(s), \psi(s)\rangle = 0$. Therefore, by Shioda [14] Lemma 8.1 and Theorem 8.4, we have

$$s \approx_{\mathbf{Q}} s_0 + F - \frac{1}{6}(5\Theta_1^{(1)} + 4\Theta_2^{(1)} + 3\Theta_3^{(1)} + 2\tilde{\sigma}\Theta_2^{(1)} + \tilde{\sigma}^*\Theta_1^{(1)}) - \frac{1}{3}(2\Theta_1^{(2)} + \tilde{\sigma}^*\Theta_1^{(2)}) - \frac{1}{2}\Theta_1^{(3)},$$

where F is a fiber of the elliptic fibration. This means

$$(\Theta_1^{(1)} + \tilde{\sigma}^*\Theta_2^{(1)} + \tilde{\sigma}^*\Theta_1^{(2)}) - \tilde{\sigma}^*(\Theta_1^{(1)} + \tilde{\sigma}^*\Theta_2^{(1)} + \tilde{\sigma}^*\Theta_1^{(2)}) \approx 3(s + \Theta_1^{(1)}) - 3\tilde{\sigma}^*(s + \Theta_1^{(1)}).$$

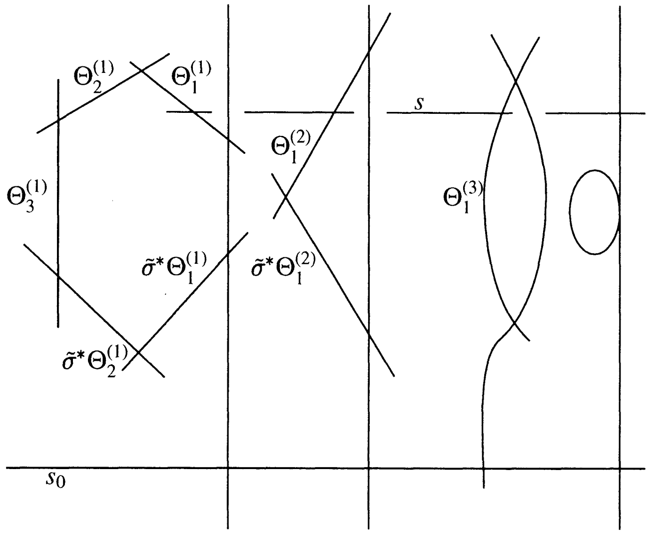

FIGURE 2

As a rational elliptic surface is simply connected, we can replace algebraic equivalence by linear equivalence. Now we put

$$D_1 = \Theta_1^{(1)} + \tilde{\sigma}^*\Theta_2^{(1)} + \tilde{\sigma}^*\Theta_1^{(2)},$$
$$D_2 = \tilde{\sigma}^*(s + \Theta_1^{(1)}),$$
$$D_3 = s + \Theta_1^{(1)},$$

and apply Proposition 0.4 in case of $n = 3$ to these three divisors on $\mathcal{E}$. Then we have the desired covering. Summing up the results in this section, we have

THEOREM 5.1. *For each case in Theorem 4.3, there exists a dihedral $\mathcal{D}_{2p}$ covering of* $\mathbf{P}^2$.

Combining Theorem 4.3 and Theorem 5.1, we have Theorem 0.9.

6. **Further examples.** In this section, we shall give examples of dihedral $\mathcal{D}_{2n}$ (n : even) covering of $\mathbf{P}^2$. Here a torsion element of the Mordell-Weil group of an elliptic surface also plays an important role. We shall use the same notation as in §4 and §5. In particular, the surfaces $\tilde{S}$, $\mathcal{E}$ and Σ mean the same surfaces as in §4.

EXAMPLE 6.1. Let Q be a reducible quartic in $\mathbf{P}^2$ consisting of two smooth conics, and the two components of Q tangent at two different points. Choose a smooth point x on Q in such a way that the tangent line at x intersects Q at two other points distinct from x. Let $f: Z \longrightarrow \mathbf{P}^2$ be a double covering branched along Q. Choosing x as the distinguished

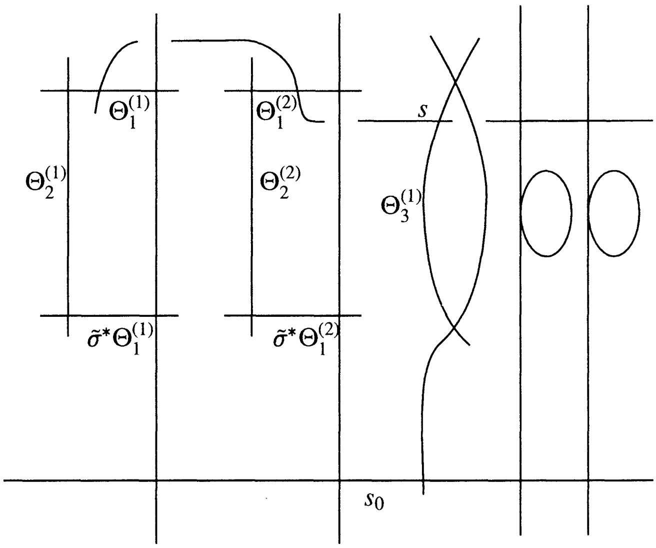

FIGURE 3

point, we get a rational elliptic surface $\mathcal{E}$ birationally equivalent to Z. The configuration of the singular fibers of $\mathcal{E}$ is I_4, I_4, I_2, I_1. We label the irreducible components of singular fibers as in Figure 3.

Consider a section s coming from the line passing through two singularities of Q. Then we have $\langle \psi(s), \psi(s) \rangle = 0$. Therefore, by the same Shioda result as before,

$$s \approx_{\mathbf{Q}} s_0 + F - \frac{1}{4}(3\Theta_1^{(1)} + 2\Theta_2^{(1)} + \tilde{\sigma}^*\Theta_1^{(1)}) - \frac{1}{4}(3\Theta_1^{(2)} + 2\Theta_2^{(2)} + \tilde{\sigma}^*\Theta_1^{(2)}) - \frac{1}{2}\Theta_1^{(3)},$$

where F is a fiber of the elliptic fibration. Therefore we have

$$4(s_0+F)-4(s+\Theta_1^{(1)}+\Theta_1^{(2)}) \approx (\tilde{\sigma}^*\Theta_1^{(1)}+\tilde{\sigma}^*\Theta_1^{(2)})-\tilde{\sigma}^*(\tilde{\sigma}^*\Theta_1^{(1)}+\tilde{\sigma}^*\Theta_1^{(2)})+2(\Theta_2^{(1)}+\Theta_2^{(2)}+\Theta_1^{(3)}).$$

As a rational elliptic surface is simply connected, we can replace algebraic equivalence by linear equivalence. Also, the number r_0 in Proposition 0.6 is 1 by Corollary 3.1. Therefore we can apply Proposition 0.6 in case of $n = 4$ to the following divisors D_1, D_2, D_3 and D_4 on $\mathcal{E}$.

$$\begin{aligned} D_1 &= \tilde{\sigma}^*\Theta_1^{(1)} + \tilde{\sigma}^*\Theta_1^{(2)}, \\ D_2 &= \Theta_2^{(1)} + \Theta_2^{(2)} + \Theta_1^{(3)}, \\ D_3 &= s + \Theta_1^{(1)} + \Theta_1^{(2)}, \\ D_4 &= s_0 + F. \end{aligned}$$

In this way, we obtain a dihedral $\mathcal{D}_8$ covering $\tilde{S}$ of Σ and this gives a dihedral $\mathcal{D}_8$ covering of $\mathbf{P}^2$ branched along Q with $\Delta\big(D(S/\mathbf{P}^2)/\mathbf{P}^2\big) = Q$.

EXAMPLE 6.2. Let Q be a quartic curve in the case (vi) in Theorem 0.9. The section s which we used in §5 to prove the existence of a dihedral $\mathcal{D}_6$ covering corresponds to a torsion element of order 6 in MW($\mathcal{E}$). Using this section, we obtain a dihedral $\mathcal{D}_{12}$ covering of $\mathbf{P}^2$ branched along Q satisfying $\Delta\big(D(S/\mathbf{P}^2)/\mathbf{P}\big) = Q$.

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Department of Mathematics
Kochi University
Kochi 780
Japan
e-mail: tokunaga@kdrs1.math.kochi-u.ac.jp