

PAPER

Semiframes: the algebra of semitopologies and actionable coalitions

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Abstract

We introduce semiframes (an algebraic structure) and investigate their duality with semitopologies (a topological one). Both semitopologies and semiframes are relatively recent developments, arising from a novel application of topological ideas to study decentralised computing systems. Semitopologies generalise topology by removing the condition that intersections of open sets are necessarily open. The motivation comes from identifying the notion of an *actionable coalition* in a distributed system – a set of participants with sufficient resources for its members to collaborate to take some action – with an *open set*, since just because two sets are actionable (have the resources to act) does not necessarily mean that their intersection is. We define notions of category and morphism and prove a categorical duality between (sober) semiframes and (spatial) semitopologies, and we investigate how key well-behavedness properties that are relevant to understanding decentralised systems transfer (or do not transfer) across the duality.

Keywords: semitopology; semiframes; decentralised systems; actionable coalitions; categorical duality

1. Introduction

This paper is about a duality between semiframes (an algebraic structure) and semitopologies (a topological one). We will define semiframes, study their duality with semitopologies and also motivate and define relevant well-behavedness conditions on semitopologies and investigate how these conditions dualise to algebraic conditions on semiframes.

Both semitopologies and semiframes are relatively recent developments, arising from a novel application of topological ideas to study decentralised computing systems. So while this paper is about pure mathematics, and it can be read as mathematics for its own sake, it would be nice to understand the motivation for why we looked at these structures in the first place.

1.1 The challenge of decentralised systems

Consider a computing system whose operation is distributed over many collaborating participants, like a blockchain. At a high level, how can we design algorithms that participants can run such that the overall system will act coherently in some appropriate sense?

We could just assume a central controller: a *dictator participant*. In this case, ‘collaborating’ just means doing as the dictator instructs, and ‘truth’ just means whatever the dictator says.

This is indeed how many distributed systems actually work, but it introduces tradeoffs of latency, scalability, availability and trust, which may or may not be acceptable depending on the



use case. Do we trust the dictator? What if it crashes or gets hacked? What if the network is slow? What if it is just busy? Sometimes, we need participants to be able, to a greater or lesser extent, to collaborate independently of central control.

This is the topic of *decentralised* systems, which are distributed systems that do *not* adopt a central controller. Decentralised systems are hard to design and implement because control itself is spread out,¹ but often this is the only architecture that is practical at scale and speed.

While blockchains are an example of decentralised systems,² these design issues arise repeatedly: for example, if we want to manage a system of drones, or satellites, or low-power devices. An ant colony is an interesting example of a (semi-)decentralised system in nature: while a queen ant exists, she does not dictate every action of every ant; for much of the time, individual ants work autonomously, yet they still manage to do so coherently and for the good of the colony. The human brain is another important example of a decentralised system. These systems are all inherently decentralised: they maintain some clear notion of global coherence (e.g., there is an identifiable thing called ‘an ant colony’ or ‘my sense of self’) all while their parts collaborate to perform significant actions independently of any explicit central control.

This leads us to consider the concept of what in Gabbay (2024, 2025) is called an *actionable coalition*: a set of participants in a decentralised system with sufficient resources to act (if the participants in the coalition choose to do so), without the involvement – that is, without the help, the permission or even the knowledge – of any other participants.³

Let us return to our ant colony and consider a column of foraging ants: what are the actionable coalitions here, where the ants’ task is to bring food back to the colony? When the column discovers some food, a relevant notion of actionable coalition is *any collection of ants with the physical strength to heave the food back to the hive*. Note that the ants do not need anybody to organise them into a group or to tell them what to do: they swarm over the food until some group of them manage to move the food – aha, an actionable coalition has assembled and agreed on their action! – and then that coalition gets on with heaving while the rest of the ants move on.

When we describe an actionable coalition as ‘a group of participants with the resources to act’, note that – this being a decentralised system – the action concerns just participants in that coalition and not the entire set of participants. Contrast with, for example, a national voting system, which is distributed but still centralised in the sense that the winning candidate wins for *everybody*, not just for the people who voted for that candidate.⁴

A union of actionable coalitions is an actionable coalition.⁵ This makes the set of all actionable coalitions look a bit like a topology, except that an intersection of actionable coalitions need not be actionable (just because sets S and S' are each collectively strong enough to act does not mean that $S \cap S'$ is; e.g., $S \cap S'$ may consist of just a single ant).

This brings us to a mathematical abstraction: *point-set semitopology*, which generalises point-set topology by removing the condition that intersections of open sets are necessarily open.

1.2 Point-set semitopologies

Notation 1.2.1. Suppose P is a set. Write $\text{pow}(P)$ for the powerset of P (the set of subsets of P).

Definition 1.2.2. A **semitopological space**, or **semitopology** for short, consists of a pair $(P, \text{Open}(P))$ of a (possibly empty) set P of **points** and a set $\text{Open}(P) \subseteq \text{pow}(P)$ of **open sets**, such that:

- (1) $\emptyset \in \text{Open}(P)$ and $P \in \text{Open}(P)$.
- (2) If $X \subseteq \text{Open}(P)$ then $\bigcup X \in \text{Open}(P)$.⁶

We may write $\text{Open}(P)$ just as Open , if P is irrelevant or understood, and we may write $\text{Open}_{\neq \emptyset}$ for the set of nonempty open sets.

Remark 1.2.3 (Justification for semitopologies). A classic text, Vickers (1989) justifies topology as follows:

- (1) Logically, open sets model *affirmations*:⁷ an open set O corresponds to an affirmation (of O) (Vickers, 1989, p. 10).
- (2) Computationally, open sets model *semidecidable properties*. See the first page of the preface in Vickers (1989).

We can justify semitopologies in a similar style as follows:

- (1) Logically, open sets model *actionable* affirmations: an open set O corresponds to an outcome that is agreed on *and can be actioned* within O .
- (2) Computationally, open sets model *actionable* outcomes.

So the distinction between topology and semitopology is this: topology is about things that can be positively said, whereas semitopology is about things that can be positively done.

Identifying actionable coalitions as a topological concept is new, but as is often the case for key concepts, with hindsight we can see them everywhere. The notion corresponds to that of a *quorum* in the classic distributed systems literature (Lamport, 1998). Social choice theorists have a similar notion called a *winning coalition* (Riker, 1962, Item 5, p. 40). In blockchains, the XRP Ledger (Schwartz et al., 2014) and the Stellar network (Lokhava et al., 2019) have explicit notions of actionable coalition, in the sense that a participant must specify information about their desired actionable coalitions when they sign up to the system. In Ethereum and other proof-of-stake blockchains, an actionable coalition is (roughly speaking) any group of participants with more than half of the voting power of the system.

1.3 Well-behavedness conditions on semitopologies

It turns out that we can understand a surprising amount about a decentralised system just by analysing its actionable coalitions; that is, by viewing it as a semitopology. And when we do, the most relevant and interesting semitopologies turn out to be ill-behaved from the usual topological viewpoint, because they are almost never Hausdorff; indeed, the most well-behaved semitopologies satisfy an *anti-Hausdorff* property of *being intertwined*, such that all of their open set intersect.

This is because if all actionable coalitions intersect, then any two actions that are taken must be compatible, in the sense that there must exist at least one participant who was able to take *both* actions. This implies a practically useful form of coherence for the overall system: while it is not the case that participants must act in synchrony (since that would not be decentralised), any actions that are taken must be compatible on some nonempty overlap; we make this formal in Theorem 3.2.3 and Corollary 3.2.4. In practice, it turns out that this property is often enough to derive important coherence properties for the overall system.

Notions of *transitive open set* (topen) and *regular point* become central to the theory, where topens are sets of pairwise intertwined participants (all their open neighbourhoods intersect), and points are called ‘regular’ when they have a topen neighbourhood. Briefly and in a nutshell: ‘good’ semitopologies have lots of regular points, and the best ones consist entirely of regular points.

The question we should ask then is: what *are* semitopologies, and what *are* well-behavedness properties like having topens and being regular?

One answer is that things are what the definitions define them to be – for example, semitopologies are Definition 1.2.2 – but the reader may know that we can use algebra to give better answers. In particular, looking at Definition 1.2.2 it might appear that semitopologies are just complete join-semilattices in sets. Well, this is not quite true, as we shall see.

1.4 Semiframes: the algebraic dual to semitopologies

There is a well-known recipe for understanding sets-based structures: we *dualise* them. That is, we decide what the morphisms should be (answer: continuous functions), we build a dual category of *semiframes*, and we study that.

It will turn out that a semiframe is a *compatible complete semilattice*. What ‘compatibility’ means is given in Definition 7.3.1, and it has to do with an algebraic abstraction of the nonempty intersection of open sets. What we can observe here is that the devil will be in the details (as always): the definition ‘compatible complete semilattice’ is not immediately obvious, and the proofs require work, and the duals to the well-behavedness conditions on semitopologies translate to algebraic structure in the world of semiframes.

Studying the dualities between semitopologies and semiframes, and how well-behavedness properties on points translate (or do not translate!) to their algebraic duals, is the topic of the paper.

Remark 1.4.1 (Semitopologies are compatible complete join-semilattices in sets). Something interesting will happen when we dualise semitopologies: we will find ourselves obliged to introduce the notion of a compatibility relation $*$ (Definition 7.2.1). This arises naturally twice:

- as a key technicality in our duality result, and also
- as an algebraic analogue of the intertwinedness property $\bowtie$ which we need to express well-behavedness properties such as regularity (Section 4).

It is interesting that $*/\bowtie$ appears for two reasons that, on the face of things, have quite distinct origins: algebraically in the duality and as a pragmatically motivated well-behavedness condition on sets.

So a message of our study of duality is to corroborate that intertwinedness is both mathematically fundamental and practically useful, and that (as noted above) semitopologies are not just complete join semilattices in sets, but something a bit richer, namely *compatible complete join-semilattices* in sets.

1.5 Map of the paper

- (1) Section 1 is the Introduction. You Are Here.
- (2) In Section 2, we show how **continuity corresponds to local agreement** (Definition 1.2.2 and Lemma 2.2.4).
- (3) In Section 3, we discuss **transitive sets, topens and intertwined points**. These are all anti-separation well-behavedness properties; for example, transitive and topen sets are guaranteed to be in agreement, in a sense made precise in Theorem 3.2.3 and Corollary 3.2.4.
- (4) In Section 4, we classify points in more detail, introducing notions of **regular, weakly regular, and quasicregular** points (Definition 4.1.4).⁸
Regular points are points contained in a topen set, and as per Theorem 3.2.3, they display good behaviour.
- (5) In Section 5, we characterise regularity as **regular = weakly regular + unconflicted**: see Theorem 5.3.4.
- (6) In Section 6, we characterise regularity as **regular = quasicregular + hypertransitive**: see Theorem 6.3.3.
This completes our treatment of point-set semitopologies, and next we dualise, as follows:

- (7) In Section 7, we introduce **semiframes**. These are the algebraic version of semitopologies, and they are to semitopologies as frames are to topologies. We discover that semiframes are not just join-semilattices; **semiframes are compatible semilattices**, which include a *compatibility relation* $*$ to abstract the property of set intersection $\cap$ (see Remark 1.4.1).
- (8) In Section 8, we introduce **semifilters**. These play a similar role as filters do in topologies, except that semifilters have a **compatibility condition** instead of closure under finite meets. We develop the notion of abstract points (completely prime semifilters) and show how to build a semitopology out of the abstract points of a semiframe.
- (9) In Section 9, we introduce **sober semitopologies** and **spatial semiframes**. The reader familiar with categorical duality may be familiar with these conditions, though some details are significantly different from the topological case (see, for instance, the discussion in Section 9.3.2).
- (10) In Section 10, we consider the **duality** between suitable categories of (sober) semitopologies and (spatial) semiframes.
- (11) In Section 11, we **dualise the well-behavedness conditions** from Section 3 to algebraic versions. The correspondence is good (Proposition 11.6.2) but also imperfect in some interesting ways (Remark 11.8.11).
- (12) In Section 12, we conclude and discuss related and future work.

1.6 Comments on the context of this research

Remark 1.6.1 (Algebraic topology $\neq$ semitopology and semiframes). Algebraic topology has been applied to the solvability of distributed-computing tasks in computational models (e.g., the impossibility of k -set consensus and the Asynchronous Computability Theorem (Herlihy and Shavit, 1993; Borowsky and Gafni, 1993; Saks and Zaharoglou, 2000); see Herlihy et al. (2013) for a survey).

This paper is not that! Algebra and topology are versatile tools: this paper is algebraic and topological, but in different senses and to different ends.

Remark 1.6.2 (Where the interesting properties are). Topology often studies spaces with strong separability properties between points, like Hausdorff separability. For our applications, the well-behavedness properties of semitopologies and corresponding properties in semiframes centre on clusters of points that *cannot* be separated.

For example, we state and discuss an ‘anti-Hausdorff’ anti-separation property which we call *being intertwined* (see Definition 3.5.1 and Remark 3.5.7). Within an intertwined set, continuity implies agreement in a particularly strong sense (see Corollary 3.5.6). This leads us to study classes of semitopologies and semiframes with various anti-separation well-behavedness conditions; see most notably *regularity* (Sections 4 and 11).

Remark 1.6.3. As the reader may know, *frames* and *locales* are the same thing: the category of locales is just the categorical opposite of the category of frames. So every time we write ‘semiframe’, the reader can safely read ‘semilocale’; these are two names for essentially the same structure up to reversing arrows.

The literature on frames and locales is huge, as indeed is the literature on topology. Classic texts are Johnstone (1986) and Mac Lane and Moerdijk (1992). More recent and very readable presentations are in Picado and Pultr (2012, 2021). This literature is a rich source of ideas for things to do with semiframes, with respect to which we cannot possibly be comprehensive in this single paper: there are many things of interest that we simply have not done yet, or perhaps have

not even (yet) realised could be done, and this is a feature, not a bug, since it reflects a wide scope for possible future research.

A partial list of possible future work is in Section 12.3; and lists of properties and non-properties of semiframes/semitopologies versus frames/topologies are in Sections 8.2.1, 8.2.2, 9.3.2 and 12.1.

Remark 1.6.4. This paper expands on selected material from a technical monograph by Gabbay (2024), and it continues where a journal paper on point-set topologies by Gabbay (2025) left off.

For this document to be meaningful and self-contained, we include some material on point-set semitopologies – just as a treatment of frames would define point-set topologies. The material has been organised, edited and streamlined specifically to provide a direct route to semiframes and their duality.

Our goal for this paper is to offer an accessible, dedicated and self-contained journal exposition on what is currently understood about semiframes and their relation to point-set semitopologies, in the context of current developments in decentralised systems.

Remark 1.6.5 (Who should read this paper?). This work is aimed at the advanced researcher interested in a new kind of point-free topology, albeit one that is grounded in the practical needs of modern decentralised systems, and at the advanced practitioner interested in seeing what they may already be doing in a new (and more abstract) mathematical light.

I hope this paper will stimulate interest in compatible complete join-semilattices and related structures as an elegant abstraction of decentralised systems.

2. Semitopology

2.1 Definition and examples

2.1.1 Semitopologies are not topologies

Recall from Definition 1.2.2 the definition of a semitopology. As a set structure, a semitopology on P is like a *topology* on P , but without the condition that the intersection of two open sets be an open set. This gives semitopologies their own distinct character, and we can see this before we have even developed much of the theory, as we discuss in Remark 2.1.1 and Lemma 2.1.2:

Remark 2.1.1. Every semitopology (P, Open) gives rise to a topology just by closing opens under intersections. But semitopologies are far more than just subbases for a corresponding topology:

- (1) Completing a semitopology to a topology by closing under intersections loses information. For example, the ‘many’, ‘all-but-one’ and ‘more-than-one’ semitopologies in Example 2.1.4 express three distinct notions of quorum, yet all three yield the discrete semitopology (Definition 2.1.3) if we close under intersections and P is infinite. See also the overview in Section 12.1.
- (2) We are explicitly interested in situations where intersections of open sets need not be open, due to our motivating interpretation of open sets as actionable coalitions (as discussed in Section 1).

Much of the difference in flavour between topologies and semitopologies comes down to this:

Lemma 2.1.2.

- (1) *In topologies, if a point p has a minimal open neighbourhood, then it is least (= unique minimal).*
- (2) *In semitopologies, a point may have multiple distinct minimal open neighbourhoods.*

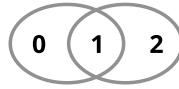


Figure 1. An example of a point with two minimal open neighbourhoods (Lemma 2.1.2).

Proof. To see that in a topology every minimal open neighbourhood is least, just note that if $p \in A$ and $p \in B$, then $p \in A \cap B$. So if A and B are two minimal open neighbourhoods, then $A \cap B$ is contained in both and, by minimality, is equal to both.

To see that in a semitopology a minimal open neighbourhood need not be least, it suffices to provide an example. Consider (P, Open) defined as follows, as illustrated in Figure 1:

- $P = \{0, 1, 2\}$
- $\text{Open} = \{\emptyset, \{0, 1\}, \{1, 2\}, \{0, 1, 2\}\}$

Note that 1 has two minimal open neighbourhoods: $\{0, 1\}$ and $\{1, 2\}$. □

2.1.2 Examples

As standard, we can make any set Val into a semitopology (indeed, it is also a topology) just by letting open sets be the powerset:

Definition 2.1.3.

- (1) Call $(P, \text{pow}(P))$ the **discrete semitopology on P** .
We may call a set with the discrete semitopology a **semitopology of values**, and when we do, we will usually call it Val . We may identify Val -the-set and Val -the-discrete-semitopology; meaning will always be clear.
- (2) When (P, Open) is a semitopology and Val is a semitopology of values, we may call a function $f : P \rightarrow \text{Val}$ a **value assignment**.
Note that a value just assigns values to points, and in particular, we do not assume a priori that it is continuous, where continuity is defined just as for topologies (see Definition 2.2.1).

Example 2.1.4. We consider further examples of semitopologies:

- (1) Every topology is also a semitopology; intersections of open sets are allowed to be open in a semitopology; they are just not constrained to be open. In particular, the discrete topology is also a discrete semitopology (Definition 2.1.3(1)).
- (2) The **initial semitopology** $(\emptyset, \{\emptyset\})$ and the **final semitopology** $(\{*\}, \{\emptyset, \{*\}\})$ are semitopologies.
- (3) An important discrete semitopological space is

$$\mathbb{B} = \{\perp, \top\} \quad \text{with the discrete semitopology} \quad \text{Open}(\mathbb{B}) = \{\emptyset, \{\perp\}, \{\top\}, \{\perp, \top\}\}.$$

We may silently treat $\mathbb{B}$ as a (discrete) semitopological space henceforth.

- (4) Take P to be any nonempty set. Let the **trivial semitopology** (this is also a topology) on P have

$$\text{Open} = \{\emptyset, P\}.$$

So (as usual) there are only two open sets: the one containing nothing and the one containing every point.⁹

The only nonempty open is P itself, reflecting a notion of actionable coalition that requires unanimous agreement.

- (5) Suppose P is a set and $\mathcal{F} \subseteq \text{pow}(P)$ is nonempty and up-closed (so if $P \in \mathcal{F}$ and $P \subseteq P' \subseteq P$ then $P' \in \mathcal{F}$, then $(P, \mathcal{F})$ is a semitopology. This is not necessarily a topology, because we do not insist that $\mathcal{F}$ is a filter (i.e., is closed under intersections).

We give four sub-examples for different choices of $\mathcal{P} \subseteq \text{pow}(P)$:

- a. Take P to be any finite nonempty set. Let the **supermajority semitopology** have

$$\text{Open} = \{\emptyset\} \cup \{O \subseteq P \mid \text{cardinality}(O) \geq \frac{2}{3} * \text{cardinality}(P)\}.$$

So O is open when it contains at least two-thirds of the points.

Two-thirds is a typical threshold used for making progress in consensus algorithms.

- b. Take P to be any nonempty set. Let the **many semitopology** have

$$\text{Open} = \{\emptyset\} \cup \{O \subseteq P \mid \text{cardinality}(O) = \text{cardinality}(P)\}.$$

For example, if $P = \mathbb{N}$ then open sets include $\text{evens} = \{2 * n \mid n \in \mathbb{N}\}$ and $\text{odds} = \{2 * n + 1 \mid n \in \mathbb{N}\}$.

Its notion of open set captures an idea that an actionable coalition is a set that may not be all of P , but does at least biject with it.

- c. Take P to be any nonempty set. Let the **all-but-one semitopology** have

$$\text{Open} = \{\emptyset, P\} \cup \{P \setminus \{p\} \mid p \in P\}.$$

This semitopology is not a topology.

The notion of actionable coalition here is that there may be at most one objector (but not two).

- d. Take P to be any set with cardinality at least 2. Let the **more-than-one semitopology** have

$$\text{Open} = \{\emptyset\} \cup \{O \subseteq P \mid \text{cardinality}(O) \geq 2\}.$$

This semitopology is not a topology.

This notion of actionable coalition reflects a security principle in banking and accounting (and elsewhere) of separation of duties, that functional responsibilities be separated such that at least two people are required to complete an action – so that errors (or worse) cannot be made without being discovered by another person.

- (6) Take $P = \mathbb{R}$ (the set of real numbers) and let open sets be generated by intervals of the form $[0, r)$ or $(-r, 0]$ for any strictly positive real number $r > 0$.

This semitopology is not a topology, since (e.g.,) $(1, 0]$ and $[0, 1)$ are open, but their intersection $\{0\}$ is not open.

- (7) Naor and Wool (1994) discuss a notion of quorum system, defined as any collection of pairwise intersecting sets. Quorum systems are key to the theory and practical implementation of consensus algorithms.

Every quorum system gives rise naturally to the least semitopology that contains it, just by closing under arbitrary unions.

To give one specific example of a quorum system from Naor and Wool (1994), consider $n \times n$ grid of cells with quorums being sets consisting of any full row and a full column; note that any two quorums must intersect in at least two points. We obtain a semitopology just by closing under arbitrary unions.

2.2 Continuity and its interpretation as agreement

Definition 2.2.1. We import standard topological notions of inverse image and continuity:

- (1) Suppose P and P' are any sets and $f : P \rightarrow P'$ is a function. Suppose $O' \subseteq P'$. Then write $f^{-1}(O')$ for the **inverse image** or **preimage** of O' , defined by

$$f^{-1}(O') = \{p \in P \mid f(p) \in O'\}.$$

- (2) Suppose (P, Open) and (P', Open') are semitopological spaces (Definition 1.2.2). Call a function $f : P \rightarrow P'$ **continuous** when the inverse image of an open set is open. In symbols:

$$\forall O' \in \text{Open}', f^{-1}(O') \in \text{Open}.$$

- (3) Call a function $f : P \rightarrow P'$ **continuous at** $p \in P$ when

$$\forall O' \in \text{Open}', f(p) \in O' \implies \exists O_{p,O'} \in \text{Open}. p \in O_{p,O'} \wedge O_{p,O'} \subseteq f^{-1}(O').$$

In words: f is continuous at p when the inverse image of every open neighbourhood of $f(p)$ contains an open neighbourhood of p .

- (4) Call a function $f : P \rightarrow P'$ **continuous on** $P \subseteq P$ when f is continuous at every $p \in P$.

Lemma 2.2.2. Suppose (P, Open) and (P', Open') are semitopological spaces (Definition 1.2.2) and suppose $f : P \rightarrow P'$ is a function. Then the following are equivalent:

- (1) f is continuous (Definition 2.2.1(2)).
- (2) f is continuous at every $p \in P$ (Definition 2.2.1(3)).

Proof. The top-down implication is immediate, taking $O = f^{-1}(O')$.

For the bottom-up implication, given p and an open neighbourhood $O' \ni f(p)$, we write

$$O = \bigcup \{O_{p,O'} \in \text{Open} \mid p \in P, f(p) \in O'\}.$$

Above, $O_{p,O'}$ is the open neighbourhood of p in the preimage of O' , which we know exists by Definition 2.2.1(3).

It is routine to check that $O = f^{-1}(O')$, and since this is a union of open sets, it is open. $\square$

Definition 2.2.3. Suppose that:

- (P, Open) is a semitopology and
- Val is a semitopology of values (Definition 2.1.3(1)) and
- $f : P \rightarrow \text{Val}$ is a value assignment (Definition 2.1.3(2); an assignment of a value to each element in P).

Then:

- (1) Call f **locally constant at** $p \in P$ when there exists $O_p \in \text{Open}$ such that

$$\forall p' \in O_p. f(p) = f(p').$$

So f is locally constant at p when it is constant on some open neighbourhood O_p of p .

- (2) Call f **locally constant** when it is locally constant at every $p \in P$.

Lemma 2.2.4. Suppose (P, Open) is a semitopology and Val is a semitopology of values and $f : P \rightarrow \text{Val}$ is a value assignment. Then the following are equivalent:

- f is locally constant/locally constant at $p \in P$ (Definition 2.2.3).
- f is continuous/continuous at $p \in P$ (Definition 2.2.1).

Proof. This is just by pushing around definitions, but we spell it out:

- Suppose f is continuous, consider $p \in P$ and write $v = f(p)$. By our assumptions, we know that $f^{-1}(v)$ is open, and $p \in f^{-1}(v)$. This is an open neighbourhood O_p on which f is constant, so we are done.
- Suppose f is locally constant, consider $p \in P$, and write $v = f(p)$. By assumption we can find $p \in O_p \in \text{Open}$ on which f is constant, so that $O_p \subseteq f^{-1}(v)$. $\square$

Remark 2.2.5 (Continuity = agreement). Lemma 2.2.4 tells us that we can view the problem of obtaining agreement across an actionable coalition (as discussed in Section 1) as being the same as obtaining a value assignment that is continuous (at least) on that coalition.

2.3 Neighbourhoods of a point

Definition 2.3.1 is standard from topology, and Lemma 2.3.2 is a (standard) characterisation of openness, which will be useful later:

Definition 2.3.1. Suppose (P, Open) is a semitopology and $p \in P$ and $O \in \text{Open}$. Then:

- (1) Call O an **open neighbourhood** of p when $p \in O$.
- (2) Define $\text{nbhd}(p) \subseteq \text{Open}$ the **neighbourhood system** of p by

$$\text{nbhd}(p) = \{O \in \text{Open} \mid p \in O\}.$$

Lemma 2.3.2. Suppose (P, Open) is a semitopology, and suppose $P \subseteq P$ is any set of points. Then the following are equivalent:

- $P \in \text{Open}$.
- Every point p in P has an open neighbourhood in P .

In symbols we can write:

$$\forall p \in P. \exists O \in \text{Open}. (p \in O \wedge O \subseteq P) \quad \text{if and only if} \quad P \in \text{Open}$$

Proof. If P is open, then P itself is an open neighbourhood for every point that it contains.

Conversely, if every $p \in P$ contains some open neighbourhood $p \in O_p \subseteq P$, then $P = \bigcup \{O_p \mid p \in P\}$, and this is open by condition 2 of Definition 1.2.2. $\square$

Remark 2.3.3. An initial inspiration for modelling collaborative action using semitopologies came from noting that the standard topological property described above in Lemma 2.3.2 corresponds to the *quorum sharing* property in (Losa et al., 2019, Property 1); the connection to topological ideas had not been noticed in Losa et al. (2019).

3. Transitive Sets & Topens

3.1 Some background on set intersection

Some notation will be convenient:

Notation 3.1.1. Suppose X, Y and Z are sets.

(1) Write

$$X \overset{\circ}{\cap} Y \quad \text{when} \quad X \cap Y \neq \emptyset.$$

When $X \overset{\circ}{\cap} Y$ holds, then we say (as standard) that X and Y **intersect**.

(2) We may chain the $\overset{\circ}{\cap}$ notation, writing, for example,

$$X \overset{\circ}{\cap} Y \overset{\circ}{\cap} Z \quad \text{for} \quad X \overset{\circ}{\cap} Y \wedge Y \overset{\circ}{\cap} Z$$

(3) We may write $X \not\overset{\circ}{\cap} Y$ for $\neg(X \overset{\circ}{\cap} Y)$, thus $X \not\overset{\circ}{\cap} Y$ when $X \cap Y = \emptyset$.

Remark 3.1.2. Note on design in Notation 3.1.1: It is uncontroversial that if $X \neq \emptyset$ and $Y \neq \emptyset$, then $X \overset{\circ}{\cap} Y$ should hold precisely when $X \cap Y \neq \emptyset$ – but there is an edge case! What truth value should $X \overset{\circ}{\cap} Y$ return when X or Y is empty?

(1) It might be nice if $X \subseteq Y$ would imply $X \overset{\circ}{\cap} Y$. This argues for setting

$$(X = \emptyset \vee Y = \emptyset) \implies X \overset{\circ}{\cap} Y.$$

(2) It might be nice if $X \overset{\circ}{\cap} Y$ were monotone on both arguments (i.e., if $X \overset{\circ}{\cap} Y$ and $X \subseteq X'$, then $X' \overset{\circ}{\cap} Y$). This argues for setting

$$(X = \emptyset \vee Y = \emptyset) \implies X \not\overset{\circ}{\cap} Y.$$

(3) It might be nice if $X \overset{\circ}{\cap} X$ always – after all, should a set *not* intersect itself? – and this argues for setting

$$\emptyset \overset{\circ}{\cap} \emptyset,$$

even if we also set $\emptyset \not\overset{\circ}{\cap} Y$ for nonempty Y .

All three choices are defensible, and they are consistent with the following nice property:

$$X \overset{\circ}{\cap} Y \implies (X \overset{\circ}{\cap} X \vee Y \overset{\circ}{\cap} Y).$$

We choose the second – if X or Y is empty then $X \not\overset{\circ}{\cap} Y$ – because it gives the simplest definition that $X \overset{\circ}{\cap} Y$ precisely when $X \cap Y \neq \emptyset$.

We list some elementary properties of $\overset{\circ}{\cap}$ from Notation 3.1.1(1):

Lemma 3.1.3.

- (1) $X \overset{\circ}{\cap} X$ if and only if $X \neq \emptyset$.
- (2) $X \overset{\circ}{\cap} Y$ if and only if $Y \overset{\circ}{\cap} X$.
- (3) $X \overset{\circ}{\cap} (Y \cup Z)$ if and only if $(X \overset{\circ}{\cap} Y) \vee (X \overset{\circ}{\cap} Z)$.
- (4) If $X \subseteq X'$ and $X \neq \emptyset$ then $X \overset{\circ}{\cap} X'$.
- (5) Suppose $X \overset{\circ}{\cap} Y$. Then $X \subseteq X'$ implies $X' \overset{\circ}{\cap} Y$, and $Y \subseteq Y'$ implies $X \overset{\circ}{\cap} Y'$.
- (6) If $X \overset{\circ}{\cap} Y$ then $X \neq \emptyset$ and $Y \neq \emptyset$.

Proof. By facts of set intersection. □

3.2 Transitive open sets and value assignments

Remark 3.2.1 (Taking stock of topens). Transitive sets are of interest because values of continuous functions are strongly correlated on them. This is Theorem 3.2.3, especially part 2 of Theorem 3.2.3.

A transitive open set – a topen – is even more important, because an open set corresponds in our semitopological model to a quorum (a collection of participants that can make progress), so a transitive open set is a collection of participants that can make progress and are guaranteed to do so in consensus, where algorithms succeed.

For this and other reasons, we very much care about finding topens and understanding when points are associated with topen sets (e.g., by having topen neighbourhoods). As we develop the maths, this will then lead us on to consider various regularity properties (Definition 4.1.4). But first, we start with transitive sets and topens:

Definition 3.2.2. Suppose (P, Open) is a semitopology. Suppose $T \subseteq P$ is any set of points.

- (1) Call T **transitive** when

$$\forall O, O' \in \text{Open}. O \bowtie T \bowtie O' \implies O \bowtie O'.$$

- (2) Call T **topen** when T is nonempty transitive and open.¹⁰

We may write

$$\text{Topen} = \{T \in \text{Open}_{\neq \emptyset} \mid T \text{ is transitive}\}.$$

- (3) Call S a **maximal topen** when S is a topen that is not a subset of any strictly larger topen.¹¹

Theorem 3.2.3 clarifies why transitivity is interesting: continuous value assignments are constant – if we think of points as participants, ‘constant function’ here means ‘in agreement’ – across transitive sets.

Theorem 3.2.3. Suppose that:

- (P, Open) is a semitopology.
- Val is a semitopology of values (a nonempty set with the discrete semitopology; see Definition 2.1.3(1)).
- $f : P \rightarrow \text{Val}$ is a value assignment (Definition 2.1.3(2)).
- $T \subseteq P$ is a transitive set (Definition 3.2.2) – in particular this will hold if T is topen – and $p, p' \in T$.

Then:

- (1) If f is continuous at p and p' then $f(p) = f(p')$.
 (2) As a corollary, if f is continuous on T , then f is constant on T .

In words we can say:

Continuous value assignments are constant across transitive sets.

Proof. Part 2 follows from part 1 since if $f(p) = f(p')$ for any $p, p' \in T$, then by definition f is constant on T . So we now just need to prove part 1 of this result.

Consider $p, p' \in T$. By continuity on T , there exist open neighbourhoods $p \in O \subseteq f^{-1}(f(p))$ and $p' \in O' \subseteq f^{-1}(f(p'))$. By construction $O \bowtie T \bowtie O'$ (because $p \in O \cap T$ and $p' \in T \cap O'$). By transitivity

ity of T , it follows that $O \not\ll O'$. Thus, there exists $p'' \in O \cap O'$, and by construction $f(p) = f(p'') = f(p')$. $\square$

Corollary 3.2.4 is an easy and useful consequence of Theorem 3.2.3:

Corollary 3.2.4. *Suppose that:*

- (P, Open) is a semitopology.
- $f : P \rightarrow \text{Val}$ is a value assignment to some set of values Val (Definition 2.1.3).
- f is continuous on topen sets $T, T' \in \text{Topen}$.

Then

$$T \not\ll T' \quad \text{implies} \quad \forall p \in T, p' \in T' . f(p) = f(p').$$

Proof. By Theorem 3.2.3, f is constant on T and T' . We assumed that T and T' intersect, and the result follows. $\square$

A converse to Theorem 3.2.3 also holds:

Proposition 3.2.5. *Suppose that:*

- (P, Open) is a semitopology.
- Val is a semitopology of values with at least two elements (to exclude a degenerate case where no functions exist, or they exist but there is only one because there is only one value to map to).
- $T \subseteq P$ is any set.

Then

- if for every $p, p' \in T$ and every value assignment $f : P \rightarrow \text{Val}$, f continuous at p and p' implies $f(p) = f(p')$,
- then T is transitive.

Proof. We prove the contrapositive. Suppose T is not transitive, so there exist $O, O' \in \text{Open}$ such that $O \not\ll T \not\ll O'$ and yet $O \cap O' = \emptyset$. We choose two distinct values $v \neq v' \in \text{Val}$ and define f to map any point in O to v and any point in $P \setminus O$ to v' .

Choose some $p \in O$ and $p' \in O'$. It does not matter which, and some such p and p' exist, because O and O' are nonempty by Lemma 3.1.3(6), since $O \not\ll T$ and $O' \not\ll T$.

We note that $f(p) = v$ and $f(p') = v'$ and f is continuous at $p \in O$ and $p' \in O' \subseteq P \setminus O$, yet $f(p) \neq f(p')$. $\square$

We can sum up what Theorem 3.2.3 and Proposition 3.2.5 mean as follows:

Remark 3.2.6. Suppose (P, Open) is a semitopology, and Val is a semitopology of values with at least two elements. Say that a value assignment $f : P \rightarrow \text{Val}$ **splits** a set $T \subseteq P$ when there exist $p, p' \in T$ such that f is continuous at p and p' and $f(p) \neq f(p')$. Then Theorem 3.2.3 and Proposition 3.2.5 together say in words that:

$T \subseteq P$ is transitive if and only if it cannot be split by a value assignment that is continuous on T .

Intuitively, transitive sets characterise areas of guaranteed agreement.

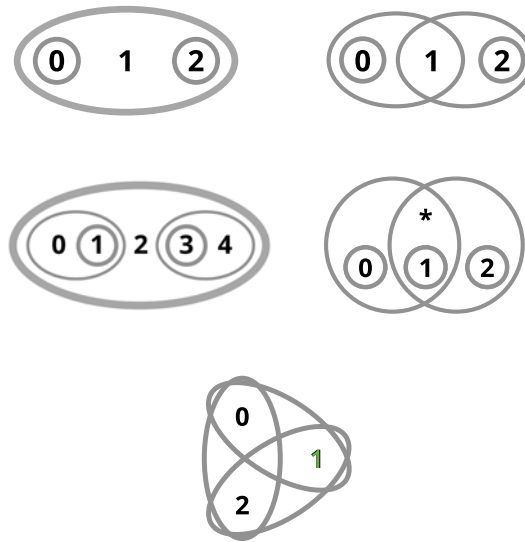


Figure 2. Examples of topens (Example 3.3.3).

3.3 Examples and discussion of transitive sets and topens

We may routinely order sets by subset inclusion, including open sets, topens, closed sets, and so on, and we may talk about maximal, minimal, greatest and least elements. We include the (standard) definition for reference:

Notation 3.3.1. Suppose $(P, \leq)$ is a poset. Then:

- (1) Call $p \in P$ **maximal** when $\forall p'. p \leq p' \implies p' = p$ and **minimal** when $\forall p'. p' \leq p \implies p' = p$.
- (2) Call $p \in P$ **greatest** when $\forall p'. p' \leq p$ and **least** when $\forall p'. p \leq p'$.

Example 3.3.2 (Examples of transitive sets).

- (1) $\{p\}$ is transitive, for any single point $p \in P$.
- (2) The empty set $\emptyset$ is (trivially) transitive. It is not topen because we insist in Definition 3.2.2(2) that topens are nonempty.
- (3) Call a set $P \subseteq P$ topologically indistinguishable when (using Notation 3.1.1) for every open set O ,

$$P \not\subseteq O \iff P \subseteq O.$$

It is easy to check that if P is topologically indistinguishable, then it is transitive.

Example 3.3.3 (Examples of topens).

- (1) Take $P = \{0, 1, 2\}$, with open sets $\emptyset, P, \{0\}$ and $\{2\}$. This has two maximal topens $\{0\}$ and $\{2\}$ as illustrated in Figure 2 (top-left diagram).
- (2) Take $P = \{0, 1, 2\}$, with open sets $\emptyset, P, \{0\}, \{0, 1\}, \{2\}, \{1, 2\}$ and $\{0, 2\}$. This has two maximal topens $\{0\}$ and $\{2\}$, as illustrated in Figure 2 (top-right diagram).
- (3) Take $P = \{0, 1, 2, 3, 4\}$, with open sets generated by $\{0, 1\}, \{1\}, \{3\}$ and $\{3, 4\}$. This has two maximal topens $\{0, 1\}$ and $\{3, 4\}$, as illustrated in Figure 2 (lower-left diagram).
- (4) Take $P = \{0, 1, 2, *\}$, with open sets generated by $\{0\}, \{1\}, \{2\}, \{0, 1, *\}$ and $\{1, 2, *\}$. This has three maximal topens $\{0\}, \{1\}$ and $\{2\}$, as illustrated in Figure 2 (lower-right diagram).

- (5) Take the all-but-one semitopology from Example 2.1.4(5c) on $\mathbb{N}$: so $P = \mathbb{N}$ with opens $\emptyset$, $\mathbb{N}$ and $\mathbb{N} \setminus \{x\}$ for every $x \in \mathbb{N}$. This has a single maximal topen $\mathbb{N}$.
- (6) The semitopology in Figure 3 has no topen sets at all ($\emptyset$ is transitive and open, but by definition in Definition 3.2.2(2) topens have to be nonempty).

Here and elsewhere, we might omit open sets that are unions of open sets that are illustrated. For example, we explicitly draw the universal open set in the left-hand diagrams above, but not in the right-hand and bottom diagrams above. Meaning is clear, and we get cleaner diagrams.

3.4 Closure properties of transitive sets and topens

Remark 3.4.1. Transitive sets have some nice closure properties which we treat in this subsection – here, we mean ‘closure’ in the sense of “the set of transitive sets is closed under various operations”, and not in the topological sense of ‘closed sets’. Topens – nonempty transitive open sets – have even better closure properties, which emanate from the requirement in Lemma 3.4.3 that at least one of the transitive sets T or T' is open.

Lemma 3.4.2. *Suppose (P, Open) is a semitopology and $T \subseteq P$. Then:*

- (1) *If T is transitive and $T' \subseteq T$, then T' is transitive.*
- (2) *If T is topen and $\emptyset \neq T' \subseteq T$ is nonempty and open, then T' is topen.*

Proof.

- (1) By Definition 3.2.2, it suffices to consider open sets O and O' such that $O \bowtie T' \bowtie O'$ and prove that $O \bowtie O'$. But this is simple: by Lemma 3.1.3(5) $O \bowtie T \bowtie O'$, so $O \bowtie O'$ follows by transitivity of T .
- (2) Direct from part 1 of this result and Definition 3.2.2(2). □

Lemma 3.4.3. *Suppose (P, Open) is a semitopology and $T, T' \subseteq P$ are transitive, and suppose that at least one of T and T' is open. Then*

$$\forall O, O' \in \text{Open}. O \bowtie T \bowtie T' \bowtie O' \implies O \bowtie O'.$$

Proof. We simplify using Definition 3.2.2 and our assumption that one of T and T' is open. We consider the case that T' is open:

$$\begin{aligned} O \bowtie T \bowtie T' \bowtie O' &\implies O \bowtie T' \bowtie O' && T \text{ transitive, } T' \text{ open} \\ &\implies O \bowtie O' && T' \text{ transitive.} \end{aligned}$$

The argument for when T is open is precisely similar. □

Proposition 3.4.4. *Suppose (P, Open) is a semitopology. Then if $\mathcal{T}$ is a set of pairwise intersecting topens, then $\bigcup \mathcal{T}$ is topen.*

Proof. $\bigcup \mathcal{T}$ is open by Definition 1.2.2(2). Also, if $O \bowtie \bigcup \mathcal{T} \bowtie O'$ then there exist $T, T' \in \mathcal{T}$ such that $O \bowtie T$ and $T' \bowtie O'$. We assumed $T \bowtie T'$, so by Lemma 3.4.3 (since T and T' are open) we have $O \bowtie O'$ as required. □

Corollary 3.4.5. *Suppose (P, Open) is a semitopology. Then every topen T is contained in a unique maximal topen.*

Proof. Consider $\mathcal{T} = \{T \cup T' \mid T' \text{ topen} \wedge T \not\sim T'\}$. By Proposition 3.4.4, this is a set of topens. By construction, they all contain T , so by our assumption that $T \neq \emptyset$, they pairwise intersect, and by Proposition 3.4.4 again $\bigcup \mathcal{T}$ is topen. It is easy to check that this is the unique maximal topen that contains T . $\square$

3.5 Intertwined points

3.5.1 The basic definition

Definition 3.5.1. Suppose (P, Open) is a semitopology and $p, p' \in P$.

- (1) Call p and p' **intertwined** when $\{p, p'\}$ is transitive. Unpacking Definition 3.2.2, this means:

$$\forall O, O' \in \text{Open}. (p \in O \wedge p' \in O') \implies O \not\sim O'.$$

By a mild abuse of notation, write

$$p \not\sim p' \text{ when } p \text{ and } p' \text{ are intertwined.}$$

- (2) Define $p_{\not\sim}$ (read ‘intertwined of p ’) to be the set of points intertwined with p . In symbols:

$$p_{\not\sim} = \{p' \in P \mid p \not\sim p'\}.$$

Example 3.5.2. We return to the examples in Example 3.3.3. There we note that:

- (1) $1_{\not\sim} = \{0, 1, 2\}$ and $0_{\not\sim} = \{0, 1\}$ and $2_{\not\sim} = \{1, 2\}$.
- (2) $1_{\not\sim} = \{1\}$ and $0_{\not\sim} = \{0\}$ and $2_{\not\sim} = \{2\}$.
- (3) $x_{\not\sim} = P$ for every x .

Lemma 3.5.3. Suppose (P, Open) is a semitopology. Then the ‘is intertwined’ relation $\not\sim$ is not necessarily transitive. That is, $p \not\sim p' \not\sim p''$ does not necessarily imply $p \not\sim p''$.

Proof. It suffices to provide a counterexample. The semitopology from Example 3.3.3(1) (illustrated in Figure 2, top-left diagram) will do. So take $P = \{0, 1, 2\}$ and $\text{Open} = \{\emptyset, P, \{0\}, \{2\}\}$. Then

$$0 \not\sim 1 \text{ and } 1 \not\sim 2, \text{ but } \neg(0 \not\sim 2).$$

$\square$

3.5.2 Pointwise characterisation of transitive sets

Lemma 3.5.4. Suppose (P, Open) is a semitopology and $T \subseteq P$. Then the following are equivalent:

- (1) T is transitive.
- (2) $p \not\sim p'$ (meaning by Definition 3.5.1 that $\{p, p'\}$ is transitive) for every $p, p' \in T$.

Proof. Suppose T is transitive. Then by Lemma 3.4.2(1), $\{p, p'\}$ is transitive for every $p, p' \in T$.

Suppose $\{p, p'\}$ is transitive for every $p, p' \in T$. Consider open sets O and O' such that $O \not\sim T \not\sim O'$. Choose $p \in O \cap T$ and $p' \in O' \cap T$. By construction $\{p, p'\} \subseteq T$ so, this is transitive. It follows that $O \not\sim O'$ as required. $\square$

Theorem 3.5.5. Suppose (P, Open) is a semitopology and $T \subseteq P$. Then the following are equivalent:

- (1) T is topen.
- (2) $T \in \text{Open}_{\neq \emptyset}$ and $\forall p, p' \in T. p \not\sim p'$.

In words we can say:

A topen is a nonempty open set of intertwined points.

Proof. By Definition 3.2.2(2), T is topen when it is nonempty, open and transitive. By Lemma 3.5.4, this last condition is equivalent to $p \not\sim p'$ for every $p, p' \in T$. $\square$

A value assignment is constant on a pair of intertwined points, where it is continuous:

Corollary 3.5.6. Suppose Val is a semitopology of values and $f : P \rightarrow \text{Val}$ is a value assignment (Definition 2.1.3) and $p, p' \in P$ and $p \not\sim p'$. Then if f continuous at p and p' then $f(p) = f(p')$.

Proof. $\{p, p'\}$ is transitive by Theorem 3.5.5; we use Theorem 3.2.3. $\square$

Remark 3.5.7 (Intertwined as ‘non-Hausdorff’). Recall that we call a topological space (P, Open) **Hausdorff** (or T_2) when any two points can be separated by pairwise disjoint open sets. Using the $\not\sim$ symbol from Notation 3.1.1, we rephrase the Hausdorff condition as

$$\forall p, p'. p \neq p' \implies \exists O, O'. (p \in O \wedge p' \in O' \wedge \neg(O \not\sim O')),$$

we can simplify to

$$\forall p, p'. p \neq p' \implies p \not\sim p',$$

and thus we simplify the Hausdorff condition just to

$$\forall p. p \not\sim = \{p\}. \quad (1)$$

Note how distinct p and p' being intertwined is the opposite of being Hausdorff: $p \not\sim p'$ when $p' \in p \not\sim$, and they cannot be separated by pairwise disjoint open sets. Thus the assertion $p \not\sim p'$ in Theorem 3.5.5 is a negation of the Hausdorff property: $\exists p. p \not\sim \neq \{p\}$.

4. Interiors, Communities & Regular Points

4.1 Community of a (regular) point

Definition 4.1.1 is standard:

Definition 4.1.1 (Open interior). Suppose (P, Open) is a semitopology and $P \subseteq P$. Define $\text{interior}(P)$ the **(open) interior of P** by

$$\text{interior}(P) = \bigcup \{O \in \text{Open} \mid O \subseteq P\}.$$

Lemma 4.1.2. Suppose (P, Open) is a semitopology and $P \subseteq P$. Then the $\text{interior}(P)$ from Definition 4.1.1 is the greatest open subset of P .

Proof. Routine by the construction in Definition 4.1.1 and closure of open sets under unions (Definition 1.2.2(2)). $\square$

Corollary 4.1.3. Suppose (P, Open) is a semitopology and $P, P' \subseteq P$. Then if $P \subseteq P'$ then $\text{interior}(P) \subseteq \text{interior}(P')$.

Proof. Routine using Lemma 4.1.2. $\square$

Definition 4.1.4 (Community of a point and regularity). Suppose (P, Open) is a semitopology and $p \in P$. Then:

- (1) Define $K(p)$ the **community of p** by

$$K(p) = \text{interior}(p_{\emptyset}).$$

The community of p is always an open set by Lemma 4.1.2.

- (2) Extend K to subsets $P \subseteq P$ by taking a sets union:

$$K(P) = \bigcup \{K(p) \mid p \in P\}.$$

- (3) Call p a **regular point** when its community is a topen neighbourhood of p . In symbols:

$$p \text{ is regular} \quad \text{when} \quad p \in K(p) \in \text{Topen}.$$

- (4) Call p a **weakly regular point** when its community is an open (but not necessarily topen) neighbourhood of p . In symbols:

$$p \text{ is weakly regular} \quad \text{when} \quad p \in K(p) \in \text{Open}.$$

- (5) Call p a **quasiregular point** when its community is nonempty. In symbols:

$$p \text{ is quasiregular} \quad \text{when} \quad \emptyset \neq K(p) \in \text{Open}.$$

- (6) If p is not regular, then we may call it an **irregular point** or just say that it is not regular.

- (7) If $P \subseteq P$ and every $p \in P$ is regular/weakly regular/quasiregular/irregular then we may call P a **regular/weakly regular/quasiregular/irregular set**, respectively (see also Definition 5.3.1(2)).

Remark 4.1.5. The notion of a regular point in Definition 4.1.4 is a key well-behavedness property. Let's remember why it matters:

A topen set is a transitive open set. We care about transitivity because it implies agreement, as per Theorem 3.2.3 (continuous value assignments are constant on transitive sets). We care about being open because we understand this as 'being actionable'. Thus, a regular point is interesting because it is a participant in a topen and thus is capable of safely making progress in algorithms we write on top of the underlying semitopology. For convenient reference, the semiframe characterisation of regularity is in Section 11.5.

Lemma 4.1.6 gives an initial overview of the relationships between the properties in Definition 4.1.4. A more detailed treatment follows, which repeats these main points and expands on them and puts them in a detailed context.

Lemma 4.1.6. Suppose (P, Open) is a semitopology and $p \in P$. Then:

- (1) If p is regular, then p is weakly regular.
- (2) If p is weakly regular, then p is quasiregular.
- (3) The converse implications need not hold.
- (4) Furthermore, it is possible for a point p to not be quasiregular.

Proof. We consider each part in turn:

- (1) If p is regular, then by Definition 4.1.4(3) $p \in K(p) \in \text{Topen}$, so certainly $p \in K(p)$ and by Definition 4.1.4(4) p is weakly regular.
- (2) If p is weakly regular, then by Definition 4.1.4(4) $p \in K(p) \in \text{Open}$, so certainly $K(p) \neq \emptyset$, and by Definition 4.1.4(5), p is quasiregular.

- (3) To see that the converse implications need not hold, note that:
 - Point 1 in Example 4.1.7(2) (illustrated in Figure 2, top-left diagram) is weakly regular ($K(1) = \{0, 1, 2\}$) but not regular ($K(1)$ is open but not topen).
 - Point $*$ in Example 4.1.7(3) (illustrated in Figure 2, lower-right diagram) is quasiregular ($K(*) = \{1\}$ is nonempty but does not contain $*$).
- (4) To see that p may not even be quasiregular, take $P = \mathbb{R}$ (real numbers), with its usual topology (which is also a semitopology). Then $x_\emptyset = \{x\}$ and $K(x) = \emptyset$ for every $x \in \mathbb{R}$. $\square$

Example 4.1.7.

- (1) In Figure 2 (bottom diagram), 0, 1 and 2 are three intertwined points, and the entire space $\{0, 1, 2\}$ consists of a single topen set. It follows that 0, 1 and 2 are all regular, and their community is $\{0, 1, 2\}$.
- (2) In Figure 2 (top-left diagram), 0 and 2 are regular, and 1 is weakly regular but not regular ($1 \in K(1) = \{0, 1, 2\}$ but $\{0, 1, 2\}$ is not topen).
- (3) In Figure 2 (lower-right diagram), 0, 1 and 2 are regular, and $*$ is quasiregular ($K(*) = \{1\}$).
- (4) In Figure 2 (top-right diagram), 0 and 2 are regular, and 1 is neither regular, weakly regular, nor quasiregular ($K(1) = \emptyset$).
- (5) In a semitopology of values $(\text{Val}, \text{pow}(\text{Val}))$ (Definition 2.1.3), every value $v \in \text{Val}$ is regular, weakly regular and unconflicted.
- (6) In $\mathbb{R}$ with its usual topology (which is also a semitopology), every point is unconflicted because the topology is Hausdorff, and by Equation (1) in Remark 3.5.7, this means precisely that $p_\emptyset = \{p\}$, so p is intertwined just with itself. Furthermore, p is not (quasi/weakly) regular, because $K(p) = \text{interior}(p_\emptyset) = \emptyset$.

4.2 Further exploration of (quasi-/weak) regularity and topen sets

Remark 4.2.1. Recall three common separation axioms from topology:

- (1) T_0 : if $p_1 \neq p_2$, then there exists some $O \in \text{Open}$ such that $(p_1 \in O) \text{ xor } (p_2 \in O)$, where xor denotes exclusive or.
- (2) T_1 : if $p_1 \neq p_2$, then there exist $O_1, O_2 \in \text{Open}$ such that $p_i \in O_j \iff i = j$ for $i, j \in \{1, 2\}$.
- (3) T_2 , or the Hausdorff condition: if $p_1 \neq p_2$, then there exist $O_1, O_2 \in \text{Open}$ such that $p_i \in O_j \iff i = j$ for $i, j \in \{1, 2\}$, and $O_1 \cap O_2 = \emptyset$. Cf. the discussion in Remark 3.5.7.

Even the weakest of the well-behavedness properties for semitopologies that we consider in Definition 4.1.4 – quasiregularity – is in some sense strongly opposed to the space being Hausdorff/ T_2 (though not to being T_1), as Lemma 4.2.2 makes formal.

Lemma 4.2.2.

- (1) Every quasiregular Hausdorff semitopology is discrete.
In more detail: if (P, Open) is a semitopology that is quasiregular (Definition 4.1.4(5)) and Hausdorff (Equation (1) in Remark 3.5.7), then $\text{Open} = \text{pow}(P)$.
- (2) There exists a (quasi)regular T_1 semitopology that is not discrete.

Proof. We consider each part in turn:

- (1) By the Hausdorff property, $p_{\varnothing} = \{p\}$. By the quasiregularity property, $K(p) \neq \emptyset$. It follows that $K(p) = \{p\}$. But by construction in Definition 4.1.4(1), $K(p)$ is an open interior. Thus $\{p\} \in \text{Open}$. The result follows.
- (2) It suffices to provide an example. We use the bottom semitopology in Figure 2. Thus $P = \{0, 1, 2\}$ and Open is generated by $\{0, 1\}$, $\{1, 2\}$ and $\{2, 0\}$. The reader can check that this is regular (since all three points are intertwined) and T_1 . □

So what is $K(p)$? We start by characterising $K(p)$ as the *greatest* topen neighbourhood of p , if this exists:

Lemma 4.2.3. *Suppose (P, Open) is a semitopology, and recall from Definition 4.1.4(3) that p is regular when $K(p)$ is a topen neighbourhood of p .*

- (1) *If $K(p)$ is a topen neighbourhood of p (i.e., if p is regular), then $K(p)$ is a maximal topen.*
- (2) *If $p \in T \in \text{Topen}$ is a maximal topen neighbourhood of p then $T = K(p)$.*

Proof.

- (1) Since p is regular, by definition, $K(p)$ is topen and is a neighbourhood of p . It remains to show that $K(p)$ is a maximal topen.
Suppose T is a topen neighbourhood of p ; we wish to prove $T \subseteq K(p) = \text{interior}(p_{\varnothing})$. Since T is open, it would suffice to show that $T \subseteq p_{\varnothing}$. By Theorem 3.5.5, $p \varnothing p'$ for every $p' \in T$, and it follows immediately that $T \subseteq p_{\varnothing}$.
- (2) Suppose T is a maximal topen neighbourhood of p .
First, note that T is open, and by Theorem 3.5.5 $T \subseteq p_{\varnothing}$, so $T \subseteq K(p)$.
Now consider any open $O \subseteq p_{\varnothing}$. Note that $T \cup O$ is an open subset of $p_{\varnothing}$, so by Theorem 3.5.5 $T \cup O$ is topen, and by maximality $T \cup O \subseteq T$ and thus $O \subseteq T$. It follows that $K(p) \subseteq T$. □

Remark 4.2.4. We can use Lemma 4.2.3 to characterise regularity in five equivalent ways: see Theorem 4.2.5 and Corollary 4.2.6.

Theorem 4.2.5. *Suppose (P, Open) is a semitopology and $p \in P$. Then the following are equivalent:*

- (1) *p is regular, or in full: $p \in K(p) \in \text{Topen}$.*
- (2) *$K(p)$ is the greatest topen neighbourhood of p .*
- (3) *$K(p)$ is a maximal topen neighbourhood of p .*
- (4) *p has a maximal topen neighbourhood.*
- (5) *p has some topen neighbourhood.*

Proof. We prove a cycle of implications:

- (1) If $K(p)$ is a topen neighbourhood of p , then it is maximal by Lemma 4.2.3(1). Furthermore, this maximal topen neighbourhood of p is necessarily the greatest, since if we have two maximal topen neighbourhoods of p , then their union is a larger topen neighbourhood of p by Proposition 3.4.4.
- (2) If $p_{\varnothing}$ is the greatest topen neighbourhood of p , then certainly it is maximal.

- (3) If $p \not\sim$ is a maximal topen neighbourhood of p , then certainly p has a maximal topen neighbourhood.
- (4) If p has a maximal topen neighbourhood then certainly p has a topen neighbourhood.
- (5) Suppose p has a topen neighbourhood T . By Corollary 3.4.5 we may assume without loss of generality that T is a maximal topen. We use Lemma 4.2.3(2). $\square$

Corollary 4.2.6. *Suppose (P, Open) is a semitopology and $p \in P$. Then the following are equivalent:*

- (1) p is regular.
- (2) p is weakly regular and $K(p) = K(p')$ for every $p' \in K(p)$.

It might be useful to look at Example 3.3.3(2) and Figure 2 (top-right diagram). In that example the point 1 is not regular, and its community $\{0, 1, 2\}$ is not a community for 0 or 2.

Proof. We prove two implications, using Theorem 4.2.5:

- Suppose p is regular. By Lemma 4.1.6(1), p is weakly regular. Now consider $p' \in K(p)$. By Theorem 4.2.5, $K(p)$ is topen, so it is a topen neighbourhood of p' . By Theorem 4.2.5, $K(p')$ is a greatest topen neighbourhood of p' . But by Theorem 4.2.5, $K(p)$ is also a greatest topen neighbourhood of p , and $K(p) \not\sim K(p')$ since they both contain p' . By Proposition 3.4.4 and maximality, they are equal.
- Suppose p is weakly regular and suppose $K(p) = K(p')$ for every $p' \in K(p)$, and consider $p', p'' \in K(p)$. Then $p' \not\sim p''$ holds, since $p'' \in K(p') = K(p)$. By Theorem 3.5.5, $K(p)$ is topen, and by weak regularity, $p \in K(p)$, so by Theorem 4.2.5, p is regular as required. $\square$

Corollary 4.2.7. *Suppose (P, Open) is a semitopology and $p, p' \in P$. Then if p is regular and $p' \in K(p)$, then p' is regular and has the same community.*

Proof. Suppose p is regular – so by Definition 4.1.4(3) $p \in K(p) \in \text{Topen}$ – and suppose $p' \in K(p)$. Then by Corollary 4.2.6 $K(p) = K(p')$, so $p' \in K(p') \in \text{Topen}$ and by Theorem 4.2.5 p' is regular. $\square$

4.3 Intersection and partition properties of regular spaces

Proposition 4.3.1 is useful for consensus in practice. Suppose we are at a regular point q and we have reached consensus with some topen neighbourhood $O \ni q$. Suppose further that our topen neighbourhood O intersects with the maximal topen neighbourhood $K(p)$ of some other regular point p . Then Proposition 4.3.1 tells us that we were inside $K(p)$ all along.

Proposition 4.3.1. *Suppose (P, Open) is a semitopology and $p \in P$ is regular and $O \in \text{Topen}$ is topen. Then*

$$O \not\sim K(p) \quad \text{if and only if} \quad O \subseteq K(p).$$

Proof. The right-to-left implication is immediate from Notation 3.1.1(1), given that topens are nonempty by Definition 3.2.2(2).

For the left-to-right implication, suppose $O \not\sim K(p)$. By Theorem 4.2.5, $K(p)$ is a maximal topen, and by Proposition 3.4.4, $O \cup K(p)$ is topen. Then $O \subseteq K(p)$ follows by maximality. $\square$

Proposition 4.3.2. *Suppose (P, Open) is a semitopology, and suppose $p, p' \in P$ are regular. Then*

$$K(p) \not\subseteq K(p') \iff K(p) = K(p')$$

Proof. We prove two implications.

- Suppose there exists $p'' \in K(p) \cap K(p')$. By Corollary 4.2.7 (p'' is regular and) $K(p) = K(p'') = K(p')$.
- Suppose $K(p) = K(p')$. By assumption $p \in K(p)$, so $p \in K(p')$. Thus $p \in K(p) \cap K(p')$. □

Corollary 4.3.3 is a simple characterisation of regular semitopological spaces:

Corollary 4.3.3. *Suppose (P, Open) is a semitopology. Then the following are equivalent:*

- (1) (P, Open) is regular.
- (2) P partitions into topen sets: there exists some set of topen sets $\mathcal{T}$ such that $T \not\subseteq T'$ for every $T, T' \in \mathcal{T}$ and $P = \bigcup \mathcal{T}$.
- (3) Every $X \subseteq P$ has a cover of topen sets: there exists some set of topen sets $\mathcal{T}$ such that $X \subseteq \bigcup \mathcal{T}$.

Proof. The proof is routine from the machinery that we already have. We prove equivalence of parts 1 and 2:

- (1) Suppose (P, Open) is regular, meaning by Definition 4.1.4(7 and 3) that $p \in K(p) \in \text{Topen}$ for every $p \in P$. We set $\mathcal{T} = \{K(p) \mid p \in P\}$. By assumption, this covers P in topens, and by Proposition 4.3.2, the cover is a partition.
- (2) Suppose $\mathcal{T}$ is a topen partition of P . By definition for every point p there exists $T \in \mathcal{T}$ such that $p \in T$ and so p has a topen neighbourhood. By Theorem 4.2.5(5 and 1), p is regular.

We prove equivalence of parts 2 and 3:

- (1) Suppose $\mathcal{T}$ is a topen partition of P , and suppose $X \subseteq P$. Then trivially $X \subseteq \bigcup \mathcal{T}$.
- (2) Suppose every $X \subseteq P$ has a cover of topen sets. Then P has a cover of topen sets; write it $\mathcal{T}$. By Corollary 3.4.5, we may assume without loss of generality that $\mathcal{T}$ is a partition, and we are done. □

Notation 4.3.4. *Call a semitopology (P, Open) **singular** when it contains a single maximal topen subset.*

Remark 4.3.5. The moral we take from the results and examples above (and those to follow) is that the world we are entering has rather different well-behavedness criteria than those familiar from the study of typical Hausdorff topologies like $\mathbb{R}$:

- (1) ‘Bad’ spaces are spaces that are not regular.
 $\mathbb{R}$ with its usual topology (which is also a semitopology) is an example of a ‘bad’ semitopology; it is not even quasiregular.
- (2) ‘Good’ spaces are spaces that are regular.
The supermajority and all-but-one semitopologies from Example 2.1.4(5a and 5c) are typical examples of ‘good’ semitopologies; both are singular regular spaces.
- (3) Corollary 4.3.3 shows that the ‘good’ spaces are just the (disjoint, possibly infinite) unions of singular regular spaces.

So to sum this up: modulo disjoint unions, the study of consensus behaviour is the study of semitopological spaces that consist of a single topen set of points that are all intertwined with one another.

5. Regular = Weakly Regular + Unconflicted

5.1 Closed sets

In Section 5.1, we check that some familiar properties of closures carry over from topologies to semitopologies. There are no technical surprises, but this in itself is a mathematical result that needs to be checked. Then, we will use this to study the relation between closures and sets of intertwined points.

Definition 5.1.1. Suppose (P, Open) is a semitopology, and suppose $p \in P$ and $P \subseteq P$. Then:

- (1) Define $|P| \subseteq P$ to be the **closure** of P to be the set of points p such that every open neighbourhood of p intersects P . In symbols using Notation 3.1.1:

$$|P| = \{p' \in P \mid \forall O \in \text{Open}. p' \in O \implies P \cap O \neq \emptyset\}.$$

- (2) As is standard, we may write $|p|$ for $\{p\}$. Unpacking definitions for reference:

$$|p| = \{p' \in P \mid \forall O \in \text{Open}. p' \in O \implies p \in O\}.$$

Lemma 5.1.2. Suppose (P, Open) is a semitopology and suppose $P, P' \subseteq P$. Then taking the closure of a set is:

- (1) *Monotone:* If $P \subseteq P'$ then $|P| \subseteq |P'|$.
- (2) *Increasing:* $P \subseteq |P|$.
- (3) *Idempotent:* $|P| = ||P||$.

Proof. By routine calculations from Definition 5.1.1. □

Lemma 5.1.3. Suppose (P, Open) is a semitopology and $P \subseteq P$ and $O \in \text{Open}$. Then

$$P \cap O \neq \emptyset \text{ if and only if } |P| \cap O \neq \emptyset.$$

Proof. Suppose $P \cap O \neq \emptyset$. Then $|P| \cap O \neq \emptyset$ using Lemma 5.1.2(2).

Suppose $|P| \cap O \neq \emptyset$. Pick $p \in |P| \cap O$. By construction of $|P|$ in Definition 5.1.1, $p \in O \implies P \cap O \neq \emptyset$. It follows that $P \cap O \neq \emptyset$ as required. □

Definition 5.1.4. Suppose (P, Open) is a semitopology and suppose $C \subseteq P$.

- (1) Call C a **closed set** when $C = |C|$.
- (2) Call C a **clopen set** when C is closed and open.
- (3) Write **Closed** for the set of **closed sets** (as we wrote **Open** for the open sets; the ambient semitopology will always be clear or understood).

Lemma 5.1.5. Suppose (P, Open) is a semitopology and suppose $P \subseteq P$. Then $|P|$ is closed and contains P . In symbols:

$$P \subseteq |P| \in \text{Closed}.$$

Proof. From Definition 5.1.4(1) and Lemma 5.1.2(2 and 3). □

Example 5.1.6.

- (1) Take $P = \{0, 1\}$ and $\text{Open} = \{\emptyset, \{0\}, \{0, 1\}\}$. Then the reader can verify that:
 - $\{0\}$ is open.
 - The closure of $\{1\}$ is $\{1\}$, and $\{1\}$ is closed.
 - The closure of $\{0\}$ is $\{0, 1\}$.
 - $\emptyset$ and $\{0, 1\}$ are the only clopen sets.
- (2) Now take $P = \{0, 1\}$ and $\text{Open} = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$.¹² Then the reader can verify that:
 - Every set is clopen.
 - The closure of every set is itself.

Remark 5.1.7. There are two standard definitions for when a set is closed: when it is equal to its closure (as per Definition 5.1.4(1)) and when it is the complement of an open set. In topology these are equivalent. We do need to check that the same holds in semitopology, but as it turns out, the proof is routine:

Lemma 5.1.8. *Suppose (P, Open) is a semitopology. Then:*

- (1) *Suppose $C \in \text{Closed}$ is closed (by Definition 5.1.4: $C = |C|$). Then $P \setminus C$ is open.*
- (2) *Suppose $O \in \text{Open}$ is open. Then $P \setminus O$ is closed (by Definition 5.1.4: $|P \setminus O| = P \setminus O$).*

Proof.

- (1) Suppose $p \in P \setminus C$. Since $C = |C|$, we have $p \in P \setminus |C|$. Unpacking Definition 5.1.1, this means precisely that there exists $O_p \in \text{Open}$ with $p \in O_p \not\subseteq C$. We use Lemma 2.3.2.
- (2) Suppose $O \in \text{Open}$. Combining Lemma 2.3.2 with Definition 5.1.1, it follows that $O \not\subseteq |P \setminus O|$ so that $|P \setminus O| \subseteq P \setminus O$. Furthermore, by Lemma 5.1.2(2), $P \setminus O \subseteq |P \setminus O|$. □

The usual duality between forming closures and interiors remains valid in semitopologies:

Lemma 5.1.9. *Suppose (P, Open) is a semitopology and $O \in \text{Open}$ and $C \in \text{Closed}$. Then:*

- (1) $O \subseteq \text{interior}(|O|)$. *The inclusion may be strict.*
- (2) $|\text{interior}(C)| \subseteq C$. *The inclusion may be strict.*
- (3) $\text{interior}(P \setminus O) = P \setminus |O|$.
- (4) $|P \setminus C| = P \setminus \text{interior}(C)$.

Proof. The reasoning is just as for topologies, but we spell out the details:

- (1) By Lemma 5.1.2(2), $O \subseteq |O|$. By Corollary 4.1.3, $\text{interior}(O) \subseteq \text{interior}(|O|)$. By Lemma 4.1.2, $O = \text{interior}(O)$, so we are done.
For an example of the strict inclusion, consider $\mathbb{R}$ with the usual topology (which is also a semitopology) and take $O = (0, 1) \cup (1, 2)$. Then $O \subsetneq \text{interior}(|O|) = (0, 2)$.
- (2) By Lemma 4.1.2, $\text{interior}(C) \subseteq C$. By Lemma 5.1.2(1), $|\text{interior}(C)| \subseteq |C|$. By Definition 5.1.4(1), (since we assumed $C \in \text{Closed}$) $|C| = C$, so we are done.
For an example of the strict inclusion, consider $\mathbb{R}$ with the usual topology and take $C = \{0\}$. Then $|\text{interior}(C)| = \emptyset \subsetneq C$.

- (3) Consider some $p' \in P$. By Definition 4.1.1, $p' \in \text{interior}(P \setminus O)$ when there exists some $O' \in \text{Open}$ such that $p' \in O' \not\subseteq O$. By definition in Definition 5.1.1(1), this happens precisely when $p' \notin |O|$.
- (4) By Definition 5.1.1(1), $p' \notin |P \setminus C|$ precisely when there exists some $O' \in \text{Open}$ such that $p' \in O' \not\subseteq P \setminus C$. By facts of sets, this means precisely that $p' \in O' \subseteq C$. By Definition 4.1.1, this means precisely that $p' \in \text{interior}(C)$. $\square$

5.2 Closed neighbourhoods and intertwined points

Definition 5.2.1. Suppose (P, Open) is a semitopology. We generalise Definition 2.3.1 as follows:

- (1) Call $P \subseteq P$ a **neighbourhood** when it contains an open set (i.e., when $\text{interior}(P) \neq \emptyset$), and call P a **neighbourhood of p** when $p \in P$ and P contains an open neighbourhood of p (i.e., when $p \in \text{interior}(P)$). In particular:
- (2) $C \subseteq P$ is a **closed neighbourhood of p** when C is closed and $p \in \text{interior}(C)$.
- (3) $C \subseteq P$ is a **closed neighbourhood** when C is closed and $\text{interior}(C) \neq \emptyset$.

Remark 5.2.2.

- (1) If C is a closed neighbourhood of p in the sense of Definition 5.2.1(2), then C is a closed neighbourhood in the sense of Definition 5.2.1(3), just because if $p \in \text{interior}(C)$, then $\text{interior}(C) \neq \emptyset$.
- (2) $p \in C$ is not enough for C to be a closed neighbourhood of p ; we require the stronger condition $p \in \text{interior}(C)$. For instance, consider the **Sierpiński space**

$$Sk = (\{0, 1\}, \{\emptyset, \{1\}, \{0, 1\}\}).$$

is a topology and so also a semitopology (cf. Remark 12.3.2), and consider $p = 0$ and $C = \{0\}$. Then $p \in C$ but $p \notin \text{interior}(C) = \emptyset$, so that C is not a closed neighbourhood of p .

Remark 5.2.3. Recall the definitions of $p_{\emptyset}$ and $|p|$:

- The set $|p|$ is the closure of p .
By Definition 5.1.1, this is the set of p' such that every open neighbourhood $O' \ni p'$ intersects with $\{p\}$.
- The set $p_{\emptyset}$ is the set of points intertwined with p .
By Definition 3.5.1(2), this is the set of p' such that every open neighbourhood $O' \ni p'$ intersects with every open neighbourhood $O \ni p$.

Lemma 5.2.4 rephrases Remark 5.2.3 more precisely by looking at it through set complements. We will use it in Lemma 11.4.4(2):

Lemma 5.2.4. Suppose (P, Open) is a semitopology and $p \in P$. Then:

- (1) $P \setminus |p| = \bigcup \{O \in \text{Open} \mid p \notin O\} \in \text{Open}$.
- (2) $P \setminus p_{\emptyset} = \bigcup \{O' \in \text{Open} \mid \exists O \in \text{Open}. p \in O \wedge O' \not\subseteq O\} \in \text{Open}$.
- (3) $P \setminus p_{\emptyset} = \bigcup \{O \in \text{Open} \mid p \notin |O|\} \in \text{Open}$.

In words, we can say that $P \setminus |p|$ is the union of the open sets such that p avoids them, and $P \setminus p_{\checkmark}$ is the union of the open sets such that p avoids their closures.

Proof.

- (1) Immediate from Definitions 3.5.1 and 5.1.1. Openness is from Definition 1.2.2(2).
- (2) By a routine argument direct from Definition 3.5.1. Openness is from Definition 1.2.2(2).
- (3) Rephrasing part 2 of this result using Definition 5.1.1(1). □

Remark 5.2.5. We can relate Lemma 5.2.4 to a concept from topology. Following standard terminology ((Bourbaki, 1998, Definition 2, p. 69) or (Engelking, 1989, p. 52)), a **cluster point** $p \in P$ of $\mathcal{O} \subseteq \text{Open}$ is one such that every open neighbourhood of p intersects every $O \in \mathcal{O}$. Then Lemma 5.2.4(2) identifies $p_{\checkmark}$ as the set of cluster points of $\text{nbhd}(p) \subseteq \text{Open}$ from Definition 2.3.1.

5.3 Regular = weakly regular + unconflicted

In Lemma 3.5.3, we asked whether the ‘is intertwined with’ relation $\checkmark$ from Definition 3.5.1(1) is transitive – answer: not necessarily.

Transitivity of $\checkmark$ is a natural condition. We now have enough machinery to study it in more detail, and this will help us gain a deeper understanding of the properties of not-necessarily-regular points.

Definition 5.3.1. Suppose (P, Open) is a semitopology.

- (1) Call a point p **conflicted** when there exist p' and p'' such that $p' \checkmark p$ and $p \checkmark p''$ yet $\neg(p' \checkmark p'')$.
- (2) If $p' \checkmark p \checkmark p''$ implies $p' \checkmark p''$ always, then call p **unconflicted**.
- (3) Continuing Definition 4.1.4(7), if $P \subseteq P$ and every $p \in P$ is conflicted/unconflicted, then we may call P **conflicted/unconflicted**, respectively.

Example 5.3.2. We consider some examples:

- (1) In Figure 2, in the top-left diagram, 0 and 2 are unconflicted and intertwined with themselves, and 1 is conflicted (being intertwined with 0, 1, and 2).
If the reader wants to know what a conflicted point looks like, it looks like 1.
- (2) In Figure 2, in the top-right diagram, 0 and 2 are unconflicted and intertwined with themselves, and 1 is conflicted (being intertwined with 0, 1, and 2).
- (3) In Figure 2, in the lower-left diagram, 0 and 1 are unconflicted and intertwined with themselves, and 3 and 4 are unconflicted and intertwined with themselves, and 2 is conflicted (being intertwined with 0, 1, 2, 3, and 4).
- (4) In Figure 2, in the lower-right diagram, all points are unconflicted, and 0 and 2 are intertwined just with themselves, and 1 and $*$ are intertwined with one another.
- (5) In Figure 3, all points are unconflicted and intertwined only with themselves. (This semitopology is useful for counterexamples, and we will use it again.)

Proposition 5.3.3. Suppose (P, Open) is a semitopology and $p \in P$. Then:

- (1) If p is regular, then it is unconflicted.
Equivalently by the contrapositive: if p is conflicted, then it is not regular.
- (2) p may be unconflicted and neither quasi-regular, weakly regular, nor regular.

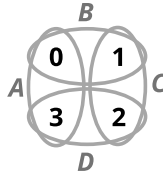


Figure 3. An unconflicted, irregular space (Proposition 5.3.3) in which every point is intertwined only with itself (Example 5.3.2).

Proof. We consider each part in turn:

- (1) So consider $q \not\sim p \not\sim q'$. We must show that $q \not\sim q'$, so consider open neighbourhoods $Q \ni q$ and $Q' \ni q'$. By assumption p is regular, so unpacking Definition 4.1.4(3) $K(p)$ is a topen (transitive and open) neighbourhood of p . By assumption $Q \not\sim K(p) \not\sim Q'$, and by transitivity of $K(p)$ (Definition 3.2.2(1)) we have $Q \not\sim Q'$ as required.
- (2) Consider the semitopology illustrated in Figure 3. Note that the point 0 is not conflicted (because it is not intertwined with any other point), but it is also neither quasiregular, weakly regular, nor regular, because its community is the empty set. □

We can combine Proposition 5.3.3 with a previous result, Lemma 4.1.6, to get a precise and attractive relation between being

- regular (Definition 4.1.4(3)),
- weakly regular (Definition 4.1.4(4)), and
- unconflicted (Definition 5.3.1),

as follows:

Theorem 5.3.4. Suppose (P, Open) is a semitopology and $p \in P$. Then the following are equivalent:

- p is regular.
- p is weakly regular and unconflicted.

More succinctly we can write: $\text{regular} = \text{weakly regular} + \text{unconflicted}$.¹³

Proof. We prove two implications:

- If p is regular, then it is weakly regular by Lemma 4.1.6 and unconflicted by Proposition 5.3.3(1).
- Suppose p is weakly regular and unconflicted. By Definition 4.1.4(4), $p \in K(p)$, and by Lemma 3.5.4, it would suffice to show that $q \not\sim q'$ for any $q, q' \in K(p)$. So consider $q, q' \in K(p)$. Now by Definition 4.1.4(1) $K(p) = \text{interior}(p_{\not\sim})$ so in particular $q, q' \in p_{\not\sim}$. Thus $q \not\sim p \not\sim q'$, and since p is unconflicted $q \not\sim q'$ as required. □

6. Regular = Quasiregular + Hypertransitive

6.1 Regular open/closed sets

We recall some simple results:

Definition 6.1.1. Suppose (P, Open) is a semitopology. Recall some standard terminology from topology (Willard, 1970, Exercise 3D, p. 29):

- (1) We call an open set $O \in \text{Open}$ a **regular open set** when $O = \text{interior}(|O|)$.
- (2) We call a closed set $C \in \text{Closed}$ a **regular closed set** when $C = |\text{interior}(C)|$.
- (3) Write Open_{reg} and $\text{Closed}_{\text{reg}}$ for the sets of regular open and regular closed sets, respectively.

Lemma 6.1.2. *Suppose (P, Open) is a semitopology and $O \in \text{Open}$ and $C \in \text{Closed}$. Then:*

- (1) $|O| = |\text{interior}(|O|)|$.
- (2) $\text{interior}(C) = \text{interior}(|\text{interior}(C)|)$.

Proof. We use Lemma 5.1.9(1 and 3) along with Lemma 5.1.2(1) and Corollary 4.1.3:

$$\begin{array}{ccccc}
 |O| & \stackrel{L5.1.9(1) \& L5.1.9(2)}{\subseteq} & |\text{interior}(|O|)| & \stackrel{L5.1.9(2)}{\subseteq} & \text{interior}(|O|) \\
 \text{interior}(C) & \stackrel{L5.1.9(1)}{\subseteq} & \text{interior}(|\text{interior}(C)|) & \stackrel{L5.1.9(2) \& C4.1.3}{\subseteq} & \text{interior}(C)
 \end{array}$$

□

The terminology ‘regular open/closed set’ is from the topological literature. It is not directly related to the terminology ‘regular point’ from Definition 4.1.4(3), which comes from semitopologies. However, it turns out that a mathematical connection does exist between these two notions:

Corollary 6.1.3. *Suppose (P, Open) is a semitopology and $O \in \text{Open}$ and $C \in \text{Closed}$. Then:*

- (1) $\text{interior}(C)$ is a regular open set.
- (2) $|O|$ is a regular closed set.

Proof. Directly from Definition 6.1.1 and Lemma 6.1.2. □

An elementary observation about open sets will be useful:

Lemma 6.1.4. *Suppose (P, Open) is a semitopology and $O, O' \in \text{Open}$. Then the following are equivalent:*

- (1) $O \not\sim O'$.
- (2) $O \not\sim \text{interior}(|O'|)$.
- (3) $\text{interior}(|O|) \not\sim \text{interior}(|O'|)$.

Proof. First we prove the equivalence of parts 1 and 2:

- (1) Suppose $O \not\sim O'$. By Lemma 5.1.9(1), $O \not\sim \text{interior}(|O'|)$.
- (2) Suppose there is some $p \in O \cap \text{interior}(|O'|)$. Then O is an open neighbourhood of p and $p \in |O'|$, so by Definition 5.1.1(1), $O \not\sim O'$ as required.¹⁴

Equivalence of parts 1 and 3 then follows easily by two applications of the equivalence of parts 1 and 2. □

Remark 6.1.5. Lemma 6.1.4 is true in topologies as well, but it is not prominent in the literature. Two standard reference works (Willard 1970; Engelking 1989) do not seem to mention it. It appears as Equation (10) in Theorem 1.37 of Koppelberg (1989) and as a lemma in π -base π -Base (thanks to the mathematics StackExchange community for the pointers). Interestingly, this result is as

true in topologies as it is in semitopologies, but somehow, it matters more in the latter than the former.

6.2 Hypertransitivity

We are now ready to define hypertransitivity.

Notation 6.2.1. Suppose (P, Open) is a semitopology and $O' \in \text{Open}$ and $\mathcal{O} \subseteq \text{Open}$.

- (1) Write $O' \not\sim \mathcal{O}$, or equivalently $\mathcal{O} \not\sim O'$, when $O' \not\sim O$ for every $O \in \mathcal{O}$. In symbols:

$$O' \not\sim \mathcal{O} \quad \text{when} \quad \forall O \in \mathcal{O}. O' \not\sim O.$$

- (2) As a special case of part 1 above, taking $\mathcal{O} = \text{nbhd}(p)$ (Definition 2.3.1), if $p \in P$, then write $O' \not\sim \text{nbhd}(p)$, or equivalently $\text{nbhd}(p) \not\sim O'$, when $O' \not\sim O$ for every $O \in \text{Open}$ such that $p \in O$.

Lemma 6.2.2. Suppose (P, Open) is a semitopology and $p \in P$ and $O' \in \text{Open}$. Then

$$p \in |O'| \quad \text{if and only if} \quad O' \not\sim \text{nbhd}(p).$$

Proof. This just rephrases Definition 5.1.1(1). □

Definition 6.2.3. Suppose (P, Open) is a semitopology. Call $p \in P$ a **hypertransitive point** when for every $O', O'' \in \text{Open}$,

$$O' \not\sim \text{nbhd}(p) \not\sim O'' \quad \text{implies} \quad O' \not\sim O''.$$

Call (P, Open) a **hypertransitive semitopology** when every $p \in P$ is hypertransitive.

Lemma 6.2.4 notes some equivalent formulations of hypertransitivity:

Lemma 6.2.4. Suppose (P, Open) is a semitopology and $p \in P$. Then the following are equivalent:

- (1) p is hypertransitive.
- (2) For every pair of open sets $O', O'' \in \text{Open}$, $p \in |O'| \cap |O''|$ implies $O' \not\sim O''$.
- (3) For every pair of regular open sets $O', O'' \in \text{Open}_{\text{reg}}$, $p \in |O'| \cap |O''|$ implies $O' \not\sim O''$.

Proof. For the equivalence of parts 1 and 2, we reason as follows:

- Suppose p is hypertransitive, and suppose $p \in |O'|$ and $p \in |O''|$. By Lemma 6.2.2 it follows that $O' \not\sim \text{nbhd}(p) \not\sim O''$. By hypertransitivity, $O' \not\sim O''$ as required.
- Suppose for every $O, O' \in \text{Open}$, $p \in |O| \cap |O'|$ implies $O' \not\sim O''$, and suppose $O' \not\sim \text{nbhd}(p) \not\sim O''$. By Lemma 6.2.2 $p \in |O| \cap |O'|$ and therefore $O' \not\sim O''$.

For the equivalence of parts 2 and 3, we reason as follows:

- Part 2 implies part 3 follows since every open regular set is also an open set.
- To show part 3 implies part 2, suppose for every pair of regular opens $O', O'' \in \text{Open}_{\text{reg}}$, $p \in |O'| \cap |O''|$ implies $O' \not\sim O''$, and suppose $O', O'' \in \text{Open}$ are two open sets that are not necessarily regular, and suppose $p \in |O'| \cap |O''|$. We must show that $O' \not\sim O''$.
Write $P' = \text{interior}(|O'|)$ and $P'' = \text{interior}(|O''|)$ and note by Corollary 6.1.3 and Lemma 5.1.5 that P' and P'' are regular open sets and $|P'| = |O'|$ and $|P''| = |O''|$. Then $|P'| \not\sim |P''|$, so $P' \not\sim P''$, and $O' \not\sim O''$ follows from Lemma 6.1.4. □

6.3 Regular = quasiregular + hypertransitive

Lemma 6.3.1. *Suppose (P, Open) is a semitopology and $p \in P$. Then:*

- (1) *If p is regular, then it is hypertransitive.*
- (2) *If p is hypertransitive, then it is unconflicted.*
- (3) *The reverse implication need not hold: it is possible for p to be unconflicted but not hypertransitive.*
- (4) *It is possible for p to be hypertransitive (and unconflicted), but not quasiregular (and thus not weakly regular or regular).*

Proof. We consider each part:

- (1) Suppose p is regular and $O, O' \in \text{Open}$ and $O \not\subseteq \text{nbhd}(p) \not\subseteq O'$. By Definition 4.1.4(3) (since p is regular), $K(p)$ is a topen (= open and transitive) neighbourhood of p . Therefore, by transitivity, $O \not\subseteq O'$ as required.
- (2) Suppose p is hypertransitive, and suppose $p', p'' \in P$ and $p' \not\subseteq p \not\subseteq p''$. Now consider $p' \in O' \in \text{Open}$ and $p'' \in O'' \in \text{Open}$. By our intertwinedness assumptions we have that $O' \not\subseteq \text{nbhd}(p) \not\subseteq O''$. But p is hypertransitive, so $O' \not\subseteq O''$ as required.
- (3) It suffices to provide a counterexample. Consider the bottom right semitopology in Figure 2, and take $p = *$ and $O' = \{1\}$ and $O'' = \{0, 2\}$. Note that:
 - $*$ is unconflicted, since it is intertwined only with itself and 1.
 - O' and O'' intersect every open neighbourhood of $*$, but $O' \not\subseteq O''$, so $*$ is not hypertransitive.
- (4) It suffices to provide an example. Consider the semitopology illustrated in Figure 2, top-right diagram; so $P = \{0, 1, 2\}$ and $\text{Open} = \{\emptyset, \{0\}, \{2\}, \{1, 2\}, \{0, 1\}, \{0, 1, 2\}\}$. The reader can check that $p = 1$ is hypertransitive, but $1 \not\subseteq \{1\}$ and $K(1) = \emptyset$ so p is not quasiregular. □

(Yet) another characterisation of being quasiregular will be helpful:

Lemma 6.3.2. *Suppose (P, Open) is a semitopology and $p \in P$. Then the following conditions are equivalent:*

- (1) *p is quasiregular (meaning by Definition 4.1.4(5) that $K(p) \neq \emptyset$).*
- (2) *$K(p) \not\subseteq \text{nbhd}(p)$ (meaning by Notation 6.2.1(2) that $K(p) \not\subseteq O$ for every $O \in \text{nbhd}(p)$).*
- (3) *$p \in |K(p)|$.*

Proof. Equivalence of parts 2 and 3 is immediate from Lemma 6.2.2.

For equivalence of parts 1 and 2, we prove two implications:

- Suppose p is quasiregular, meaning by Definition 4.1.4(5) that $K(p) \neq \emptyset$. Pick some $p' \in K(p)$ (it does not matter which). It follows by construction in Definitions 3.5.1(2) and 4.1.4(1) and Lemma 4.1.2 that $p' \not\subseteq p$, so that $p' \in K(p)$. Using Definition 3.5.1(1), it follows that $K(p) \not\subseteq O$ for every $O \in \text{nbhd}(p)$, as required.
- Suppose $K(p) \not\subseteq \text{nbhd}(p)$. Then in particular $K(p) \not\subseteq P$ (because $p \in P \in \text{Open}$), and by Notation 3.1.1(1) it follows that $K(p) \neq \emptyset$. □

Compare and contrast Theorem 6.3.3 with Theorem 5.3.4:

Theorem 6.3.3. Suppose (P, Open) is a semitopology and $p \in P$. Then the following are equivalent:

- (1) p is regular.
- (2) p is quasiregular and hypertransitive.

Proof. We consider two implications:

- Suppose p is regular.

Then p is quasiregular by Lemma 4.1.6(1 and 2) and hypertransitive by Lemma 6.3.1(1).

- Suppose p is quasiregular and hypertransitive.

By Lemma 6.3.1(2), p is unconflicted. If we can prove that p is weakly regular (meaning by Definition 4.1.4(4) that $p \in K(p)$), then by Theorem 5.3.4 it would follow that p is regular as required. Thus, it would suffice to show that $p \in K(p)$, thus that there is an open neighbourhood of points with which p is intertwined.

Write $O'' = \text{interior}(P \setminus K(p))$. We have two subcases to consider:

- Suppose $\text{nbhd}(p) \not\subseteq O''$.

By Lemma 6.3.2 (since p is quasiregular) we have that $K(p) \not\subseteq \text{nbhd}(p)$. Thus $K(p) \not\subseteq \text{nbhd}(p) \not\subseteq O''$, and by hypertransitivity of p it follows that $K(p) \not\subseteq O''$. But this contradicts the construction of O'' as being a subset of $P \setminus K(p)$, so this case is impossible.

- Suppose $\text{nbhd}(p) \subseteq O''$. Then there exists some $O \in \text{nbhd}(p)$ such that $O \not\subseteq O''$, and it follows that $O \subseteq K(p)$ so that $p \in K(p)$ as required.

Thus, p is weakly regular, as required. $\square$

Remark 6.3.4. Recall from Definition 4.1.4(3) the notion of a regular point, which in the theory of semitopologies is a canonical well-behavedness property because it ensures that the point is part of a community of other regular points, all in a single maximal intertwined actionable coalition (i.e., open set). We have obtained two nice characterisations of regularity:

- (1) Regular = weakly regular + unconflicted, by Theorem 5.3.4.
- (2) Regular = quasiregular + hypertransitive, by Theorem 6.3.3.

We are now ready to dualise everything – including semitopologies, regularity, weak regularity, quasiregularity, and being unconflicted and hypertransitive – and thus we will investigate the algebraic structures which correspond to the set structures above:

7. Semiframes: Compatible Complete Semilattices

7.1 Complete join-semilattices and morphisms between them

We recall some (mostly standard) definitions and facts:

Definition 7.1.1.

- (1) A **poset** $(X, \leq)$ is a set X of **elements** and a relation $\leq \subseteq X \times X$ that is transitive, reflexive, and antisymmetric.
- (2) A poset $(X, \leq)$ is a **complete join-semilattice** when every $X \subseteq X$ (X may be empty or equal to all of X) has a least upper bound – or **join** – $\bigvee X \in X$.

All the semilattices we consider will be join (rather than meet) semilattices, so we may omit the word ‘join’ and just call this a complete semilattice henceforth.

- (3) If $(X, \leq)$ is a complete semilattice, then we may write

$$\perp_X = \bigvee \emptyset.$$

By the least upper bound property, $\perp_X \leq x$ for every $x \in X$.

- (4) If $(X, \leq)$ is a complete semilattice, then we may write

$$\top_X = \bigvee X.$$

By the least upper bound property, $x \leq \top_X$ for every $x \in X$.

Definition 7.1.2. Suppose $(X', \leq')$ and $(X, \leq)$ are complete join-semilattices. Define a **morphism** $g: (X', \leq') \rightarrow (X, \leq)$ to be a function $X' \rightarrow X$ that commutes with joins and sends $\top_{X'}$ to $\top_X$. That is:

- (1) If $X' \subseteq X'$, then $g(\bigvee X') = \bigvee_{x' \in X'} g(x')$.
- (2) $g(\top_{X'}) = \top_X$.

Remark 7.1.3. In Definition 7.1.1(2), we insist that $g(\top_{X'}) = \top_X$; that is, we want our notion of morphism to preserve the top element.

This does not follow from Definition 7.1.1(1), because g need not be surjective onto X , so we need to add this as a separate condition. Contrast with $g(\perp_X) = \perp_{X'}$, which does follow from Definition 7.1.1(1), because $\perp_X$ is the least upper bound of $\emptyset$.

We want $g(\top_{X'}) = \top_X$ because our intended model is that $(X, \leq) = (\text{Open}, \subseteq)$ is the semilattice of open sets of a semitopology (P, Open) , and similarly for $(X', \leq')$, and g is equal to f^{-1} where $f: (P, \text{Open}) \rightarrow (P', \text{Open}')$ is a continuous function.

We recall a standard result:

Lemma 7.1.4. Suppose $(X, \leq)$ is a complete join-semilattice. Then:

- (1) If $x_1, x_2 \in X$, then $x_1 \leq x_2$ if and only if $x_1 \vee x_2 = x_2$.
- (2) If $f: (X, \leq) \rightarrow (X', \leq')$ is a semilattice morphism (Definition 7.1.2), then f is a **monotone morphism**: if $x_1 \leq x_2$, then $f(x_1) \leq f(x_2)$, for every $x_1, x_2 \in X$.

Proof. We consider each part in turn:

- (1) Suppose $x_1 \leq x_2$. By the definition of a least upper bound, this means precisely that x_2 is a least upper bound for $\{x_1, x_2\}$. It follows that $x_1 \vee x_2 = x_2$. The converse implication follows just by reversing this reasoning.
- (2) Suppose $x_1 \leq x_2$. By part 1 of this result $x_1 \vee x_2 = x_2$, so $f(x_1 \vee x_2) = f(x_2)$. By Definition 7.1.2, $f(x_1) \vee f(x_2) = f(x_2)$. By part 1 of this result, $f(x_1) \leq f(x_2)$. □

Remark 7.1.5. As the reader may know, frames and locales are the same thing: the category of locales is just the categorical opposite of the category of frames. So every time we write ‘semiframe’, the reader can safely read ‘semilocale’; these are two names for essentially the same structure up to reversing arrows. The literature on frames and locales is huge: the interested reader can consult two classic texts (Johnstone, 1986; Mac Lane and Moerdijk, 1992); more recent (and very readable) presentations include Picado and Pultr (2012, 2021).

7.2 The compatibility relation

Definition 7.2.1 is a simple idea, but so far as we are aware, it is novel:

Definition 7.2.1. Suppose $(X, \leq)$ is a complete semilattice. A **compatibility relation** $* \subseteq X \times X$ is a relation on X such that:

- (1) $*$ is symmetric, so if $x, x' \in X$ then

$$x * x' \text{ if and only if } x' * x.$$

- (2) $*$ is a **properly reflexive relation**,¹⁵ by which we mean

$$\forall x \in X \setminus \{\perp_X\}. x * x.$$

Note that it will follow from the axioms of a compatibility relation that $x * x \iff x \neq \perp_X$; see Lemma 7.3.7(2).

- (3) $*$ satisfies a **distributive law**, that if $x \in X$ and $X' \subseteq X$ then

$$x * \bigvee X' \iff \exists x' \in X'. x * x'.$$

Thus, we can say:

a compatibility relation $* \subseteq X \times X$ is a symmetric properly reflexive completely distributive relation on X .

When $x * x'$ holds, we may call x and x' **compatible elements**.

Remark 7.2.2. The compatibility relation $*$ is what it is, but we take a moment to discuss some intuitions, and to put it in the context of some natural generalisations:

- (1) We can think of $*$ as an abstract intersection.

It lets us observe whether x and x' intersect – but without having to explicitly represent this intersection as a meet $x \wedge x'$ in the semilattice itself.

We call $*$ a compatibility relation following an intuition of $x, x' \in X$ as observations, and $x * x'$ holds when there is some possible world at which it is possible to observe x and x' together. More on this in Example 7.3.3.

- (2) We can think of $*$ as a generalised intersection; so our notion of semiframe in Definition 7.3.1 is an instance of a frame with a generalised meet.

We will concentrate on the case where $x * x'$ measures whether x and x' intersect, but there are other possibilities. Here are some natural ways to proceed:

- $(X, \leq)$ is a complete join-semilattice and $* : (X \times X) \rightarrow X$ is any commutative distributive map. For concreteness, we can set $x * x' \in \{\perp_X, \top_X\} \subseteq X$.
- $(X, \leq)$ is a complete join-semilattice and $* : (X \times X) \rightarrow \mathbb{N}$ is any commutative distributive map. We think of $x * x'$ as returning the size of the intersection of x and x' .
- Any complete join-semilattice $(X, \leq)$ is of course a (generalised) semiframe by taking $x * x' = \bigvee \{x'' \mid x'' \leq x, x'' \leq x'\}$.
- We can generalise further, in more than one direction. We would take $(X, \leq)$ and $(X', \leq')$ to be complete join-semilattices and $* : (X \times X) \rightarrow X'$ to be any commutative distributive map (which generalises the above). We could also take X to be a cocomplete symmetric monoidal category (Mac Lane, 1971, Section VII): a category with all colimits and with a (symmetric) monoid action $*$ that distributes over (commutes with) colimits.
- The compatibility relation $*$ is binary: as noted in part 1 of this Remark, it abstracts two open sets having a nonempty open intersection.

However, the literature on distributed systems is rich in well-behavedness conditions based on intersections of sets: for example, ones corresponding to nonempty ternary intersections between open sets, and on intersections between open and closed sets, and on other types of intersections. These are not thought of in the literature in topological/algebraic terms – that way of looking at things is a contribution of this paper and the recent Gabbay (2025) – but they could be.

This suggests that $*$ may be a (canonical simplest) representative of a design space of ‘nonempty n -ary intersection’ operators, which remains to be explored.

More details, with some references, are in Remark 12.3.6.

Lemma 7.2.3. *Suppose $(X, \leq)$ is a complete semilattice and suppose $* \subseteq X \times X$ is a compatibility relation on X . Then:*

- (1) *$*$ is monotone on both arguments.
That is: if $x_1 * x_2$ and $x_1 \leq x'_1$ and $x_2 \leq x'_2$, then $x'_1 * x'_2$.*
- (2) *If $x_1, x_2 \in X$ have a non- $\perp$ lower bound $\perp_X \lneq x \leq x_1, x_2$, then $x_1 * x_2$.
In words we can write: $*$ reflects non- $\perp$ lower bounds.*
- (3) *The converse implication to part 2 need not hold: it may be that $x_1 * x_2$ (x_1 and x_2 are compatible) but the greatest lower bound of $\{x_1, x_2\}$ is $\perp$.*

Proof. We consider each part in turn:

- (1) We argue much as for Lemma 7.1.4(1). Suppose $x_1 * x_2$ and $x_1 \leq x'_1$ and $x_2 \leq x'_2$. By Lemma 7.1.4 $x_1 \vee x'_1 = x'_1$ and $x_2 \vee x'_2 = x'_2$. It follows using distributivity and commutativity (Definition 7.2.1(3 and 1)) that $x_1 * x_2$ implies that $(x_1 \vee x'_1) * (x_2 \vee x'_2)$, and thus that $x'_1 * x'_2$ as required.
- (2) Suppose $\perp_X \neq x \leq x_1, x_2$, so x is a non- $\perp_X$ lower bound. By assumption $*$ is properly reflexive (Definition 7.2.1(2)) so (since $x \neq \perp_X$) $x * x$. By part 1 of this result it follows that $x_1 * x_2$ as required.
- (3) It suffices to provide a counterexample. Define $(X, \leq, *)$ by:
 - $X = \{\perp, 0, 1, \top\}$.
 - $\perp \leq 0, 1 \leq \top$ and $\neg(0 \leq 1)$ and $\neg(1 \leq 0)$.
 - $x * x'$ for every $\perp \neq x, x' \in X$.

We note that $0 * 1$ but the greatest lower bound of $\{0, 1\}$ is $\perp$. We will revisit a slightly more elaborate version of this counterexample in Figure 5. □

7.3 The definition of a semiframe

Definition 7.3.1. A **semiframe** is a tuple $(X, \leq, *)$ such that

- (1) $(X, \leq)$ is a complete semilattice (Definition 7.1.1), and
- (2) $*$ is a compatibility relation on it (Definition 7.2.1).

Slightly abusing terminology, we can say that

semiframe = compatible complete semilattice.

Semiframes are new, so far as we know, but they are a natural idea. We consider some elementary ways to generate examples, starting with arguably the simplest possible instance:

Example 7.3.2 (The empty semiframe). Suppose $(X, \leq, *)$ is a semiframe.

- (1) If X is a singleton set, so that $X = \{\bullet\}$ for some element $\bullet$, then we call $(X, \leq, *)$ the **empty semiframe** or **singleton semiframe**. Then necessarily $\bullet = \perp_X = \top_X$ and $\bullet \leq \bullet$ and $\neg(\bullet * \bullet)$.
- (2) If X has more than one element then we call $(X, \leq, *)$ a **nonempty semiframe**. Then necessarily $\perp_X \neq \top_X$.

Thus, $(X, \leq, *)$ is nonempty if and only if $\perp_X \neq \top_X$. We call a singleton semiframe empty, because this corresponds to the semiframe of open sets of the empty topology, which has no points and one open set, $\emptyset$.

Example 7.3.3 continues Remark 7.2.2:

Example 7.3.3.

- (1) Suppose (P, Open) is a semitopology. Then the reader can check that the semiframe of open sets $(P, \subseteq, \cap)$ is a semiframe. We will study this example in detail; see Definition 7.3.4 and Lemma 7.3.5.
- (2) Suppose $(X, \leq, \perp, \top)$ is a frame (a complete lattice such that meets distribute over arbitrary joins). Then $(X, \leq, *)$ is a semiframe, where $x * x'$ when $x \wedge x' \neq \perp$.¹⁶
- (3) Take $X = \{\perp, 0, 1, \top\}$ with $\perp \leq 0 \leq \top$ and $\perp \leq 1 \leq \top$ (so 0 and 1 are incomparable). There are two possible semiframe structures on this, characterised by choosing $0 * 1$ or $\neg(0 * 1)$.
- (4) See also the semiframes used in Lemmas 8.2.7.

Definition 7.3.4 is just an example of semiframes for now, though we will see much more of it later:

Definition 7.3.4 (Semitopology $\rightarrow$ semiframe). Suppose (P, Open) is a semitopology. Define the **semiframe of open sets** $\text{Fr}(P, \text{Open})$ (cf. Example 7.3.3(1)) by:

- (1) $\text{Fr}(P, \text{Open})$ has elements open sets $O \in \text{Open}$.
- (2) $\subseteq$ is subset inclusion.
- (3) $*$ is $\cap$ (set intersection).

We may write

$(\text{Open}, \subseteq, \cap)$ as a synonym for $\text{Fr}(P, \text{Open})$.

Lemma 7.3.5. Suppose (P, Open) is a semitopology. Then $(\text{Open}, \subseteq, \cap)$ is indeed a semiframe.

Proof. As per Definition 7.3.1 we must show that Open is a complete semilattice (Definition 7.1.1) and $\cap$ is a compatibility relation (Definition 7.2.1) – symmetric, properly reflexive, and distributive and satisfies a distributive law that if $O \cap \bigcup \mathcal{O}'$ then $O \cap O'$ for some $O' \in \mathcal{O}'$. These are all facts of sets. $\square$

Remark 7.3.6. Definition 7.3.4 and Lemma 7.3.5 are the start of our development. Once we have built more machinery, we will have a pair of translations:

- Definition 7.3.4 and Lemma 7.3.5 go from semitopologies to semiframes.
- Definition 8.4.3 and Lemma 8.4.4 go from semiframes to semitopologies.

These translations are part of a dual pair of functors between categories of semitopologies and semiframes, as described in Definitions 10.1.1 and 10.2.1 and Proposition 10.3.7.

Semitopologies are (relatively) concrete: we have concrete points and open sets that are sets of points. Semiframes are more abstract: we have a join-complete semilattice, and a compatibility relation. The duality we are about to build will show how these two worlds interact and reflect each other.

We conclude with a simple technical lemma:

Lemma 7.3.7. *Suppose $(X, \leq, *)$ is a semiframe (a complete semilattice with a compatibility relation) and $x \in X$. Then:*

- (1) $\neg(x * \perp_X)$ and in particular $\neg(\perp_X * \perp_X)$.
- (2) $x * x$ if and only if $x \neq \perp_X$.
- (3) $x * \top_X$ if and only if $x \neq \perp_X$.
- (4) $\top_X * \top_X$ holds precisely if X is nonempty (Example 7.3.2).

Proof. We consider each part in turn:

- (1) Recall from Definition 7.1.1(3) that $\perp_X = \bigvee \emptyset$. By distributivity (Definition 7.2.1(3))

$$x * \perp_X \iff \exists x' \in \emptyset. x * x' \iff \perp.$$

- (2) We just combine part 1 of this result with Definition 7.2.1(2).
- (3) Suppose $x \neq \perp_X$. Then $\perp_X \leq x \leq x \leq \top_X$, and by Lemma 7.2.3(2) $x * \top_X$.
Suppose $x = \perp_X$. Then $\neg(x * \top_X)$ by combining commutativity of $*$ (Definition 7.2.1(1)) with part 1 of this result.
- (4) If X is nonempty then by Example 7.3.2 $\perp_X \neq \top_X$ and so $\top_X * \top_X$ holds by part 2 of this result. However, in the degenerate case that X has one element then $\top_X = \perp_X$ and $\top_X * \top_X$ does not hold. □

Remark 7.3.8. Recall from (Davey and Priestley, 2002, Definition 5.22, p. 128) that if X is a lattice, then the **pseudocomplement** to $x \in X$ is $x^c = \bigvee \{x' \in X \mid x' \wedge x = \perp\}$. A semiframe $(X, \leq, *)$ naturally supports a notion of pseudocomplement for $x \in X$, given by

$$x^c = \bigvee \{x' \in X \mid \neg(x' * x)\}.$$

It is easy to prove that $\neg(x^c * x)$, arguing by contradiction: if $x^c * x$ then $\bigvee \{x' \mid \neg(x' * x)\} * x$, and by distributivity (Definition 7.2.1(3)) there exists $x' \in X$ such that $x' * x$ and $\neg(x' * x)$, a contradiction.

Note that it may be that $(x^c)^c \leq x$. For example, in the semiframe illustrated in Figure 5, $0^c = \bigvee \{1, 2, 3\} = \top$ and $(0^c)^c = \perp \leq 0$ (this behaviour will be familiar to the reader who has seen, e.g., double negation in intuitionistic logic).

x^c and related constructions will be useful later, in Definition 11.4.2 and Lemma 11.4.4.

8. Semifilters & Abstract Points

8.1 The basic definition, and discussion

Definition 8.1.1. Suppose $(X, \leq, *)$ is a semiframe and suppose $F \subseteq X$. Then:

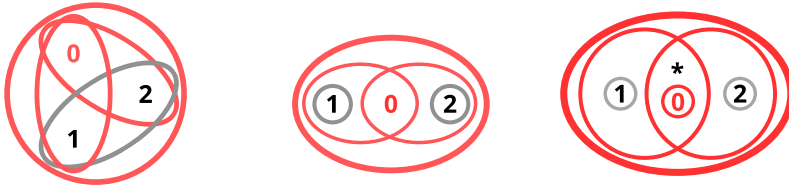


Figure 4. Examples of open neighbourhoods (Remarks 8.1.3 and 9.2.1).

- (1) Call F **prime** when for every $x, x' \in X$,

$$x \vee x' \in F \text{ implies } x \in F \vee x' \in F.$$

- (2) Call F **completely prime** when for every (possibly empty) $X \subseteq X$,

$$\bigvee X \in F \text{ implies } \exists x \in X. x \in F.$$

(This condition is used in Lemma 8.3.2, which is needed for Lemma 8.4.2.)

- (3) Call F **up-closed** when $x \in F$ and $x \leq x'$ implies $x' \in F$.
 (4) Call F **compatible** when its elements are **pairwise compatible**, by which we mean that $x * x'$ for every $x, x' \in F$.
 (5) A **semifilter** is a nonempty, up-closed, compatible subset $F \subseteq X$.
 (6) Call $F \subseteq X$ a **maximal semifilter** when it is a semifilter and is contained in no strictly greater semifilter.
 (7) An **abstract point** is a completely prime semifilter.
 (8) Write

$$\text{Points}(X, \leq, *)$$

for the set of abstract points of $(X, \leq, *)$.

Notation 8.1.2. We will generally write $F \subseteq X$ for a subset of X that is intended to be a semifilter, or for which in most cases of interest F is a semifilter. We will generally write $P \subseteq X$ when the subset is intended to be an abstract point, or when in most cases of interest P is an abstract point.

Remark 8.1.3. Note on design: The notion of semifilter from Definition 8.1.1 is, obviously, based on the standard notion of filter (Johnstone, 1986, I.2.2, p. 12). We just replace the closure under binary meets condition

$$\text{'if } x, x' \in F \text{ then } x \wedge x' \in F'$$

with a weaker compatibility condition

$$\text{'if } x, x' \in F \text{ then } x * x'.$$

This is in keeping with our move from frames to semiframes, which weakens from $\wedge$ to the compatibility relation $*$.

Note that a semifilter or abstract point need not be directed:

- (1) Consider $\text{nbhd}(0)$ in the (semiframes of open sets of the) semitopologies in the left-hand and middle examples in Figure 4. In both cases, $\{0, 1\}, \{0, 2\} \in \text{nbhd}(0)$ but $\{0\} \notin \text{nbhd}(0)$ because $\{0\}$ is not an open set.

- (2) Consider $\{0, 1, 2\}$ with the discrete semitopology (so every set is open). Then the set of all two- or three-element subsets $\{\{0, 1\}, \{1, 2\}, \{2, 0\}, \{0, 1, 2\}\}$ is a semifilter, but it is not closed under set intersections because it does not contain $\{0\}$, $\{1\}$, or $\{2\}$.

This second example is particularly interesting. As the reader may know, the intuition of a filter in topology is a set of approximations. But this example is clearly not approximating anything – after all, we are in the discrete semitopology and there is no need to approximate anything since we can just take a singleton set! This suggests that a better intuition for semiframe is a set of collaborations; in this case, of 0 with 1, 1 with 2, and 2 with 0.

Thus in particular, the standard result in frames that a proper finite filter¹⁷ has a non- $\perp$ least element (obtained as the finite meet of all the elements in the filter), does not hold for semifilters in semiframes. See also Remark 8.1.6 and Proposition 8.2.8(1).

Example 8.1.4. Suppose $(X, \leq, *)$ is a semiframe. We recall some (standard) facts about abstract points, which carry over from topologies and frames:

- (1) Suppose (P, Open) is a semitopology and $(X, \leq, *) = (\text{Open}, \subseteq, \cap)$. By Lemma 7.3.5, $(X, \leq, *)$ is a semiframe.
If $p \in P$ then

$$\text{nbhd}(p) = \{O \in \text{Open} \mid p \in O\}$$

from Definition 2.3.1 is an abstract point: see Proposition 9.2.2. Intuitively, $\text{nbhd}(p)$ abstractly represents p as the set of all of its open approximations in Open .

- (2) Suppose (P, Open) is a semitopology. Then $(\text{Open}, \subseteq, \cap)$ could contain an abstract point that is not the neighbourhood semifilter $\text{nbhd}(p)$ of a point $p \in P$.

Set $X = \{\mathbb{Q}\} \cup \{(\pi - q, \pi + q) \subseteq \mathbb{Q} \mid q \in \mathbb{Q}_{\geq 0}\}$ (the set of all symmetric open intervals around π in the rational numbers $\mathbb{Q}$), and set $\leq = \subseteq$ and $* = \cap$.

Set $P = X \setminus \{\emptyset\}$ to be the set of all nonempty symmetric open intervals around π . Note that $\pi \notin \mathbb{Q}$, but P is a set of open sets ‘approximating’ π .

- (3) We mention one more (standard) example. Consider $\mathbb{N}$ with the **final segment semitopology** such that opens are either $\emptyset$ or sets $n_{\geq} = \{n' \in \mathbb{N} \mid n' \geq n\}$. Then $\{n_{\geq} \mid n \in \mathbb{N}\}$ is an abstract point. Intuitively, this approximates a point at infinity, which we can understand as ω .

Lemma 8.1.5. Suppose $(X, \leq, *)$ is a semiframe and suppose $F \subseteq X$ is compatible. Then $\perp_X \notin F$.

Proof. By compatibility, $x * x$ for every $x \in F$. We use Lemma 7.3.7(1). □

Remark 8.1.6. We continue Remark 8.1.3.

As the reader may know, a semiframe still has greatest lower bounds, because we can build them as $x \wedge x' = \bigvee \{x'' \mid x'' \leq x, x'' \leq x'\}$. It is just that this greatest lower bound may be unhelpful. To see why, consider again the examples in Figure 4. In the left-hand and middle examples in Figure 4, the greatest lower bound of $\{0, 1\}$ and $\{0, 2\}$ exists in the semiframe of open sets: but it is $\emptyset$ the empty set in the left-hand and middle example, not $\{0\}$. In the right-hand example, the greatest lower bound of $\{0, *, 1\}$ and $\{0, *, 2\}$ is $\{0\}$, not $\{0, *\}$.

So the reader could ask whether perhaps we should add the following weakened meet-closure condition to the definition of semifilters (and thus to abstract points):

$$\text{If } x, x' \in F \text{ and } x \wedge x' \neq \perp \text{ then } x \wedge x' \in F.$$

Intuitively, this insists that semifilters are closed under *non- $\perp$* greatest lower bounds. However, there are two problems with this:

- It would break our categorical duality proof in the construction of g° in Lemma 10.3.3; see the discussion in Remark 10.3.4. This technical difficulty may be superable, but . . .
- . . . the condition is probably not what we want anyway. It would mean that the set of open neighbourhoods of $*$ in the right-hand example of Figure 4, would not be a semifilter, because it contains $\{0, *, 1\}$ and $\{0, *, 2\}$ but not its (non- $\emptyset$) greatest lower bound $\{0\}$.

8.2 Properties of semifilters

8.2.1 Things that are familiar from filters

Lemma 8.2.1. *Suppose $(X, \leq, *)$ is a semiframe and $F \subseteq X$ is a semifilter. Then:*

- (1) $\top_X \in F$.
- (2) $\perp_X \notin F$.

Proof. We consider each part in turn:

- (1) By nonemptiness (Definition 8.1.1(7)) F is nonempty, so there exists some $x \in F$. By definition $x \leq \top_X$. By up-closure (Definition 8.1.1(3)) $\top_X \in F$ follows.
- (2) By assumption in Definition 8.1.1(4) elements in F are pairwise compatible (so $x * x$ for every $x \in F$). We use Lemma 8.1.5. □

Lemma 8.2.2. *Suppose $(X, \leq, *)$ is a semiframe. It is possible for a semifilter $F \subseteq X$ to be completely prime but not maximal.*

Proof. We give a standard example (which also works for frames and filters). Take $P = \{0, 1\}$ and $\text{Open} = \{\emptyset, \{0\}, \{0, 1\}\}$. Then $P' = \{\{0, 1\}\}$ is an abstract point, but it is not a maximal semifilter (it is not even a maximal abstract point) since P' is contained in the strictly larger semifilter $\{\{0\}, \{0, 1\}\}$ (which is itself also a strictly larger abstract point). □

Lemma 8.2.3. *If $(X, \leq, *)$ is a finite semiframe (meaning that X is finite) then the properties of*

- *being a prime semifilter (Definition 8.1.1(1)) and*
 - *being a completely prime semifilter (Definition 8.1.1(2)),*
- coincide.*

Proof. This is almost trivial, except that if $X = \emptyset$ in the condition for being completely prime then we get that $\perp_X \notin P$ – but we know that anyway from Lemma 8.2.1(2), from the compatibility condition on semifilters. □

Lemma 8.2.4. *Suppose $(X, \leq, *)$ is a semiframe. Then:*

- (1) *The union of an ascending chain of semifilters in X , is a semifilter in X .*
- (2) *As a corollary, every semifilter $F \subseteq X$ is contained in some maximal semifilter $F' \subseteq X$ (assuming Zorn's lemma).*

Proof. We consider each part in turn:

- (1) By a straightforward verification of the conditions of being a semifilter from Definition 8.1.1(5).
- (2) Direct application of Zorn's lemma. □

Remark 8.2.5.

- (1) Lemma 8.2.1(2) has a small twist to it. In the theory of filters, it does not follow from the property of being nonempty, up-closed, and closed under finite meets, that $\perp_X \notin F$; this must be added as a distinct condition if required.
In contrast, we see in the proof of Lemma 8.2.1(2) that for semifilters, $\perp_X \notin F$ follows from the compatibility condition.
- (2) Lemma 8.2.3 matters in particular to us here, because we are particularly interested in abstracting the behaviours of finite semitopologies, because our original motivation for looking at both of these structures comes from looking at real networks, which are finite.¹⁸

8.2.2 Things that are different from filters

Remark 8.2.6. Obviously, by definition semifilters are necessarily compatible but not necessarily closed under meets. But aside from this fact, we have so far seen semiframes and semifilters behave more-or-less like frames and filters, modulo small details like that mentioned in Remark 8.2.5(1).

But there are also differences, as we will now briefly explore. In the theory of (finite) frames, the following facts hold:

- (1) Every proper filter F has a greatest lower bound x , and $F = x^\leq = \{x' \mid x \leq x'\}$.
Just take $x = \bigwedge F$ the meet of all of its (finitely many) elements. This is not $\perp$, by the filter's finite intersection property.
- (2) Every proper filter can be extended to a maximal filter.¹⁹
Just extend using Zorn's lemma (as in Lemma 8.2.4).
- (3) Every maximal filter is completely prime.
It is a fact of finite frames that a maximal filter is prime,²⁰ and since we assume the frame is finite, it is also completely prime.
- (4) Every non- $\perp$ element $x \neq \perp_X$ in a finite frame is contained in some abstract point.
Just form $\{x' \mid x \leq x'\}$, observe it is a filter, form a maximal filter above it, and we get an abstract point.
- (5) As a corollary, if the frame is nonempty (so $\perp \neq \top$; see Example 7.3.2) then it has at least one abstract point.

In Lemma 8.2.7 and Proposition 8.2.8 we consider some corresponding non-properties of (finite) semiframes.

Lemma 8.2.7. *Suppose $(X, \leq, *)$ is a semiframe. It is possible for $\text{Points}(X, \leq, *)$ to be empty, even if $(X, \leq, *)$ is nonempty (Example 7.3.2(2)). This is possible even if X is finite, and even if X is infinite.*

Proof. It suffices to provide an example. We define a semiframe as below, and as illustrated in Figure 5:

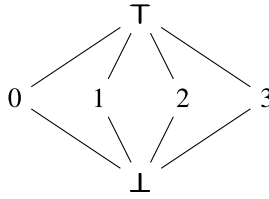


Figure 5. A semiframe with no abstract points (Lemma 8.2.7).

- $X = \{\perp, 0, 1, 2, 3, T\}$.
- Let $x \leq x'$ when $x = x'$ or $x = \perp$ or $x' = T$.
- Let $x * x'$ when $x \wedge x' \neq \perp$.²¹

Then $(X, \leq, *)$ has no abstract points.

For suppose P is one such. By Lemma 8.2.1 $T \in P$. Note that $T = 0 \vee 1 = 2 \vee 3$. By assumption P is completely prime, we know that $0 \in P \vee 1 \in P$, and also $2 \in P \vee 3 \in P$. But this is impossible because 0, 1, 2, and 3 are not compatible.

For the infinite case, we just increase the width of the semiframe by taking $X = \{\perp\} \cup \mathbb{N} \cup \{T\}$. $\square$

Proposition 8.2.8. Suppose $(X, \leq, *)$ is a semiframe and $F \subseteq X$ is a semifilter. Then:

- (1) It is not necessarily the case that F has a non- $\perp$ greatest lower bound (even if X is finite).
- (2) Every semifilter can be extended to a maximal semifilter, but . . .
- (3) . . .this maximal semifilter is not necessarily prime (even if X is finite).
- (4) There may exist a non- $\perp$ element $x \neq \perp_X$ that is contained in no abstract point.

Proof. We consider each part in turn:

- (1) Consider $(\text{pow}(\{0, 1, 2\}), \subseteq, \emptyset)$ and take

$$F = \{\{0, 1\}, \{1, 2\}, \{0, 2\}, \{0, 1, 2\}\}.$$

The greatest lower bound of F is $\emptyset$.

- (2) This is Lemma 8.2.4.
- (3) F from part 1 of this result is maximal, and it cannot be extended to a point $P \supseteq F$.

Figure 5 gives another counterexample, and in a rather interesting way: the semitopology has four maximal semifilters $\{i, T\}$ for $i \in \{0, 1, 2, 3\}$, but by Lemma 8.2.7 it has no prime semifilters at all.²²

- (4) We just take $x = 0 \in X$ from the example in Lemma 8.2.7 (see Figure 5). Since this semiframe has no abstract points at all, there is no abstract point that contains x . $\square$

Remark 8.2.9. For now, we will just read Proposition 8.2.8 as a caution not to assume that semiframes and semifilters behave like frames and filters. Sometimes they do, and sometimes they don't; we have to check.

We now proceed to build our categorical duality, culminating with Theorem 10.4.2. Once that machinery is constructed, we will continue our study of the fine structure of semifilters in Section 11.

8.3 Sets of abstract points

Definition 8.3.1. Suppose $(X, \leq, *)$ is a semiframe and recall $\text{Points}(X, \leq, *)$ from Definition 8.1.1(7). Define a map $Op : X \rightarrow \text{pow}(\text{Points}(X, \leq, *))$ by

$$Op(x) = \{P \in \text{Points}(X, \leq, *) \mid x \in P\}.$$

Lemma 8.3.2. Suppose $(X, \leq, *)$ is a semiframe and $X \subseteq X$. Then

$$Op(\bigvee X) = \bigcup_{x \in X} Op(x).$$

In words: we can say that Op commutes with joins, and that Op commutes with taking least upper bounds.

Proof. Suppose $P \in \text{Points}(X, \leq, *)$. We reason as follows:

$$\begin{aligned} P \in Op(\bigvee X) &\iff \bigvee X \in P && \text{Definition 8.3.1} \\ &\iff \exists x \in X. x \in P && \text{Definition 8.1.1(2)} \\ &\iff P \in \bigcup_{x \in X} Op(x) && \text{Definition 8.3.1} \end{aligned}$$

□

Proposition 8.3.3. Suppose $(X, \leq, *)$ is a semiframe and $x, x' \in X$. Then:

- (1) If $x \leq x'$ then $Op(x) \subseteq Op(x')$.
- (2) If $Op(x) \not\subseteq Op(x')$ then $x * x'$.
- (3) $Op(\top_X) = \text{Points}(X, \leq, *)$ and $Op(\perp_X) = \emptyset$.
- (4) $Op(\bigvee X) = \bigcup_{x \in X} Op(x)$ for $X \subseteq X$.

Proof. We consider each part in turn:

- (1) We prove that $x \leq x'$ implies $Op(x) \subseteq Op(x')$.
Suppose $x \leq x'$, and consider some abstract point $P \in Op(x)$. By Definition 8.3.1 $x \in P$, and by up-closure of P (Definition 8.1.1(3)) $x' \in P$, so by Definition 8.3.1 $P \in Op(x')$. P was arbitrary, and it follows that $Op(x) \subseteq Op(x')$.
- (2) We prove that $Op(x) \not\subseteq Op(x')$ implies $x * x'$.
Suppose there exists an abstract point $P \in Op(x) \cap Op(x')$. By Definition 8.3.1 $x, x' \in P$, and by compatibility of P (Definition 8.1.1(4)) $x * x'$.
- (3) Unpacking Definition 8.3.1, it suffices to show that $\top_X \in P$ and $\perp_X \notin P$, for every abstract point $P \in \text{Points}(X, \leq, *)$. This is from Lemma 8.2.1(1).
- (4) This is just Lemma 8.3.2.

□

Remark 8.3.4. Proposition 8.3.3 carries a clear suggestion that $(\{Op(x) \mid x \in X\}, \subseteq, \not\subseteq)$ is trying, in some sense, to be an isomorphic copy of $(X, \leq, *)$. Lemma 8.3.5 notes that it may not quite manage this, because there may not be enough points (indeed, there may not be any abstract points at all). This will (just as for topologies and frames) lead us to the notion of a *spatial* semiframe in Definition 9.1.2 and Proposition 9.1.4.

Lemma 8.3.5. The converse implications in Proposition 8.3.3(1 and 2) need not hold. That is:

- (1) There exists a semiframe $(X, \leq, *)$ and $x, x' \in X$ such that $Op(x) \subseteq Op(x')$ yet $x \not\leq x'$.
- (2) There exists a semiframe $(X, \leq, *)$ and $x, x' \in X$ such that $x * x'$ yet $Op(x) \not\subseteq Op(x')$.

Proof. The example from Lemma 8.2.7 (as illustrated in Figure 5) is a counterexample for both cases:

- $Op(0) \subseteq Op(1)$ because both are equal to the empty set, yet $0 \not\leq 1$; and
- $\top * \top$ yet $Op(\top) \not\subseteq Op(\top)$.

□

8.4 The semitopology of abstract points

Recall from Definition 8.1.1(7) that an abstract point in a semiframe $(X, \leq, *)$ is a nonempty up-closed compatible completely prime subset of X , and recall from Definition 8.3.1 that

$$Op(x) = \{P \in \text{Points}(X, \leq, *) \mid x \in P\},$$

or in words: $Op(x)$ is the set of abstract points that contain x .

Definition 8.4.1. Suppose $(X, \leq, *)$ is a semiframe. Then define $Op(X, \leq, *)$ by

$$Op(X, \leq, *) = \{Op(x) \mid x \in X\}.$$

Lemma 8.4.2. Suppose $(X, \leq, *)$ is a semiframe. Then:

- (1) $Op(X, \leq, *)$ from Definition 8.4.1 is closed under arbitrary sets union.
- (2) As a corollary, $(Op(X, \leq, *), \subseteq)$ (in words: $Op(X, \leq, *)$ ordered by subset inclusion) is a complete join-semilattice.

Proof. Part 1 is just Lemma 8.3.2. The corollary part 2 is just a fact, since $Op(X, \leq, *) \subseteq \text{pow}(\text{Points}(X, \leq, *))$, and sets union is the join (least upper bound) in the powerset lattice. □

Recall from Definition 7.3.4 and Lemma 7.3.5 that we showed how to go from a semitopology (P, Open) to a semiframe $(\text{Open}, \subseteq, \cap)$. We now show how to go in the other direction:

Definition 8.4.3 (Semiframe $\rightarrow$ semitopology). Suppose $(X, \leq, *)$ is a semiframe. Define the **semitopology of abstract points** $\text{St}(X, \leq, *)$ by

$$\text{St}(X, \leq, *) = (\text{Points}(X, \leq, *), Op(X, \leq, *)).$$

Unpacking this a little:

- (1) The set of points of $\text{St}(X, \leq, *)$ is the set of abstract points $\text{Points}(X, \leq, *)$ from Definition 8.1.1(7) – namely, the completely prime nonempty up-closed compatible subsets of X .²³
- (2) Open sets $\text{Opens}(X, \leq, *)$ are the $Op(x)$ from Definition 8.3.1:

$$Op(x) = \{P \in \text{Points}(X, \leq, *) \mid x \in P\}.$$

Lemma 8.4.4. Suppose $(X, \leq, *)$ is a semiframe. Then $\text{St}(X, \leq, *)$ from Definition 8.4.3 is indeed a semitopology.

Proof. From conditions 1 and 2 of Definition 1.2.2, we need to check that $Op(X, \leq, *)$ contains $\emptyset$ and $\text{Points}(X, \leq, *)$ and is closed under arbitrary unions. This is from Proposition 8.3.3(3 and 4). □

Recall from Definitions 8.4.3 and 7.3.4 that $\text{St}(X, \leq, *)$ is a semitopology, and $\text{Fr St}(X, \leq, *)$ is a semiframe each of whose elements is the set of abstract points of $(X, \leq, *)$ that contain some $x \in X$:

Lemma 8.4.5. Suppose $(X, \leq, *)$ is a semiframe. Then $Op : (X, \leq, *) \rightarrow \text{Fr St}(X, \leq, *)$ is surjective.

Proof. Direct from Definition 8.4.3(2). □

We conclude with Definition 8.4.6 and Proposition 8.4.7, which are standard properties of the construction in Definition 8.4.3.

Definition 8.4.6. Suppose (P, Open) is a semitopology and $p, p' \in P$. Define $p = p'$ by

$$p = p' \quad \text{when} \quad \forall O \in \text{Open}. p \in O \iff p' \in O.$$

We recall some standard terminology from topology:

- (1) Call p and p' **topologically indistinguishable** when $p = p'$.
- (2) Call p and p' **topologically distinguishable** when $\neg(p = p')$ (so there exists some $O \in \text{Open}$ such that $p \in O \wedge p' \notin O$ or $p \notin O \wedge p' \in O$).
- (3) Call (P, Open) a T_0 **space** when points are topologically indistinguishable precisely when they are equal, or in symbols: $\doteq = =$.

Proposition 8.4.7. Suppose $(X, \leq, *)$ is a semiframe. Then $\text{St}(X, \leq, *)$ (Definition 8.4.3) is a T_0 space.

Proof. Suppose $P, P' \in \text{Points}(X, \leq, *)$. Unpacking Definition 8.1.1(7), this means that P and P' are completely prime nonempty up-closed compatible subsets of X .

It is immediate that $P = P'$ implies $P \doteq P'$.

Now suppose $P \doteq P'$ in $\text{St}(X, \leq, *)$; to prove $P = P'$ it would suffice to show that $x \in P \iff x \in P'$, for arbitrary $x \in X$. By Definition 8.4.3(2), every open set in $\text{St}(X, \leq, *)$ has the form $Op(x)$ for some $x \in X$. We reason as follows:

$$\begin{aligned} x \in P &\iff P \in Op(x) && \text{Definition 8.3.1} \\ &\iff P' \in Op(x) && P, P' \text{ top. indisting.} \\ &\iff x \in P' && \text{Definition 8.3.1} \end{aligned}$$

Since x was arbitrary and $P, P' \subseteq \text{Open}$, it follows that $P = P'$ as required. □

9. Spatial Semiframes & Sober Semitopologies

9.1 Definition of spatial semiframes

Remark 9.1.1. We continue Remark 8.3.4. We saw in Example 8.1.4(2&3) that there may be more abstract points than there are concrete points, and in Remark 8.3.4 that there may also be fewer.

In the theory of frames, the condition of being *spatial* means that the abstract points and concrete points correspond. We imitate this terminology for a corresponding definition on semiframes:

Definition 9.1.2 (Spatial semiframe). Call a semiframe $(X, \leq, *)$ **spatial** when:

- (1) $Op(x) \subseteq Op(x')$ implies $x \leq x'$, for every $x, x' \in X$.
- (2) $x * x'$ implies $Op(x) \not\subseteq Op(x')$, for every $x, x' \in X$.

Remark 9.1.3. Not every semiframe is spatial, just as not every frame is spatial. Lemma 8.2.7 gives an example of a semiframe that is not spatial because it has no points at all, as illustrated in Figure 5.

We check that the conditions in Definition 9.1.2 correctly strengthen the implications in Proposition 8.3.3 to become logical equivalences:

Proposition 9.1.4. Suppose $(X, \leq, *)$ is a spatial semiframe and $x, x' \in X$. Then:

- (1) $x \leq x'$ if and only if $Op(x) \subseteq Op(x')$.
- (2) $x * x'$ if and only if $Op(x) \not\subseteq Op(x')$.
- (3) $x = x'$ if and only if $Op(x) = Op(x')$.
- (4) $Op(\top_X) = \text{Points}(X, \leq, *)$ and $Op(\perp_X) = \emptyset$.
- (5) $Op(\bigvee X) = \bigcup_{x \in X} Op(x)$ for $X \subseteq X$.

Proof. We consider each part in turn:

- (1) We prove that $x \leq x'$ if and only if $Op(x) \subseteq Op(x')$.
The right-to-left implication is direct from Definition 9.1.2(1). The left-to-right implication is Proposition 8.3.3(1).
- (2) We prove that $x * x'$ if and only if $Op(x) \not\subseteq Op(x')$.
The left-to-right implication is direct from Definition 9.1.2(2). The right-to-left implication is Proposition 8.3.3(2).
- (3) We prove that $x = x'$ if and only if $Op(x) = Op(x')$.
If $x = x'$ then $Op(x) = Op(x')$ is immediate. If $Op(x) = Op(x')$ then $Op(x) \subseteq Op(x')$ and $Op(x') \subseteq Op(x)$. By part 1 of this result (or direct from Definition 9.1.2(1)) $x \leq x'$ and $x' \leq x$. By antisymmetry of $\leq$ it follows that $x = x'$.
- (4) This is just Proposition 8.3.3(3)
- (5) This is just Lemma 8.3.2. □

Definition 9.1.5 will be useful in a moment.²⁴

Definition 9.1.5. Suppose $(X, \leq, *)$ and $(X', \leq', *')$ are semiframes. Then an **isomorphism** between them is a function $g : X \rightarrow X'$ such that:

- (1) g is a bijection between X and X' .
- (2) $x_1 \leq x_2$ if and only if $g(x_1) \leq g(x_2)$.
- (3) $x_1 * x_2$ if and only if $g(x_1) * g(x_2)$.

Lemma 9.1.6. Suppose $(X, \leq, *)$ and $(X', \leq', *')$ are semiframes and $g : X \rightarrow X'$ is an isomorphism between them. Then $g(\perp_X) = g(\perp_{X'})$ and $g(\top_X) = \top_{X'}$.

Proof. By construction $\perp_X \leq x$ for every $x \in X$. It follows from Definition 9.1.5(2) that $g(\perp_X) \leq g(x)$ for every $x \in X$; but g is a bijection, so $g(\perp_X) \leq x'$ for every $x' \in X'$. It follows that $g(\perp_X) = \perp_{X'}$.
By similar reasoning we conclude that $g(\top_X) = \top_{X'}$. □

Remark 9.1.7. Suppose $(X, \leq, *)$ is a semiframe and recall from Definition 8.4.1 that $Op(X, \leq, *) = \{Op(x) \mid x \in X\}$. Then the intuitive content of Proposition 9.1.4 is that a semiframe

$(X, \leq, *)$ is spatial when $(X, \leq, *)$ is isomorphic (in the sense made formal by Definition 9.1.5) to $(\text{Op}(X, \leq, *), \subseteq, \cap)$.

And, because $\text{Op}(\mathcal{T}_X) = \text{Points}(X, \leq, *)$ we can write a slogan:

A semiframe is spatial when it is (up to isomorphism) generated by its abstract points.

We will go on to prove in Proposition 9.2.5 that every semitopology generates a spatial semiframe – and in Theorem 10.4.2 we will tighten and extend the slogan above to a full categorical duality.

9.2 The neighbourhood semifilter $\text{nbhd}(p)$

9.2.1 The definition and basic lemma

Recall from Definition 2.3.1 that we define $\text{nbhd}(p) = \{O \in \text{Open} \mid p \in O\}$.

Remark 9.2.1. If (P, Open) is a topology, then $\text{nbhd}(p)$ is a filter (a nonempty up-closed down-directed set) and is often called the neighbourhood filter of p .

We are working with semitopologies, so Open is not necessarily closed under intersections, and $\text{nbhd}(p)$ is not necessarily a filter (it is still a compatible set, because every $O \in \text{nbhd}(p)$ contains p). Figure 4 illustrates examples of this: for example in the left-hand example $\{0, 1\}, \{0, 2\} \in \text{nbhd}(0)$ but $\{0\} \notin \text{nbhd}(0)$ (because this is not an open set).

Recall from Definition 2.3.1 that $\text{nbhd}(p) = \{O \in \text{Open} \mid p \in O\}$.

Proposition 9.2.2. Suppose (P, Open) is a semitopology and $p \in P$ and $O \in \text{Open}$. Then:

- (1) $\text{nbhd}(p)$ (Definition 2.3.1) is an abstract point (a completely prime semifilter) in the semiframe $\text{Fr}(P, \text{Open})$ (Definition 7.3.4). In symbols:

$$\text{nbhd} : P \rightarrow \text{Points}(\text{Fr}(P, \text{Open})).$$

- (2) The following are equivalent:

$$\text{nbhd}(p) \in \text{Op}(O) \iff O \in \text{nbhd}(p) \iff p \in O.$$

- (3) We have an equality:

$$\text{nbhd}^{-1}(\text{Op}(O)) = O.$$

Proof. We consider each part in turn:

- (1) From Definition 8.1.1(7), we must check that $\text{nbhd}(p)$ is a nonempty, completely prime, up-closed, and compatible subset of Open when considered as a semiframe as per Definition 7.3.4. All properties are by facts of sets; we give brief details:
 - $\text{nbhd}(p)$ is nonempty because $p \in P \in \text{Open}$.
 - $\text{nbhd}(p)$ is completely prime because it is a fact of sets that if $P \subseteq \text{Open}$ and $p \in \bigcup P$ then $p \in O$ for some $O \in P$.
 - $\text{nbhd}(p)$ is up-closed because it is a fact of sets that if $p \in O$ and $O \subseteq O'$ then $p \in O'$.
 - $\text{nbhd}(p)$ is compatible because if $p \in O$ and $p \in O'$ then $O \cap O' \in \text{nbhd}(p)$.
- (2) By Definition 8.3.1, $\text{Op}(O)$ is precisely the set of abstract points P that contain O , and by part 1 of this result $\text{nbhd}(p)$ is one of those points. By Definition 2.3.1, $\text{nbhd}(p)$ is precisely the set of open sets that contain p . The equivalence follows.

(3) We reason as follows:

$$\begin{aligned} p \in \text{nbhd}^{-1}(\text{Op}(O)) &\iff \text{nbhd}(p) \in \text{Op}(O) && \text{Fact of function inverse} \\ &\iff p \in O && \text{Part 2 of this result} \end{aligned}$$

□

Corollary 9.2.3. *Suppose (P, Open) is a semitopology and $O, O' \in \text{Open}$. Then:*

- (1) $\text{Op}(O) \subseteq \text{Op}(O')$ if and only if $O \subseteq O'$.
- (2) $\text{Op}(O) \not\subseteq \text{Op}(O')$ if and only if $O \not\subseteq O'$.
- (3) As a corollary, $\text{nbhd}^{-1}(\emptyset) = \emptyset$ and $\text{nbhd}^{-1}(\text{Points}(\text{Open}, \subseteq, \not\subseteq)) = P$; that is nbhd^{-1} maps the bottom/top element to the bottom/top element.

Proof. We consider each part in turn:

- (1) If $\text{Op}(O) \subseteq \text{Op}(O')$ then $\text{nbhd}^{-1}(\text{Op}(O)) \subseteq \text{nbhd}^{-1}(\text{Op}(O'))$ by facts of inverse images, and $O \subseteq O'$ follows by Proposition 9.2.2(3).
If $O \subseteq O'$ then $\text{Op}(O) \subseteq \text{Op}(O')$ by Proposition 8.3.3(1).
- (2) If $O \not\subseteq O'$ then there exists some point $p \in P$ with $p \in O \cap O'$. By Proposition 9.2.2(1) $\text{nbhd}(p)$ is an abstract point, and by Proposition 9.2.2(2) $\text{nbhd}(p) \in \text{Op}(O) \cap \text{Op}(O')$; thus $\text{Op}(O) \not\subseteq \text{Op}(O')$.
If $\text{Op}(O) \not\subseteq \text{Op}(O')$ then $O \not\subseteq O'$ by Proposition 8.3.3(2).
- (3) Routine from Proposition 8.3.3(3) (or from Lemma 9.1.6).

□

9.2.2 Application to semiframes of open sets

Proposition 9.2.4. *Suppose (P, Open) is a semitopology. Then:*

- (1) nbhd^{-1} bijects open sets of $\text{St}(\text{Open}, \subseteq, \not\subseteq)$ (as defined in Definition 8.4.3(2)), with open sets of (P, Open) , taking $\text{Op}(O)$ to O .
- (2) nbhd^{-1} is an isomorphism between the semiframe of open sets of $\text{St}(\text{Open}, \subseteq, \not\subseteq)$, and the semiframe of open sets of (P, Open) (Definition 9.1.5).

Proof. We consider each part in turn:

- (1) Unpacking Definition 8.4.3(2), an open set in $\text{St Fr}(P, \text{Open})$ has the form $\text{Op}(O)$ for some $O \in \text{Open}$. By Proposition 9.2.2(3) $\text{nbhd}^{-1}(\text{Op}(O)) = O$, and so nbhd^{-1} is surjective and injective.
- (2) Unpacking Definition 9.1.5 it suffices to check that:
 - nbhd^{-1} is a bijection, and maps $\text{Op}(O)$ to O .
 - $\text{Op}(O) \subseteq \text{Op}(O')$ if and only if $O \subseteq O'$.
 - $\text{Op}(O) \not\subseteq \text{Op}(O')$ if and only if $O \not\subseteq O'$.

The first condition is part 1 of this result; the second and third are from Corollary 9.2.3. □

Proposition 9.2.5. *Suppose (P, Open) is a semitopology. Then the semiframe $\text{Fr}(P, \text{Open}) = (\text{Open}, \subseteq, \not\subseteq)$ from Definition 7.3.4 is spatial.*

Proof. The properties required by Definition 9.1.2 are that $Op(O) \subseteq Op(O')$ implies $O \subseteq O'$, and $O \not\subseteq O'$ implies $Op(O) \not\subseteq Op(O')$. Both of these are immediate from Proposition 9.2.4(2). $\square$

9.2.3 Application to characterise T_0 spaces

Lemma 9.2.6. Suppose (P, Open) is a semitopology and $p, p' \in P$. Then the following are equivalent:

- (1) $nbhd(p) = nbhd(p')$ (cf. also Lemma 11.7.1)
- (2) $\forall O \in \text{Open}. p \in O \iff p' \in O$
- (3) $p = p'$ (Definition 8.4.6: p and p' are topologically indistinguishable in (P, Open)).
- (4) $nbhd(p) = nbhd(p')$ ($nbhd(p)$ and $nbhd(p')$ are topologically indistinguishable as – by Proposition 9.2.2(1) – abstract points in $\text{St Fr}(P, \text{Open})$).

Proof. Equivalence of parts 1 and 2 is direct from Definition 2.3.1. Equivalence of parts 2 and 3 is just Definition 8.4.6(1). Equivalence of parts 4 and 1 is from Proposition 8.4.7. $\square$

Corollary 9.2.7. Suppose (P, Open) is a semitopology. Then the following are equivalent:

- (1) (P, Open) is T_0 (Definition 8.4.6(3)).
- (2) $nbhd : P \rightarrow \text{Points}(\text{Open}, \subseteq, \not\subseteq)$ is injective.

Proof. Suppose (P, Open) is T_0 , and suppose $nbhd(p) = nbhd(p')$. By Lemma 9.2.6(1 and 3) $p = p'$. By Definition 8.4.6(2) $p = p'$. Since p and p' were arbitrary, $nbhd$ is injective.

Suppose $nbhd$ is injective. Reversing the reasoning of the previous paragraph, we deduce that (P, Open) is T_0 . $\square$

9.3 Sober semitopologies

Recall from Proposition 9.2.5 that if we go from a semitopology (P, Open) to a semiframe $(\text{Open}, \subseteq, \not\subseteq)$, then the result is not just any old semiframe – it is a *spatial* one.

We now investigate what happens when we go from a semiframe to a semitopology using Definition 8.4.3.

9.3.1 The definition and a key result

Definition 9.3.1. Call a semitopology (P, Open) **sober** when every abstract point P of $\text{Fr}(P, \text{Open})$ – that is every completely prime nonempty up-closed compatible set of open sets – is equal to the neighbourhood semifilter $nbhd(p)$ of some unique $p \in P$.

Equivalently, (P, Open) is sober when $nbhd : P \rightarrow \text{Points}(\text{Fr}(P, \text{Open}))$ (Definition 8.1.1(7)) is a bijection.

Remark 9.3.2. A bijection is a map that is injective and a surjective. We noted in Corollary 9.2.7 that a space is T_0 when $nbhd$ is injective. So the sobriety condition can be thought of as having two parts:

- $nbhd$ is injective and the space is T_0 , so it intuitively contains no ‘unnecessary’ duplicates of points;
- $nbhd$ is surjective, so the space contains ‘enough’ points that there is (precisely) one concrete point for every abstract point.²⁵

We start with a very simple example of sober semitopologies:

Lemma 9.3.3. *Suppose P is any set. Then the discrete semitopology $(P, \text{pow}(\text{Open}))$ is sober.*

Proof. Consider an abstract point $P \subseteq \text{Open}$ (completely prime nonempty up-closed and compatible, as per Definition 8.1.1(7)). Then $P \in P$ and $P = \bigvee \{\{p\} \mid p \in P\}$. Since P is completely prime, $\{p\} \in P$ for some $p \in P$. It follows easily that $P = \text{nbhd}(p)$. $\square$

Example 9.3.4. We give some more examples of sober and non-sober semitopologies.

- (1) Take $P = \{0, 1\}$ and $\text{Open} = \{\emptyset, \{0, 1\}\}$. This has one abstract point $P = \{\{0, 1\}\}$ but two concrete points 0 and 1. It is therefore not sober.
- (2) Take $P = \{0, 1\}$ and $\text{Open} = \{\emptyset, \{1\}, \{0, 1\}\}$. This has two abstract points

$$\{\{1\}, \{0, 1\}\} \quad \text{and} \quad \{\{0, 1\}\}$$

corresponding to two concrete points 0 and 1. It is sober.

- (3) Take $P = \mathbb{N}$ with the final topology; so $O \in \text{Open}$ when $O = \emptyset$ or $O = n_{\geq}$ for some $n \in \mathbb{N}$, where $n_{\geq} = \{n' \in \mathbb{N} \mid n' \geq n\}$. Take $P = \{n_{\geq} \mid n \in \mathbb{N}\}$. The reader can check that this is an abstract point (up-closed, completely prime, compatible); however P is not the neighbourhood semifilter of any $n \in \mathbb{N}$. Thus this space is not sober.

- (4) $\mathbb{R}$ with its usual topology (which is also a semitopology) is sober. This is a known result for topologies, but Remark 8.1.3 (and also the later Remark 9.3.11) caution us that we cannot take this for granted, so we sketch the proof.

Suppose P is an abstract point; we wish to exhibit a unique $p \in \mathbb{R}$ such that $P = \text{nbhd}(p)$.

We cover $\mathbb{R}$ with open intervals of radius 1 by writing $\mathbb{R} = \bigcup \{(r-0.5, r+0.5) \mid r \in \mathbb{R}\}$, and we use the completely prime property to find (at least one) such open interval that is in P ; write it $O_1 \in P$. We then cover O_1 with open intervals of radius at most 1/2 by writing $O_1 = \bigcup \{O_1 \cap (r-0.25, r+0.25) \mid r \in O_1\}$, and we iterate to obtain a sequence $(O_i \mid i \in \mathbb{N}) \subseteq P$. This converges to some unique $p \in \mathbb{R}$. We check that $P = \text{nbhd}(p)$:

- Suppose $O \in \text{Open}$ is such that $p \in O$. Because $p \in O$, there exists some ϵ such that $(p-\epsilon, p+\epsilon) \subseteq O$. It follows that for $i > 1/\epsilon$ we have $O_i \subseteq O$ and thus $O \in P$ by up-closedness.
- Suppose $O' \in \text{Open}$ is such that $p \notin O'$. For a sufficiently large i we have $O_i \not\subseteq O'$, so by compatibility it follows that $O' \notin P$.

- (5) $\mathbb{Q}$ is sober. The argument is much as for $\mathbb{R}$ above. We have to work just a little harder because the p we obtain need not be rational, but the arguments on open intervals remain valid.

Proposition 9.3.5. *Suppose $(X, \leq, *)$ is a semiframe. Then $\text{St}(X, \leq, *)$ from Definition 8.4.3 is a sober semitopology.*

Proof. We know from Lemma 8.4.4 that $\text{St}(X, \leq, *)$ is a semitopology. The issue is whether it is sober; thus by Definition 9.3.1 we wish to exhibit every abstract point P of $\text{Fr St}(X, \leq, *)$ as a neighbourhood semifilter $\text{nbhd}(p)$ for some unique abstract point p of $(X, \leq, *)$. The calculations to do so are routine, but we give details.

Fix some abstract point P of $\text{Fr St}(X, \leq, *)$. By Definition 8.1.1(7), P is a completely prime nonempty up-closed set of intersecting open sets in the semitopology $\text{St}(X, \leq, *)$, and by Definition 8.4.3(2) each open set in $\text{St}(X, \leq, *)$ has the form $Op(x) = \{p \in \text{Points}(X, \leq, *) \mid x \in p\}$ for some $x \in X$.

We define $p \subseteq X$ as follows:

$$p = \{x \in X \mid Op(x) \in P\} \subseteq X.$$

By construction we have that $x \in p$ if and only if $Op(x) \in P$, and so

$$\begin{aligned} nbhd(p) &= \{Op(x) \mid p \in Op(x)\} && \text{Definition 2.3.1} \\ &= \{Op(x) \mid x \in p\} && \text{Definition 8.3.1} \\ &= \{Op(x) \mid Op(x) \in P\} && \text{Construction of } p \\ &= P && \text{Fact.} \end{aligned}$$

Now P is completely prime, nonempty, up-closed, and compatible and it follows by elementary calculations using Proposition 9.1.4 that p is also completely prime, nonempty, up-closed, and compatible – so p is an abstract point of $(X, \leq, *)$.

So we have that

$$p \in \text{Point}(X, \leq, *) \quad \text{and} \quad P = nbhd(p).$$

To prove uniqueness of p , suppose p' is any other abstract point such that $P = nbhd(p')$. We follow the definitions: $Op(x) \in nbhd(p') \iff Op(x) \in nbhd(p)$, and thus by Definition 2.3.1 $p' \in Op(x) \iff p \in Op(x)$, and thus by Definition 8.3.1 $x \in p' \iff x \in p$, and thus $p' = p$. $\square$

9.3.2 Sober topologies contrasted with sober semitopologies

We will need Notation 9.3.6 for Remark 9.3.7:

Notation 9.3.6. Call a closed set **irreducible** when it cannot be written as the union of two proper closed subsets.

Remark 9.3.7. Topology has a wealth of separation actions. Three of them are: T_0 (distinct points have distinct neighbourhood (semi)filters); T_1 (distinct points have distinct open neighbourhoods); and T_2 , also known as the Hausdorff condition (distinct points have disjoint open neighbourhoods) – see Remark 4.2.1 for formal statements. For topologies, the following is known of sobriety:

- (1) Every finite T_0 (and thus every finite T_1) topological space is sober.
- (2) Every T_2 /Hausdorff space (including infinite ones) is sober (Mac Lane and Moerdijk, 1992, p. 475, Theorem 3).
- (3) A topological space is sober if and only if every nonempty irreducible closed set is the closure of a unique point (Mac Lane and Moerdijk, 1992, p. 475).

The situation for semitopologies is different, as we explore in the rest of this Subsection.

Lemma 9.3.8.

- (1) It is not necessarily the case that a finite T_0 semitopology (or even a finite T_1 semitopology) is sober (Definition 9.3.1).
- (2) It is not necessarily the case that if every nonempty irreducible closed set is the closure of a unique point, then a semitopology is sober.

These non-implications hold even if the semitopology is regular (so $p \in K(p) \in \text{Topen}$ for every p ; see Definition 4.1.4(3)).



Figure 6. Two counterexamples for sobriety: (a) finite T_0 (and also T_1) semitopology that is not sober (Lemma 9.3.8); (b) Hausdorff semitopology that is not sober (Lemma 9.3.10).

Proof. We provide a semitopology that is a counterexample for both parts.

Consider the left-hand semitopology illustrated in Figure 6, so that:

- $P = \{0, 1, 2\}$, and
- $\text{Open} = \{\emptyset, \{0, 1\}, \{1, 2\}, \{0, 2\}, \{0, 1, 2\}\}$.

We note that:

- (P, Open) is T_0 and T_1 .
- (P, Open) is regular because all points are intertwined, so that $K(p) = P$ for every $p \in P$.
- The nonempty irreducible closed sets are $\{0\}$ (which is the complement of $\{1, 2\}$), $\{1\}$, and $\{2\}$. Since these are singleton sets, they are certainly the closures of unique points.

So (P, Open) is T_0 , regular, and irreducible closed sets are the closures of unique points.

We take as our semifilter $P = \text{Open} \setminus \{\emptyset\}$. The reader can check that P is completely prime (Definition 8.1.1(2)), nonempty, up-closed, and compatible (P is also the greatest semifilter); but, P is not the neighbourhood semifilter of 0, 1, or 2 in P . Thus, (P, Open) is not sober. $\square$

Remark 9.3.9. The counterexample used in Lemma 9.3.8 generalises, as follows: the reader can check that the all-but-one semitopology from Example 2.1.4(3) on three or more points (so open sets are generated by $P \setminus \{p\}$ for every $p \in P$) has similar behaviour.

In topology, every Hausdorff space is sober. In semitopologies, this implication does not hold, and in a rather strong sense:

Lemma 9.3.10.

- (1) *It is not necessarily the case that if a semitopology is Hausdorff, then it is sober.*
- (2) *Every quasiregular Hausdorff semitopology is (discrete and therefore) sober.*

Proof. We consider each part in turn:

- (1) It suffices to give a counterexample. Consider the right-hand semitopology illustrated in Figure 6 (which we also used, for different purposes, in Figure 3), so that:

- $P = \{0, 1, 2, 3\}$, and
- Open is generated by $X = \{\{3, 0\}, \{0, 1\}, \{1, 2\}, \{2, 3\}\}$.

This is Hausdorff, but it is not sober: the reader can check that the up-closure $\{3, 0\}^{\leq} \subseteq \text{Open}$ is nonempty, up-closed, compatible, and completely prime, but it is not the neighbourhood filter of any $p \in P$.

- (2) From Lemmas 4.2.2 (quasiregular Hausdorff is discrete) and 9.3.3 (discrete is sober). $\square$



Figure 7. Soberifications of the examples in Figure 6 (Remark 9.3.12).

Remark 9.3.11. A bit more discussion of Lemma 9.3.10.

- (1) The space used in the counterexample for part 1 is Hausdorff, T_1 , and unconflicted (Definition 5.3.1(2)). It is not quasiregular (Definition 4.1.4(5)) because the community of every point is empty; see Proposition 5.3.3.
- (2) The implication holds if we add quasiregularity as a condition: every quasiregular Hausdorff space is sober. But, this holds for very bad reasons, because by Lemma 4.2.2 every quasiregular Hausdorff space is discrete.
- (3) Thus, the non-implication discussed in Lemma 9.3.10 is informative and tells us something interesting about semitopological sobriety. Semitopological sobriety is not just a weak form of topological sobriety, and it has its own distinct personality.

Remark 9.3.12. We can inject the examples illustrated in Figure 6 (used in Lemmas 9.3.8 and 9.3.10) into soberified versions of the spaces that are sober and have an isomorphic lattice of open sets, as illustrated in Figure 7:

- (1) The left-hand semitopology has abstract points (completely prime semifilters; see Definition 8.1.1(2)) generated as the $\subseteq$ -up-closures of the following sets: $\{A\}$, $\{B\}$, $\{C\}$, $\{A, B\}$, $\{B, C\}$, $\{C, A\}$, and $\{A, B, C\}$. Of these, $\{A, B\}^\subseteq = nbhd(0)$, $\{B, C\}^\subseteq = nbhd(1)$, and $\{C, A\}^\subseteq = nbhd(2)$. The other completely prime semifilters are not generated as the neighbourhood semifilters of any point in the original space, so we add points as illustrated using $\bullet$ in the left-hand diagram in Figure 7. This semitopology is sober, and has the same semiframe of open sets.
- (2) For the right-hand example, we again add a $\bullet$ point for every abstract point in the original space that is not already the neighbourhood semifilter of a point in the original space. These abstract points are generated as the $\subseteq$ -up-closures of $\{A\}$, $\{B\}$, $\{C\}$, and $\{D\}$. There is no need to add a $\bullet$ for the abstract point generated as the $\subseteq$ -up-closure of $\{A, B\}$, because $\{A, B\}^\subseteq = nbhd(0)$. Similarly $\{B, C\}^\subseteq = nbhd(1)$, $\{C, D\}^\subseteq = nbhd(2)$, and $\{D, A\}^\subseteq = nbhd(3)$. Note that $\{A, B, C\}$ does not generate an abstract point because it is not compatible: $A \not\subseteq C$. Similarly for $\{B, C, D\}$, $\{C, D, A\}$, $\{D, A, C\}$, and $\{A, B, C, D\}$.

These soberified spaces are instances of a general construction described in Theorem 10.1.4. And, continuing the observation made in Remark 9.3.11, note that neither of these spaces, with their extra points, are Hausdorff.

10. Four Categories & Functors Between Them

10.1 The categories *SemiTop/Sober* of semitopologies/sober semitopologies

Definition 10.1.1.

- (1) Suppose (P, Open) and (P', Open') are semitopologies and $f : P \rightarrow P'$ is any function. Then call f a **morphism of semitopologies** when f is continuous, by which we mean (as standard)

that

$$O' \in \text{Open}' \text{ implies } f^{-1}(O') \in \text{Open}.$$

- (2) Define **SemiTop** the **category of semitopologies** such that:
 - objects are semitopologies, and
 - arrows are morphisms of semitopologies (continuous maps on points).²⁶
- (3) Write **Sober** for the **category of sober semitopologies** and continuous functions between them. By construction, **Sober** is the full subcategory of **SemiTop**, on its sober semitopologies.

Remark 10.1.2. For convenience reading Theorem 10.1.4 we recall some facts:

- (1) The semiframe

$$\text{Fr}(P, \text{Open}) = (\text{Open}, \subseteq, \cap)$$

from Definition 7.3.4 has elements open sets $O \in \text{Open}$, preordered by subset inclusion and with a compatibility relation given by set intersection.

It is spatial, by Proposition 9.2.5.

- (2) An abstract point P in $\text{Points}(\text{Fr}(P, \text{Open}))$ is a completely prime nonempty up-closed compatible subset of Open .
- (3) $\text{St Fr}(P, \text{Open})$ is by Definition 8.4.3 a semitopology such that:
 - a. Its set of points is $\text{Points}(\text{Fr}(P, \text{Open}))$; the set of abstract points in $\text{Fr}(P, \text{Open}) = (\text{Open}, \subseteq, \cap)$, the semilattice of open sets of (P, Open) , and
 - b. Its open sets are given by the $\text{Op}(O)$, for $O \in \text{Open}$.
 It is sober, by Proposition 9.3.5.

Continuing Remark 10.1.2, a notation will be useful:

Notation 10.1.3. Suppose (P, Open) is a semitopology. Then define

$$\text{Soberify}(P, \text{Open}) = \text{St Fr}(P, \text{Open}).$$

We may use $\text{Soberify}(P, \text{Open})$ and $\text{St Fr}(P, \text{Open})$ interchangeably, depending on whether we want to emphasise “this is a sober semitopology obtained from (P, Open) ” or “this is St acting on $\text{Fr}(P, \text{Open}) = (\text{Open}, \subseteq, \cap)$ ”.

Theorem 10.1.4. Suppose (P, Open) is a semitopology. Then

- (1) $\text{nbhd} : P \rightarrow \text{Points}(\text{Fr}(P, \text{Open}))$ is a morphism of semitopologies from (P, Open) to $\text{St Fr}(P, \text{Open}) = \text{Soberify}(P, \text{Open})$
- (2) taking (P, Open) to a sober semitopology $\text{Soberify}(P, \text{Open})$, such that
- (3) nbhd^{-1} induces a bijection on open sets by mapping $\text{Op}(O)$ to O , and furthermore this is an isomorphism of the semiframes of open sets, in the sense of Definition 9.1.5.

Proof. We consider each part in turn:

- (1) Following Definition 10.1.1 we must show that nbhd is continuous (inverse images of open sets are open) from (P, Open) to $\text{Soberify}(P, \text{Open})$. So following Definition 8.4.3(2), consider $\text{Op}(O) \in \text{Open}(\text{Soberify}(P, \text{Open}))$. By Proposition 9.2.2(3)

$$\text{nbhd}^{-1}(\text{Op}(O)) = O \in \text{Open}.$$

Continuity follows.

- (2) Soberify(P, Open) is sober by Proposition 9.3.5.
- (3) This is Proposition 9.2.4. □

Remark 10.1.5. We can summarise Theorem 10.1.4 as follows:

- (1) By construction, the kernel of the *nbhd* function (the relation determined by which points it maps to equal elements) is topological indistinguishability $\doteq$.
- (2) We can think of $\text{St Fr}(P, \text{Open})$ as being obtained from (P, Open) by
 - a. quotienting topologically equivalent points to obtain a T_0 space, and then
 - b. adding extra points to make it sober.
 See also the discussion in Remark 9.3.2 about what it means to have ‘enough’ points.
- (3) This is done without affecting the semiframe of open sets (up to isomorphism), with the semiframe bijection given by nbhd^{-1} .

In this sense, we can view $\text{St Fr}(P, \text{Open})$ as a **soberification** of (P, Open) .

10.2 The categories *SemiFrame*/*Spatial of semiframes*/*spatial semiframes*

Definition 10.2.1.

- (1) Suppose $(X, \leq, *)$ and $(X', \leq', *')$ are semiframes (Definition 7.3.1) and $g: X \rightarrow X'$ is any function. Then call g a **morphism of semiframes** when:
 - a. g is a morphism of complete semilattices (Definition 7.1.2).
 - b. g is **compatible**, by which we mean that $g(x') * g(x'')$ implies $x' * x''$ for every $x', x'' \in X'$.
- (2) We define *SemiFrame* the **category of semiframes** such that:
 - objects are semiframes, and
 - arrows are morphisms of semiframes.
- (3) Write *Spatial* for the **category of spatial semiframes** and semiframe morphisms between them. By construction, *Spatial* is the full subcategory of *SemiFrame*, on its spatial semiframes (Definition 9.1.2).

Lemma 10.2.2. Suppose $(X, \leq, *)$ is a semiframe. Then $Op: (X, \leq, *) \rightarrow \text{Fr St}(X, \leq, *)$ is a morphism of semiframes and is surjective on underlying sets.

Proof. Following Definition 10.2.1(1) we must show that

- Op is a semilattice morphism (Definition 7.1.2) – commutes with joins and maps T_X to $\text{Points}(X, \leq, *)$ – and
- is compatible with the compatibility relation $*$, and
- we must show that Op is surjective.

We consider each property in turn:

- Op is a semilattice morphism.
 $Op(\bigvee X) = \bigvee_{x \in X} Op(x)$ by Lemma 8.3.2, and $Op(T_X) = \text{Points}(X, \leq, *)$ by Proposition 8.3.3(3).

- Op is compatible with $*$.

Unpacking Definition 10.2.1(2), we must show that $Op(x) \checkmark Op(x')$ implies $x * x'$. We use Proposition 8.3.3(2).

- Op is surjective . . . by Lemma 8.4.5.

□

10.3 Functoriality of the maps

Definition 10.3.1. Suppose $g : (X', \leq', *) \rightarrow (X, \leq, *) \in \text{SemiFrame}$. Define a mapping $g^\circ : \text{St}(X, \leq, *) \rightarrow \text{St}(X', \leq', *)$ by

$$\begin{aligned} g^\circ : \text{Points}(X, \leq, *) &\longrightarrow \text{Points}(X', \leq', *) \\ P &\longmapsto P' = \{x' \in X' \mid g(x') \in P\}. \end{aligned}$$

Remark 10.3.2. We will show that g° from Definition 10.3.1 is an arrow in SemiTop . We will need to prove the following:

- If $P \in \text{Points}(X, \leq, *)$ then $g^\circ(P) \in \text{Points}(X', \leq', *)$.
- g° is a morphism of semitopologies.

We do this in Lemmas 10.3.3 and 10.3.6 respectively.

Lemma 10.3.3 (g° well-defined). Suppose $g : (X', \leq', *) \rightarrow (X, \leq, *) \in \text{SemiFrame}$ and suppose $P \in \text{Points}(X, \leq, *)$. Then $g^\circ(P)$ from Definition 10.3.1 is indeed in $\text{Points}(X', \leq', *)$ – and thus g° is well-defined function from $\text{Points}(X, \leq, *)$ to $\text{Points}(X', \leq', *)$.

Proof. For brevity write

$$P' = \{x' \in X' \mid g(x') \in P\}.$$

We must check that P' is a completely prime nonempty up-closed compatible subset of X' . We consider each property in turn:

- (1) P' is completely prime.

Consider some $X' \subseteq P'$ and suppose $g(\bigvee X') \in P$. By Definition 10.2.1(1) g is a semilattice homomorphism, so by Definition 7.1.1(2) $g(\bigvee X') = \bigvee_{x' \in X'} g(x')$. Thus $\bigvee_{x' \in X'} g(x') \in P$. By assumption P is completely prime, so $g(x') \in P$ for some $x' \in X'$. Thus $x' \in P'$ for that x' . Since X' was arbitrary, it follows that P' is completely prime.

- (2) P' is nonempty.

By assumption g is an arrow in SemiFrame (i.e. a semiframe morphism) and unpacking Definition 10.2.1(1) it follows that it is a semilattice homomorphism. In particular by Definition 7.1.1(2) $g(\top_{X'}) = \top_X$, and by Lemma 8.2.1(1) $\top_X \in P$. Thus $\top_{X'} \in P'$, so P' is nonempty.

- (3) P' is up-closed.

Suppose $x' \in P'$ and $x' \leq x''$. By construction $g(x') \in P$. By Lemma 7.1.4 (because g is a semilattice morphism by Definition 10.2.1(1)) g is monotone, so $g(x') \leq g(x'')$. By assumption in Definition 8.1.1(3) P is up-closed, so that $g(x'') \in P$ and thus $x'' \in P'$ as required.

- (4) P' is compatible.

Suppose $x', x'' \in P'$. Thus $g(x'), g(x'') \in P$. By assumption in Definition 8.1.1(4) P is compatible, so $g(x') * g(x'')$. By compatibility of g (Definition 10.2.1(2)) it follows that $x' * x''$. Thus P' is compatible.

□

Remark 10.3.4. Note on design: If we want to impose further conditions on being an abstract point (such as those discussed in Remark 8.1.6) then Lemma 10.3.3 would need to be extended to show that these further conditions are preserved by the g° operation, so that for $P \in \text{Points}(X, \leq, *)$ an abstract point in $(X, \leq, *)$, $g^\circ(P) = \{x' \in X' \mid g(x') \in P\}$ is an abstract point in $(X', \leq', *')$.

For example: consider what would happen if we add the extra condition on semifilters from Remark 8.1.6. Then the P' defined in the proof of Lemma 10.3.3 above might not be closed under this additional condition (it will be if g is surjective). This could be mended by closing P' under greatest lower bounds that are not $\perp$, but that in turn might compromise the property of being completely prime. These comments are not a proof that the problems would be insuperable; but they suggest that complexity would be added. For now, we prefer to keep things simple!

Lemma 10.3.5. Suppose $g : (X', \leq', *) \rightarrow (X, \leq, *) \in \text{SemiFrame}$, and suppose $x' \in X'$. Then

$$(g^\circ)^{-1}(Op(x')) = Op(g(x')).$$

Proof. Consider an abstract point $P \in \text{Point}(\text{Gr}(X', \leq', *'))$. We just chase definitions:

$$\begin{aligned} P \in (g^\circ)^{-1}(Op(x')) &\iff g^\circ(P) \in Op(x') && \text{Fact of inverse image} \\ &\iff x' \in g^\circ(P) && \text{Definition 8.3.1} \\ &\iff g(x') \in P && \text{Definition 10.3.1} \\ &\iff P \in Op(g(x')). && \text{Definition 8.3.1} \end{aligned}$$

The choice of P was arbitrary, so $(g^\circ)^{-1}(Op(x')) = Op(g(x'))$ as required. $\square$

Lemma 10.3.6 (g° continuous). Suppose $g : (X', \leq', *) \rightarrow (X, \leq, *) \in \text{SemiFrame}$. Then $g^\circ : \text{St}(X, \leq, *) \rightarrow \text{St}(X', \leq', *')$ is continuous:

$$(g^\circ)^{-1}(\mathcal{O}') \in \text{Open}(\text{St}(X, \leq, *))$$

for every $\mathcal{O}' \in \text{Open}(\text{St}(X', \leq', *'))$.

Proof. By Definition 8.4.3, $\mathcal{O}' = Op(x')$ for some $x' \in X'$. By Lemma 10.3.5 $(g^\circ)^{-1}(Op(x')) = Op(g(x'))$. By Definition 8.4.3(2) $Op(g(x')) \in \text{Open}(\text{St}(X, \leq, *))$. $\square$

Proposition 10.3.7 (Functoriality).

- (1) Suppose $f : (P, \text{Open}) \rightarrow (P', \text{Open}') \in \text{SemiTop}$ (so f is a continuous map on underlying points). Then f^{-1} is an arrow $\text{Fr}(P', \text{Open}') \rightarrow \text{Fr}(P, \text{Open})$ in SemiFrame .
- (2) Suppose $g : (X', \leq', *) \rightarrow (X, \leq, *) \in \text{SemiFrame}$. Then g° from Definition 10.3.1 is an arrow $\text{St}(X, \leq, *) \rightarrow \text{St}(X', \leq', *')$ in SemiTop .
- (3) The assignments $f \mapsto f^{-1}$ and $g \mapsto g^\circ$ are **functorial** – they map identity maps to identity maps, and commute with function composition.

Proof. We consider each part in turn:

- (1) Following Definition 10.2.1, we must check that f^{-1} is a morphism of semiframes. We just unpack what this means and see that the required properties are just facts of taking inverse images:
 - f^{-1} commutes with joins, that is with $\bigcup$.
This is a fact of inverse images.

- f^{-1} maps $\mathsf{T}_{\mathsf{Fr}(P', \mathsf{Open}')} = P'$ to $\mathsf{T}_{\mathsf{Fr}(P, \mathsf{Open})} = P$.
This is a fact of inverse images.
- f^{-1} is compatible, meaning that $f^{-1}(O') \bowtie f^{-1}(O'')$ implies $O' \bowtie O''$.
This is a fact of inverse images.
- (2) We must check that g° is continuous. This is Lemma 10.3.6.
- (3) Checking functoriality is routine, but we sketch the reasoning anyway:
 - Consider the identity function id on some semitopology (P, Open) . Then id^{-1} should be the identity function on $(\mathsf{Open}, \subseteq, \bowtie)$. It is.
 - Consider $f : (P, \mathsf{Open}) \rightarrow (P', \mathsf{Open}')$ and $f' : (P', \mathsf{Open}') \rightarrow (P'', \mathsf{Open}'')$. Then $(f' \circ f)^{-1}$ should be equal to $f^{-1} \circ (f')^{-1}$. It is.
 - Consider the identity function id on $(X, \leq, *)$. Then id° should be the identity function on $\mathsf{Points}(X, \leq, *)$. We look at Definition 10.3.1 and see that this amounts to checking that $P = \{x \in X \mid id(x) \in P\}$. It is.
 - Consider $g : (X, \leq, *) \rightarrow (X', \leq', *')$ and $g' : (X', \leq', *') \rightarrow (X'', \leq'', *'')$ and consider some $P'' \in \mathsf{Points}(X'', \leq'', *'')$. Then $(g' \circ g)^\circ(P'')$ should be equal to $(g^\circ \circ (g')^\circ)(P'')$. We look at Definition 10.3.1 and see that this amounts to checking that $\{x \in X \mid g'(g(x)) \in P''\} = \{x \in X \mid g(x) \in P'\}$ where $P' = \{x' \in X' \mid g'(x') \in P''\}$. Unpacking these definitions, we see that the equality does indeed hold. □

10.4 Sober semitopologies are dual to spatial semiframes

Remark 10.4.1. A categorical duality between two categories $\mathbb{C}$ and $\mathbb{D}$ is a categorical equivalence between $\mathbb{C}$ and $\mathbb{D}^{op}$. See (Mac Lane, 1971, IV.4) or (Riehl, 2016, Section 1.5 & Exercise 4.2.i).²⁷ It suffices to provide an adjoint pair of functors whose unit and counit are natural isomorphisms.²⁸

There are many duality results in the literature. The duality between topologies and frames is described (e.g.,) in Mac Lane and Moerdijk (1992, p. 479, Corollary 4). A duality between distributive lattices and coherent spaces is in Johnstone (1986, p. 66). There is the classic duality by Stone between Boolean algebras and compact Hausdorff spaces with a basis of clopen sets (Stone (1936); Johnstone (1986)). An encyclopedic treatment is in Caramello (2011), with a rather good overview in Example 2.9 on p. 17.

Theorem 10.4.2 states a duality result between Sober and Spatial, thus appending another item to this extensive canon.

It also constructively moves us forward in studying semitopologies, because it gives us an algebraic treatment of semitopologies, and a formal framework for studying morphisms between semitopologies. For instance: taking morphisms to be continuous functions is sensible not just because this is also how things work for topologies, but also because this is what is categorically dual to the $\leq/$ -homomorphisms between semiframes (Definition 10.2.1). And of course, if we become interested in different notions of semitopology morphism (a flavour of these is in Remark 12.3.1) then the algebraic framework gives us a distinct mathematical light with which to inspect and evaluate them.

Theorem 10.4.2. *The maps St (Definition 8.4.3) and Fr (Definition 7.3.4), with actions on arrows as described in Proposition 10.3.7, form a categorical duality between the categories*

- Sober of sober semitopologies (Definition 9.3.1) and continuous compatible morphisms between them; and
- Spatial of spatial semiframes and morphisms between them (Definition 10.2.1(3)).

Proof. There are various things to check:

- Proposition 9.3.5 shows that St maps spatial semiframes to sober semitopologies.
- Proposition 9.2.5 shows that Fr maps sober semitopologies to spatial semiframes.
- By Proposition 10.3.7 the maps $f \mapsto f^{-1}$ (inverse image) and $g \mapsto g^\circ$ (Definition 10.3.1) are functorial.
- The equivalence morphisms are given by the bijections Op and $nbhd$:
 - Op is from Definition 8.3.1. By Lemma 10.2.2 Op is a morphism $(X, \leq, *) \rightarrow \text{Fr St}(X, \leq, *)$ in Spatial that is surjective on underlying sets. Injectivity is from Proposition 9.1.4(3).
 - $nbhd$ is from Definition 2.3.1. By Theorem 10.1.4 $nbhd$ is a morphism $(P, \text{Open}) \rightarrow \text{St Fr}(P, \text{Open})$ in Sober . It is a bijection on underlying sets by the sobriety condition in Definition 9.3.1.

Finally, we must check naturality of Op and $nbhd$, which means (as standard) checking commutativity of the following diagrams:

$$\begin{array}{ccc}
 (P, \text{Open}) & \xrightarrow{nbhd} & \text{St Fr}(P, \text{Open}) & & (X, \leq, *) & \xrightarrow{Op} & \text{Fr St}(X, \leq, *) \\
 \downarrow f & & \downarrow (f^{-1})^\circ & & \downarrow g & & \downarrow (g^\circ)^{-1} \\
 (P', \text{Open}') & \xrightarrow{nbhd} & \text{St Fr}(P', \text{Open}') & & (X', \leq', *') & \xrightarrow{Op} & \text{Fr St}(X', \leq', *')
 \end{array}$$

We proceed as follows:

- Suppose $g : (X', \leq', *) \rightarrow (X, \leq, *)$ in Spatial , so that $g^\circ : \text{St}(X, \leq, *) \rightarrow \text{St}(X', \leq', *)$ in Sober and $(g^\circ)^{-1} : \text{Fr St}(X', \leq', *) \rightarrow \text{Fr St}(X, \leq, *)$ in Spatial . To prove naturality we must check that

$$(g^\circ)^{-1}(Op(x)) = Op(g(x))$$

for every $x \in X$. This is just Lemma 10.3.5.

- Suppose $f : (P, \text{Open}) \rightarrow (P', \text{Open}')$ in SemiTop , so that $f^{-1} : \text{Fr}(P', \text{Open}') \rightarrow \text{Fr}(P, \text{Open})$ in Spatial and $(f^{-1})^\circ : \text{St Fr}(P, \text{Open}) \rightarrow \text{St Fr}(P', \text{Open}')$ in SemiTop . To prove naturality we must check that

$$(f^{-1})^\circ(nbhd(p)) = nbhd(f(p)).$$

We just chase definitions, for an open set $O' \in \text{Open}'$:

$$\begin{aligned}
 O' \in (f^{-1})^\circ(nbhd(p)) &\iff f^{-1}(O') \in nbhd(p) && \text{Definition 10.3.1} \\
 &\iff p \in f^{-1}(O') && \text{Definition 2.3.1} \\
 &\iff f(p) \in O' && \text{Inverse image} \\
 &\iff O' \in nbhd(f(p)) && \text{Definition 2.3.1.}
 \end{aligned}$$

□

Remark 10.4.3. We review the background to Theorem 10.4.2:

- (1) A semitopology (P, Open) is a set of points P and a set of open sets $\text{Open} \subseteq \text{pow}(P)$ that contains P and is closed under arbitrary (possibly empty) unions (Definition 1.2.2).
- (2) A morphism between semitopologies is a continuous function, just as for topologies (Definition 10.1.1(1)).

- (3) A semiframe $(X, \leq, *)$ is a complete join-semilattice $(X, \leq)$ with a properly reflexive distributive compatibility relation $*$ (Definition 7.3.1).
- (4) A morphism between semiframes is a morphism of complete join-semilattices with $\top$ that is compatible with the compatibility relation (Definition 10.2.1(1)).
- (5) An abstract point of a semitopology (P, Open) is a completely prime nonempty up-closed compatible subset $P \subseteq \text{Open}$ (Definition 8.1.1(7)).
- (6) A semitopology is sober when the neighbourhood semifilter map $p \in P \mapsto \text{nbhd}(p) = \{O \in \text{Open} \mid p \in O\}$ is injective and surjective between the points of P and the abstract points of P (Definition 9.3.1).
- (7) By Theorem 10.1.4, and as discussed in Remark 10.1.5, every (possibly non-sober) semitopology (P, Open) maps into its soberification $\text{St Fr}(P, \text{Open})$, which has an isomorphic semiframe of open sets. So even if our semitopology (P, Open) is not sober, there is a standard recipe to make it so.
- (8) A semiframe is spatial when $x \in X \mapsto \text{Op}(x) = \{P \in \text{Point} \mid x \in P\}$ respects $\leq$ and $*$ in senses make formal in Definition 9.1.2 and Proposition 9.1.4.
- (9) Sober semitopologies and continuous functions between them, and spatial semiframes and semiframe morphisms between them, are categorically dual (Theorem 10.4.2).

Remark 10.4.4. Note what Theorem 10.4.2 does not do: it does not give a duality between all semitopologies and all semiframes; it gives a duality between sober semitopologies and spatial semiframes. This in itself is nothing new – the topological duality is just the same – but what is interesting is that our motivation for studying semitopologies comes from practical network systems. These tend to be (finite) non-sober semitopologies – non-sober, because a guarantee of sobriety cannot be enforced, and anyway it is precisely the point of the exercise to achieve coordination, without explicitly requiring every possible constellation of cooperating agents to be explicitly represented by a point.

It is true that by Theorem 10.1.4 every non-sober T_0 semitopology can be embedded into a sober one without affecting the semiframe of open sets, but this makes the system to which it corresponds larger, by adding points. Thus the duality in Theorem 10.4.2 is a mathematical statement, but not necessarily a practical one – and this is as expected, because we knew that this is an abstract result. nbhd maps a point to a set of (open) sets; and Op maps an (open) set of points to a set of sets of (open) sets. Of course these might not be computationally optimal.

11. Well-Behavedness Conditions, Dually

We now study how properties of semifilters and abstract points correspond to the well-behavedness properties which we found useful in studying semitopologies – for example *topens*, *regularity*, and being *unconflicted* (Definitions 3.2.2, 4.1.4 and 5.3.1).

11.1 (Maximal) semifilters and transitive elements

Remark 11.1.1 (Semifilters are not filters). We know that semifilters do not necessarily behave like filters. For instance:

- (1) It is possible for a finite semifilter to have more than one minimal element, because the closure under binary meets condition of filters is replaced by a weaker compatibility condition (see also Remarks 8.1.3 and 8.1.6).
- (2) There are more semifilters than proper filters – even if the underlying space is a topology. For example, the discrete semitopology on $\{0, 1, 2\}$ (whose open sets are all subsets of the

space) is a topology. Every proper filter in this space is a semifilter, but it also has a semifilter $\{\{0, 1\}, \{1, 2\}, \{2, 0\}, \{0, 1, 2\}\}$ (see the top-left diagram in Figure 6) and this is not a filter.

More on this in Section 8.2.2.

In summary: semifilters are different and we cannot necessarily take their behaviour for granted without checking it. We now examine them in more detail.

We start with some easy definitions and results:

Notation 11.1.2. Suppose $(X, \leq, *)$ is a semiframe and $X, Y \subseteq X$ and $x \in X$. Then we generalise $x * y$ to $x * Y$, $X * y$, and $X * Y$ as follows:

- (1) Write $x * Y$ for $\forall y \in Y. x * y$.
- (2) Write $X * y$ for $\forall x \in X. x * y$.
- (3) Write $X * Y$ for $\forall x \in X. \forall y \in Y. x * y$.

We read $x * Y$ as ‘ x is **compatible** with Y ’, and similarly for $X * y$ and $X * Y$.

Remark 11.1.3. We will see later on in Lemma 11.7.1 that $X * X'$ generalises $p \dot{\vee} p'$, in the sense that if $X = \text{nbhd}(p)$ and $X' = \text{nbhd}(p')$, then $p \dot{\vee} p'$ if and only if $\text{nbhd}(p) * \text{nbhd}(p')$.

Lemma 11.1.4 (Characterisation of maximal semifilters). Suppose $(X, \leq, *)$ is a semiframe and $F \subseteq X$ is a semifilter. Then the following conditions are equivalent:

- (1) F is maximal.
- (2) For every $x \in X$, $x * F$ if and only if $x \in F$.

Proof. We prove two implications:

- Suppose F is a maximal semifilter.
Suppose $x \in F$. Then $x * F$ is immediate from Notation 11.1.2(1) and semifilter compatibility (Definition 8.1.1(4)).
Suppose $x * F$; thus by Notation 11.1.2(1) x is compatible with (every element of) F . We note that the $\leq$ -up-closure of $\{x\} \cup F$ is a semifilter (nonempty, up-closed, compatible). By maximality, $x \in F$.
- Suppose $x * F$ if and only if $x \in F$, for every $x \in X$.
Suppose F' is a semifilter and $F \subseteq F'$. Consider $x' \in F'$. Then $x' * F$ by compatibility of F' , and so $x' \in F$. Thus, $F' \subseteq F$. □

Definition 11.1.5. Suppose $(X, \leq, *)$ is a semiframe and $x \in X$. Call x **transitive** when:

- (1) $x \neq \perp_X$.
- (2) $x' * x * x''$ implies $x' * x''$, for every $x', x'' \in X$.

‘Being open’ in semitopologies (Definition 3.2.2(2)) corresponds to ‘being transitive’ in semiframes (Definition 11.1.5):

Lemma 11.1.6 (Characterisation of open sets). Suppose (P, Open) is a semitopology and $O \in \text{Open}$. Then the following are equivalent:

- (1) O is *topen* in (P, Open) in the sense of Definition 3.2.2(2).
- (2) O is *transitive* in $(\text{Open}, \subseteq, \emptyset)$ in the sense of Definition 11.1.5.²⁹

Proof. We unpack the definitions and note that the condition for being *topen* – being a nonempty open set that is transitive for $\emptyset$ – is identical to the condition for being transitive in $(\text{Open}, \subseteq, \emptyset)$ – being a non- $\perp_{\text{Open}}$ element that is transitive for $* = \emptyset$. $\square$

11.2 The compatibility system x^*

Definition 11.2.1. Suppose $(X, \leq, *)$ is a semiframe and $x \in X$. Then define x^* the **compatibility system** of x by

$$x^* = \{x' \in X \mid x' * x\}.$$

Lemma 11.2.2. Suppose $(X, \leq, *)$ is a semiframe and $X \subseteq X$. Then $(\bigvee X)^* = \bigcup_{x \in X} x^*$.

Proof. We just follow the definitions:

$$\begin{aligned} y \in (\bigvee X)^* &\iff y * \bigvee X && \text{Definition 11.2.1} \\ &\iff \exists x \in X. y * x && \text{Definition 7.2.1(3)} \\ &\iff \exists x \in X. y \in x^* && \text{Definition 11.2.1} \\ &\iff y \in \bigcup_{x \in X} x^* && \text{Fact of sets} \end{aligned}$$

$\square$

Lemma 11.2.3. Suppose $(X, \leq, *)$ is a semiframe and $x \in X$ is transitive. Then the following are equivalent for every $y \in X$:

$$y * x \iff y \in x^* \iff y * x^*.$$

Proof. We prove a cycle of implications:

- Suppose $y * x$. Then $y \in x^*$ is direct from Definition 11.2.1.
- Suppose $y \in x^*$. Then $y * x^*$ – meaning by Notation 11.1.2(1) that $y * x'$ for every $x' \in x^*$ – follows by transitivity of x .
- Suppose $y * x^*$. By proper reflexivity of $*$ (Definition 7.2.1(2); since $x \neq \perp_X$) $x \in x^*$, and $y * x$ follows. $\square$

Proposition 11.2.4. Suppose $(X, \leq, *)$ is a semiframe and suppose $\perp_X \neq x \in X$. Then the following are equivalent:

- (1) x is transitive.
- (2) x^* is a completely prime semifilter (i.e. an abstract point).
- (3) x^* is a semifilter.
- (4) x^* is compatible.
- (5) x^* is a maximal semifilter.

Proof. We first prove a cycle of implications between parts 1–4:

- (1) Suppose x is transitive. We need to check that x^* is nonempty, up-closed, compatible, and completely prime. We consider each property in turn:
 - $x * x$ by proper reflexivity of $*$ (Definition 7.2.1(2); since $x \neq \perp_X$), so $x \in x^*$.
 - It follows from monotonicity of $*$ (Lemma 7.2.3(1)) that if $x' \leq x''$ and $x * x'$ then $x * x''$.
 - Suppose $x' * x * x''$. By transitivity of x (Definition 11.1.5), $x' * x''$.
 - Suppose $x * \bigvee X'$; then by distributivity of $*$ (Definition 7.2.1(3)) $x * x'$ for some $x' \in X'$.
- (2) If x^* is a completely prime semifilter, then it is certainly a semifilter.
- (3) If x^* is a semifilter, then it is compatible (Definition 8.1.1(5 and 4)).
- (4) Suppose x^* is compatible (Definition 8.1.1(4)) and suppose $x' * x * x''$. By Lemma 11.2.3 $x', x'' \in x^*$, and by compatibility of x^* we have $x' * x''$. Thus, x is transitive.

To conclude, we prove two implications between parts 4 and 5:

- Suppose x^* is a semifilter. By equivalence of parts 3 and 1 of this result, x is transitive, and so using Lemma 11.2.3 $x' * x^*$ if and only if $x' \in x^*$. By Lemma 11.1.4, x^* is maximal.
- Clearly, if x^* is a maximal semifilter then it is a semifilter. □

11.3 The compatibility system F^*

11.3.1 Basic definitions and results

Definition 11.3.1. Suppose $(X, \leq, *)$ is a semiframe and suppose $F \subseteq X$ (F may be a semifilter, but the definition does not depend on this). Define F^* the **compatibility system** of F by

$$F^* = \{x' \in X \mid x' * F\}$$

Unpacking Notation 11.1.2(1), and combining with Definition 11.2.1, we can write:

$$F^* = \{x' \in X \mid x' * F\} = \{x' \in X \mid \forall x \in F. x' * x\} = \bigcap \{x^* \mid x \in F\}.$$

Lemma 11.3.2 presents one easy and useful example of Definition 11.3.1:

Lemma 11.3.2. Suppose (P, Open) is a semitopology and suppose $p \in P$ and $O' \in \text{Open}$. Then:

$$\begin{aligned} O' \in \text{nbhd}(p)^* &\iff \forall O \in \text{Open}. p \in O \implies O' \not\leq O \\ O' \notin \text{nbhd}(p)^* &\iff \exists O \in \text{Open}. p \in O \wedge O' \not\leq O. \end{aligned}$$

Proof. We just unpack Definitions 2.3.1 and 11.3.1. □

Lemma 11.3.3. Suppose $(X, \leq, *)$ is a semiframe and $F \subseteq X$. Then F^* is up-closed.

Proof. This is just from Definition 11.3.1 and monotonicity of $*$ (Lemma 7.2.3(1)). □

Lemma 11.3.4. Suppose $(X, \leq, *)$ is a semiframe and $F \subseteq X$ is a semifilter. Then:

- (1) If $x \in F$ then $F \subseteq x^*$.
- (2) As a corollary, $F \subseteq F^*$.

Proof. Suppose $x \in F$. By compatibility of F (Definition 8.1.1(4)), $x' * x$ for every $x' \in F$. It follows from Definition 11.2.1 that $F \subseteq x^*$. The corollary is immediate from Definition 11.3.1. $\square$

We can use Lemma 11.3.4 and Definition 11.3.1 to give a more succinct rendering of Lemma 11.1.4:

Corollary 11.3.5. *Suppose $(X, \leq, *)$ is a semiframe and $F \subseteq X$ is a semifilter. Then the following are equivalent:*

- (1) F is maximal.
- (2) $F^* = F$.
- (3) $F^* \subseteq F$.

Proof. Equivalence of parts 1 and 2 just repeats Lemma 11.1.4 using Definition 11.3.1. To prove equivalence of parts 2 and 3 we use Lemma 11.3.4(2). $\square$

11.3.2 Strong compatibility: when F^* is a semifilter

Proposition 11.2.4 relates good properties of x (transitivity) to good properties of its compatibility system x^* (e.g. being compatible). It will be helpful to ask similar questions of F^* . What good properties are of interest for F^* , and what conditions can we impose on F to guarantee them?

Definition 11.3.6. Suppose $(X, \leq, *)$ is a semiframe. Then:

- (1) Call $F \subseteq X$ **strongly compatible** when F^* is nonempty and compatible.
- (2) Call $(X, \leq, *)$ **strongly compatible** when every abstract point (completely prime semifilter) $P \subseteq X$ is strongly compatible.

Remark 11.3.7. For the reader's convenience we unpack Definition 11.3.6.

- (1) By Definition 8.1.1(4), F^* is compatible when $x * x'$ for every $x, x' \in F^*$. Combining this with Definition 11.3.1 and Notation 11.1.2, F^* is compatible when $x * F * x'$ implies $x * x'$, for every $x, x' \in X$. Thus, F is strongly compatible when

$$\forall x, x' \in X. x * F * x' \implies x * x'.$$

- (2) $(X, \leq, *)$ is strongly compatible when every abstract point $P \in \text{Point}(X, \leq, *)$ is strongly compatible in the sense just given above.

Lemma 11.3.8. *Suppose (P, Open) is a semitopology and $p \in P$. Recall from Definition 7.3.4(3) and Lemma 7.3.5 that $(\text{Open}, \subseteq, \cap)$ is a semiframe. Then the following are equivalent:*

- (1) The point $p \in P$ is hypertransitive in the sense of Definition 6.2.3.
- (2) The semifilter $\text{nbhd}(p) \subseteq \text{Open}$ is strongly compatible in the sense of Definition 11.3.6.

Proof. Remark 11.3.7 notes that the condition in Definition 6.2.3 is precisely the condition for $\text{nbhd}(p)$ to be strongly compatible. $\square$

Remark 11.3.9. Given Lemma 11.3.8, the reader might ask why we do not just call a strongly compatible semifilter 'hypertransitive'.

There is a case for doing so, but caution is required: strong compatibility of semiframes is not quite the same thing as hypertransitivity of points. Every point p generates a semifilter $\text{nbhd}(p)$, but there may be more semifilters than there are points, and this makes the strong compatibility condition subtly different from the hypertransitivity condition. We shall see the effects of this in Lemma 11.5.3(2), and in Theorem 11.5.4 (see Remark 11.5.5 for a brief discussion), and then again in Definition 11.8.8 where we define a notion of strongly compatible semitopology (essentially: all of its semifilters are strongly compatible), which is not the same thing as the space being hypertransitive (essentially: all of its points are hypertransitive).

Therefore, we maintain a terminological distinction: points are hypertransitive, semiframes are strongly compatible. The notions are related, but not quite the same thing.

Lemma 11.3.10. *Suppose $(X, \leq, *)$ is a semiframe and suppose $F \subseteq X$ is nonempty. Then the following are equivalent:*

- (1) F^* is a semifilter.
- (2) F^* is compatible.
- (3) F is strongly compatible.

Proof. Equivalence of parts 2 and 3 is just Definition 11.3.6. For equivalence of parts 1 and 2 we prove two implications:

- Suppose F^* is a semifilter. Then F^* is compatible by assumption in Definition 8.1.1(5).
- Suppose F^* is compatible. Then F^* is up-closed by Lemma 11.3.3, and nonempty by Lemma 11.3.4(2) (since F is nonempty). Thus, by Definition 8.1.1(5) F^* is a semifilter. $\square$

Lemma 11.3.11. *Suppose $(X, \leq, *)$ is a semiframe and suppose $F \subseteq X$. Then it is not necessarily the case that F^* is a semifilter.*

*This non-implication holds even in strong well-behavedness conditions: that $(X, \leq, *)$ is spatial and F is an abstract point (a completely prime semifilter).*

Proof. It suffices to provide a counterexample. Let $(P, \text{Open}) = (\{0, 1, 2\}, \{\emptyset, \{0\}, \{2\}, P\})$, as illustrated in the top-left semitopology in Figure 2. Take $(X, \leq, *) = (\text{Open}, \subseteq, \cap)$ (which is spatial by Proposition 9.2.5) and set $F = \text{nbhd}(1) = \{0, 1, 2\}$. Then $\text{nbhd}(1)^* = \{\{0\}, \{2\}, \{0, 1, 2\}\}$, and this is not compatible because $\{0\} \not\subseteq \{2\}$.³⁰ $\square$

Remark 11.3.12. Lemma 11.3.11 gives an example of a semifilter F that is not strongly compatible (i.e. such that F^* is not a semifilter). Note that in this example both the space and F are well-behaved. This raises the question of finding sufficient (though perhaps not necessary) criteria for strong compatibility. We conclude with Proposition 11.3.13 which provides one such criterion; it will be useful later in Lemma 11.5.3 and Theorem 11.8.10.

Proposition 11.3.13 bears a family resemblance to Theorem 4.2.5 (if a point has a topen neighbourhood then it is regular):

Proposition 11.3.13. *Suppose $(X, \leq, *)$ is a semiframe and $F \subseteq X$ is a semifilter. Then:*

- (1) *If F contains a transitive element then F is strongly compatible.*
- (2) *The converse implication need not hold: it may be that F is strongly compatible yet F contains no transitive element.*

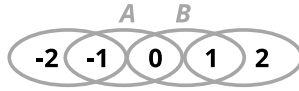


Figure 8. Strongly compatible filter that contains no transitive element.

Proof. We consider each part in turn:

- (1) Suppose $x \in F$ is transitive. By Lemma 11.3.10 it would suffice to show that F^* is compatible (Definition 8.1.1(4)). So consider $y * F * y'$. Then $y * x * y'$ and by transitivity $y * y'$. Thus F^* is compatible.
- (2) It suffices to provide a counterexample. We take, as illustrated in Figure 8,
 - $P = \{-2, -1, 0, 1, 2\}$ and
 - we let Open be generated by $\{i, i+1\}$ for $-2 \leq i \leq 1$ (unordered pairs of adjacent numbers).

Write $A = \{-1, 0\}$ and $B = \{0, 1\}$ and let F be the up-closure of $\{A, B\}$. Note that A and B are not transitive (i.e. not topen). The reader can check that $F^* = F$ (e.g. $\{1, 2\} \notin F^*$ because $\{1, 2\} \not\subseteq \{-1, 0\} \in F$), but F contains no transitive element. $\square$

11.4 Semiframe characterisation of community

Remark 11.4.1. We saw the notion of $K(p)$ the community of a point in Definition 4.1.4(1). In this Subsection we construct an analogue to it in semiframes. We will give two characterisations: one in Definition 11.4.5, and another in Proposition 11.4.7.

We will mostly be interested in Definition 11.4.2 when F is a semifilter, but the definition does not require this:

Definition 11.4.2. Suppose $(X, \leq, *)$ is a semiframe and $F \subseteq X$ and $x \in X$. Then define $F^c \in X$, $F^{*c} \in X$, and $x^{*c} \in X$ by

$$F^c = \bigvee \{y \in X \mid y \notin F\}, \quad F^{*c} = (F^*)^c, \quad \text{and} \quad x^{*c} = (x^*)^c.$$

Remark 11.4.3. We unpack the definitions of F^{*c} and x^{*c} :

$$\begin{aligned} F^{*c} &= (F^*)^c && \text{Definition 11.4.2} \\ &= \bigvee \{y \in X \mid y \notin F^*\} && \text{Definition 11.4.2} \\ &= \bigvee \{y \in X \mid \neg(y * F)\} && \text{Definition 11.3.1} \\ \\ x^{*c} &= (x^*)^c && \text{Definition 11.4.2} \\ &= \bigvee \{y \in X \mid y \notin x^*\} && \text{Definition 11.4.2} \\ &= \bigvee \{y \in X \mid \neg(y * x)\}. && \text{Definition 11.2.1} \end{aligned}$$

Lemma 11.4.4 will be useful, and gives some intuition for $(-)^c$ and $(-)^{*c}$ by unpacking their concrete meaning in the special case of a semiframe of open sets of a semitopology:

Lemma 11.4.4. *Suppose (P, Open) is a semitopology and $p \in P$ and $O \in \text{Open}$. Then:*

- (1) $\text{nbhd}(p)^c = P \setminus |p|$.
- (2) $\text{nbhd}(p)^{*c} = P \setminus p_{\emptyset}$.
- (3) $O^{*c} = P \setminus |O| = \text{interior}(P \setminus O)$.

Proof. We consider each part in turn:

- (1) It is a fact of Definition 5.1.1 that $P \setminus |p| = \bigcup \{O' \in \text{Open} \mid p \notin O'\}$. By Proposition 9.2.2(2) $p \notin O'$ if and only if $O' \notin \text{nbhd}(p)$.
- (2) It is a fact of Definition 3.5.1, which is spelled out in Lemma 5.2.4(2), that $P \setminus p_{\emptyset} = \bigcup \{O' \in \text{Open} \mid \exists O \in \text{Open}. p \in O \wedge O' \not\ll O\}$. By Lemma 11.3.2 $\exists O \in \text{Open}. p \in O \wedge O' \not\ll O$ precisely when $O' \notin \text{nbhd}(p)^*$.
- (3) By Definitions 11.4.2 and 11.3.1 we have

$$O^c = (O^*)^{*c} = \bigcup \{O' \in \text{Open} \mid O' \notin O^*\} = \bigcup \{O' \in \text{Open} \mid O' \not\ll O\}$$

The result then follows by routine reasoning on closures (Definition 5.1.1). □

Definition 11.4.5. Suppose $(X, \leq, *)$ is a semiframe and $F \subseteq X$. Then define $k(F) \in X$ the **abstract community** of F by

$$k(F) = (F^{*c})^{*c} \in X.$$

(For a more direct characterisation, see Proposition 11.4.7.)

Proposition 11.4.6. *Suppose (P, Open) is a semitopology and $p \in P$. Then*

$$\text{knbhd}(p) = K(p).$$

In words: the abstract community of the abstract point $\text{nbhd}(p)$ in $(\text{Open}, \subseteq, \emptyset)$, is identical to the community of p .

Proof. We reason as follows:

$$\begin{aligned} \text{knbhd}(p) &= (\text{nbhd}(p)^{*c})^{*c} && \text{Definition 11.4.5} \\ &= (P \setminus p_{\emptyset})^{*c} && \text{Lemma 11.4.4(2)} \\ &= \text{interior}(P \setminus (P \setminus p_{\emptyset})) && \text{Lemma 11.4.4(3)} \\ &= \text{interior}(p_{\emptyset}) && \text{Fact of sets} \\ &= K(p) && \text{Definition 4.1.4(1)} \end{aligned}$$
□

We can also give a more direct characterisation of the abstract community from Definition 11.4.5:

Proposition 11.4.7. *Suppose $(X, \leq, *)$ is a semiframe and $F \subseteq X$. Then*

$$k(F) = \bigvee \{x \in X \mid x^* \subseteq F^*\},$$

and $k(F)$ is the greatest element in X such that $k(F)^ \subseteq F^*$.*

Proof. We follow the definitions:

$$\begin{aligned}
 (F^{*c})^{*c} &= \bigvee \{x \in X \mid \neg(x * F^{*c})\} && \text{Remark 11.4.3} \\
 &= \bigvee \{x \in X \mid \neg(x * \bigvee \{y \mid \neg(y * F)\})\} && \text{Remark 11.4.3} \\
 &= \bigvee \{x \in X \mid \neg \exists y \in X. (x * y \wedge \neg(y * F))\} && \text{Definition 7.2.1(3)} \\
 &= \bigvee \{x \in X \mid \forall y \in X. x * y \implies y * F\} && \text{Fact of logic} \\
 &= \bigvee \{x \in X \mid x^* \subseteq F^*\} && \text{Definitions 11.2.1 \& 11.3.1}
 \end{aligned}$$

To see that $k(F)$ is the greatest element such that $k(F)^* \subseteq F^*$, we note from Lemma 11.2.2 that

$$k(F)^* = \bigcup \{x^* \mid x \in X, x^* \subseteq F^*\}. \quad \square$$

11.5 Semiframe characterisation of regularity

We now have enough to generalise the notions of quasiregularity, weak regularity, and regularity from semitopologies (Definition 4.1.4 parts 5, 4, and 3) to semiframes:

Definition 11.5.1. Suppose $(X, \leq, *)$ is a semiframe and $F \subseteq X$ is a semifilter.

- (1) Call F **quasiregular** when $k(F) \neq \perp_X$.
Thus, there exists some $x \in X$ such that $x^* \subseteq F^*$.
- (2) Call F **weakly regular** when $k(F) \in F$.
- (3) Call F **regular** when $k(F) \in F$ and $k(F)$ is transitive.

Lemma 11.5.2 does for semiframes what Lemma 4.1.6 does for semitopologies:

Lemma 11.5.2. Suppose $(X, \leq, *)$ is a semiframe and $F \subseteq X$ is a semifilter. Then:

- (1) If F is regular then it is weakly regular.
- (2) If F is weakly regular then it is quasiregular.

(The converse implications need not hold, and it is possible for F to not be quasiregular: it is convenient to defer the proofs to Corollary 11.6.3.)

Proof. The proofs are easy: If $k(F) \in F$ and $k(F)$ is transitive, then certainly $k(F) \in F$. If $k(F) \in F$ then by Lemma 8.2.1(2) $k(F) \neq \perp_X$. $\square$

Lemma 11.5.3. Suppose $(X, \leq, *)$ is a semiframe and $F \subseteq X$ is a semifilter. Then:

- (1) If F is quasiregular and strongly compatible then $k(F)$ is transitive.
- (2) The converse implication need not hold: it is possible for F to be quasiregular and $k(F)$ to be transitive, yet F is not strongly compatible.
- (3) If F is weakly regular and $k(F)$ is transitive then F is strongly compatible.
- (4) If F is weakly regular, then $k(F)$ is transitive if and only if F is strongly compatible.

Proof. We consider each part in turn:

- (1) Suppose F is quasiregular and strongly compatible.

By quasiregularity $\perp_X \neq k(F)$. By Proposition 11.4.7 $k(F)^* \subseteq F^*$. By strong compatibility F^* is a semifilter and so in particular F^* is compatible. It follows from Proposition 11.2.4(1 and 4) that $k(F)$ is transitive, as required.

- (2) It suffices to provide a counterexample. Let $(\mathbb{R}, \text{Open})$ be the real numbers with their usual topology, and let $(\mathbb{R}, \text{Open}')$ be the topology generated by $\text{Open} \cup \{\{0\}\}$ – in words: we add $\{0\}$ as an open set.

Let F be the semifilter of all Open -open neighbourhoods of 0. F^* is the set of Open' -open sets that intersect every Open -open neighbourhood of 0. This is not compatible, because it contains $(0,)$ (the set of numbers strictly less than 0) and $(, 0)$ (the set of numbers strictly greater than 0), and these do not intersect. Using Proposition 11.4.7, we calculate that $k(F) = \{0\}$; this is transitive because it is a singleton set.

So F is quasiregular, $k(F)$ is transitive, yet F is not strongly compatible.

- (3) Suppose $k(F)$ is transitive and suppose F is weakly regular, so $k(F) \in F$. By Proposition 11.3.13 F is strongly compatible.
- (4) From parts 1 and 3 of this result, noting from Lemma 11.5.2 that if F is weakly regular then it is quasiregular. □

Theorem 11.5.4. *Suppose $(X, \leq, *)$ is a semiframe and $F \subseteq X$ is a semifilter. Then F is regular if and only if F is weakly regular and strongly compatible. We can write this succinctly as follows:*

Regular = weakly regular + strongly compatible.

(Compare this slogan with the version for semitopologies in Theorem 5.3.4.)

Proof. Suppose F is weakly regular and strongly compatible. By Lemma 11.5.3(4) $k(F)$ is transitive, and by Definition 11.5.1(3) F is regular.

For the converse implication we just reverse the reasoning above. □

Remark 11.5.5. In Theorem 6.3.3 we characterised regularity of points in terms of quasiregularity and being hypertransitive. In view of Lemma 11.3.8 we might expect Theorem 11.5.4 to read ‘regular = quasiregular + strongly compatible’. But this is false, as per the discussion in Remark 11.3.9 and the counterexample in Lemma 11.5.3(2). Thus, the semiframes results are subtly different from those governing point-set semitopologies.

11.6 Semiframe characterisation of (quasi/weak)regularity

The direct translation in Definition 11.5.1 of parts 5, 4, and 3 of Definition 4.1.4, along with the machinery we have now built, makes Lemma 11.6.1 easy to prove:

Lemma 11.6.1. *Suppose (P, Open) is a semitopology and $p \in P$. Recall from Definition 2.3.1 and Proposition 9.2.2(1) that $\text{nbhd}(p) = \{O \in \text{Open} \mid p \in O\}$ is a (completely prime) semifilter. Then:*

- (1) *p is quasiregular in the sense of Definition 4.1.4(5) if and only if $\text{nbhd}(p)$ is quasiregular in the sense of Definition 11.5.1(1).*
- (2) *p is weakly regular in the sense of Definition 4.1.4(4) if and only if $\text{nbhd}(p)$ is weakly regular in the sense of Definition 11.5.1(2).*
- (3) *p is regular in the sense of Definition 4.1.4(3) if and only if $\text{nbhd}(p)$ is regular in the sense of Definition 11.5.1(3).*

Proof. We consider each part in turn:

- (1) Suppose p is quasiregular. By Definition 4.1.4(5) $K(p) \neq \emptyset$. By Proposition 11.4.6 $knbhd(p) \neq \emptyset = \perp_{\text{Open}}$. By Definition 11.5.1(1) $nbhd(p)$ is quasiregular. The reverse implication follows just reversing the reasoning above.
- (2) Suppose p is weakly regular. By Definition 4.1.4(4) $p \in K(p)$. By Definition 2.3.1 $K(p) \in nbhd(p)$. By Proposition 11.4.6 $knbhd(p) \in nbhd(p)$ as required. The reverse implication follows just reversing the reasoning above.
- (3) Suppose p is regular. By Definition 4.1.4(3) $p \in K(p) \in \text{Topen}$. By Definition 2.3.1 and Proposition 11.4.6 $knbhd(p) \in nbhd(p)$. By Proposition 11.4.6 and Lemma 11.1.6 $knbhd(p)$ is transitive. The reverse implication follows just reversing the reasoning above. $\square$

Proposition 11.6.2. *Suppose (P, Open) is a semitopology and $p \in P$. Then*

- p is quasiregular / weakly regular / regular in (P, Open) in the sense of Definition 4.1.4 if and only if
- $nbhd(p)$ is quasiregular / weakly regular / regular in $\text{Soberify}(P, \text{Open})$ in the sense of Definition 11.5.1.

Proof. We consider just the case of regularity; quasiregularity and weak regularity are no different.

Suppose p is regular. By Definition 4.1.4(3) $p \in K(p) \in \text{Topen}$. It follows from Lemma 11.1.6 that $K(p)$ is transitive in $(\text{Open}, \subseteq, \breve{})$, and from Proposition 9.2.2(2) that $K(p) \in nbhd(p)$. It follows from Proposition 11.4.6 that $nbhd(p)$ is regular in the sense of Definition 11.5.1(3). $\square$

Corollary 11.6.3. *Suppose $(X, \leq, *)$ is a semiframe and $F \subseteq X$ is a semifilter. Then the converse implications in Lemma 11.5.2 need not hold: F may be quasiregular but not regular, and it may be weakly regular but not regular, and it may not even be quasiregular.*

Proof. It suffices to provide counterexamples. We easily obtain these by using Proposition 11.6.2 to consider $nbhd(p)$ for $p \in P$ as used in Lemma 4.1.6. $\square$

11.7 Characterisation of being intertwined

This Subsection continues Remark 11.1.3.

The notion of points being intertwined from Definition 3.5.1(1) generalises in semiframes to the notion of semifilters being compatible:

Lemma 11.7.1. *Suppose (P, Open) is a semitopology and $p, p' \in P$. Then*

$$p \breve{} p' \iff nbhd(p) * nbhd(p') \iff nbhd(p) \breve{} nbhd(p').$$

For clarity and precision we unpack this. The following are equivalent:

- (1) $p \breve{} p'$ in the semitopology (P, Open) (Definition 3.5.1(1)).
In words: the point p is intertwined with the point p' .
- (2) $nbhd(p) * nbhd(p')$ in the semiframe $(\text{Open}, \subseteq, \breve{})$ (Notation 11.1.2(3)).
In words: the abstract point $nbhd(p)$ is compatible with the abstract point $nbhd(p')$.
- (3) $nbhd(p) \breve{} nbhd(p')$ in the semitopology $\text{St}(\text{Open}, \subseteq, \breve{})$ (Definition 3.5.1(1)). In words: the point $nbhd(p)$ is intertwined with the point $nbhd(p')$.

Proof. We unpack definitions:

- By Definition 3.5.1(1) $p \not\sim p'$ when for every pair of open neighbourhoods $p \in O$ and $p' \in O'$ we have $O \not\sim O'$.
- By Notation 11.1.2(3) $nbhd(p) * nbhd(p')$ when for every $O \in nbhd(p)$ and $O' \in nbhd(p')$ we have $O * O'$.
By Proposition 9.2.2(2) we can simplify this to: $p \in O$ and $p' \in O'$ implies $O * O'$.
- By Definition 3.5.1(1) and Theorem 10.1.4, $nbhd(p) \not\sim nbhd(p')$ when: for every pair of open neighbourhoods $nbhd(p) \in Op(O)$ and $nbhd(p') \in Op(O')$ we have $Op(O) \not\sim Op(O')$.
By Proposition 9.2.2(2) we can simplify this to: $p \in O$ and $p' \in O'$ implies $Op(O) \not\sim Op(O')$.
By Proposition 8.3.3(2) we can simplify this further to: $p \in O$ and $p' \in O'$ implies $O * O'$.

But by definition, the compatibility relation $*$ of $(Open, \subseteq, \not\sim)$ is $\not\sim$, so $O * O'$ and $O \not\sim O'$ are the same assertion. The equivalences follow. $\square$

The property of being intertwined is preserved and reflected when we use $nbhd$ to map to the soberified space:

Corollary 11.7.2. *Suppose $(P, Open)$ is a semitopology and $p, p' \in P$. Then $p \not\sim p'$ in $(P, Open)$ if and only if $nbhd(p) \not\sim nbhd(p')$ in $Soberify(P, Open)$.*

Proof. This just reiterates the equivalence of parts 1 and 3 in Lemma 11.7.1. $\square$

Proposition 11.7.3. *Suppose $(P, Open)$ is a semitopology. Then:*

- (1) *It may be that $(P, Open)$ is unconflicted (meaning that it contains no conflicted points), but the semitopology $Soberify(P, Open)$ contains a conflicted point.*
- (2) *It may further be that $(P, Open)$ is unconflicted and $p \in P$ is such that $nbhd(p)$ is conflicted in the semitopology $Soberify(P, Open)$.*

We can summarise the two assertions above as follows:

- (1) *Soberifying a space might introduce a conflicted point, even if none was originally present.*
- (2) *Soberifying a space can make a point that was unconflicted, into a point that is conflicted.*³¹

Proof. It suffices to provide counterexamples.

- (1) Consider the right-hand semitopology in Figure 6; this is unconflicted because every point is intertwined only with itself. The soberification of this space is illustrated in the right-hand semitopology in Figure 7. Each of the extra points is intertwined with the two numbered points next to it; for example the extra point in the open set A – write it $\bullet_A$ (in-between 3 and 0) – is intertwined with 0 and 3; so $3 \not\sim \bullet_A \not\sim 0$. However, the reader can check that $3 \not\sim 0$. Thus, $\bullet_A$ is conflicted.
- (2) We define $(P, Open)$ by:
 - $P = (-1, 1)$ (real numbers between -1 and 1 exclusive).
 - $Open$ is generated by:
 - All open intervals that do not contain 0; so this is open intervals (r_1, r_2) where $-1 \leq r_1 < r_2 \leq 0$ or $0 \leq r_1 < r_2 \leq 1$.
 - All of the open intervals $(-1/n, 1/n)$, for $n \geq 2$.

The reader can check that:

- Points in this semitopology are intertwined only with themselves.
- The soberification includes four additional points, corresponding to completely prime semifilters $-1/0$ generated by $\{(-1/n, 0) \mid n \geq 2\}$ and $+1/0$ generated by $\{(0, 1/n) \mid n \geq 2\}$, and to the endpoints -1 and $i+1$.
- $-1/0$ and $+1/0$ are intertwined with 0 , but are not intertwined with one another.

Thus, 0 is conflicted in $\text{Soberify}(P, \text{Open})$ but not in (P, Open) . □

Remark 11.7.4. Proposition 11.7.3 may seem surprising in view of Corollary 11.7.2, but the key observation is that the soberified space may add points to the original space. These points can add conflicting behaviour that is ‘hidden’ in the completely prime semifilters of the original space.

Thus, Proposition 11.7.3 shows that the property of ‘being unconflicted’ cannot be characterised purely in terms of the semiframe of open sets – if it could be, then soberification would make no difference, by Theorem 10.1.4(3).

There is nothing wrong with that, except that we are interested in well-behavedness conditions on semiframes. We can now look for some other condition – but one having to do purely with open sets – that might play a similar role in the theory of (weak/quasi)regularity of semiframes, as being unconflicted does in theory of (weak/quasi)regularity of semitopologies.

We already saw a candidate for this in Theorem 11.5.4: strong compatibility. We examine this next.

11.8 Strong compatibility in semitopologies

Remark 11.8.1. Note that:

- (1) Theorem 11.5.4 characterises ‘regular’ for semiframes as ‘weakly regular + strongly compatible’.
- (2) Theorem 5.3.4 characterises ‘regular’ for semitopologies as ‘weakly regular + unconflicted’.

We know from results like Lemma 11.6.1 and Corollary 11.7.2 that there are accurate correspondences between notions of regularity in semiframes and semitopologies. This is by design, for example in Definition 11.5.1; we designed the semiframe definitions so that semiframe regularity and semitopological regularity would match up closely.

Yet there are differences too, since Theorem 11.5.4 uses strong compatibility, and Theorem 5.3.4 uses being unconflicted. What is the difference here, and why does it arise?

One answer is given by Proposition 11.7.3, which illustrates that the condition of ‘unconflicted’ (which comes from semitopologies) does not sit comfortably with the ‘pointless’ semiframe definitions. This raises the question of how strong compatibility (which comes from semiframes) translates into the context of semitopologies; and how this relates to being (un)conflicted?

We look into this now; see Remark 11.8.11 for a summary.

We can translate the notion of *strongly compatible filter* (Definition 11.3.6(1)) to semitopologies in the natural way, just applying it to the neighbourhood semifilter $\text{nbhd}(p)$ of a point (Definition 2.3.1):

Definition 11.8.2. Suppose (P, Open) is a semitopology. Then call $p \in P$ **strongly compatible** when the (by Example 8.1.4(1)) abstract point $\text{nbhd}(p)$ is strongly compatible (Definition 11.3.6) as a semifilter in $(\text{Open}, \subseteq, \cap)$.

We unpack what Definition 11.8.2 means concretely:

Lemma 11.8.3. *Suppose (P, Open) is a semitopology and $p \in P$. Then the following are equivalent:*

- (1) p is hypertransitive (Definition 6.2.3).
- (2) $\text{nbhd}(p)$ is strongly compatible.
- (3) $\text{nbhd}(p)^*$ is compatible.
- (4) For every $O', O'' \in \text{Open}$, if $O' * \text{nbhd}(p) * O''$ then $O' \not\leq O''$.

(Above, $O' * \text{nbhd}(p)$ follows Notation 11.1.2(1) and means that $O' \not\leq O$ for every $p \in O \in \text{Open}$, and similarly for $\text{nbhd}(p) * O''$.)

Proof. Equivalence of parts 1 and 2 is just Lemma 11.3.8. Equivalence of parts 2 and 3 is Definition 11.3.6(1). For the equivalence of parts 3 and 4, we just unpack what it means for $\text{nbhd}(p)^*$ to be compatible (see Remark 11.3.7). $\square$

' $p \in P$ is strongly compatible' is a strictly stronger condition than ' $p \in P$ is unconflicted':

Lemma 11.8.4. *Suppose (P, Open) is a semitopology and $p \in P$. Then:*

- (1) If p is strongly compatible then it is unconflicted.
- (2) If p is conflicted then: p is not strongly compatible, $\text{nbhd}(p)$ is not strongly compatible, and $\text{nbhd}(p)^*$ is not compatible.
- (3) The reverse implication need not hold, even if (P, Open) is sober:³² it is possible for p to be unconflicted but not strongly compatible.

Proof. We consider each part in turn:

- (1) Suppose p is strongly compatible and suppose $p' \not\leq p \not\leq p''$; we must show that $p' \not\leq p''$. Consider open neighbourhoods $p' \in O'$ and $p'' \in O''$. By assumption $p' \not\leq p$ and so by Lemma 11.7.1(1 and 2) $\text{nbhd}(p') * \text{nbhd}(p)$. Since $O' \in \text{nbhd}(p')$, it follows that $O' * \text{nbhd}(p)$, and similarly it follows that $\text{nbhd}(p) * O''$. Then by strong compatibility, $O' \not\leq O''$ as required.
- (2) We take the contrapositive of part 1 of this result, and use Lemma 11.8.3.
- (3) It suffices to provide a counterexample. Consider the bottom right semitopology in Figure 2, and take $p = *$ and $O' = \{1\}$ and $O'' = \{0, 2\}$. Note that:
 - $*$ is unconflicted, since it is intertwined only with itself and 1.
 - O' and O'' intersect every open neighbourhood of $*$, but $O' \not\leq O''$, so $*$ is not strongly compatible.

This space is sober: the only completely prime filters are the neighbourhood semifilters of $*$, 0, 1, and 2. $\square$

Example 11.8.5. Continuing Lemma 11.8.4(3), it is possible for a point to be strongly compatible (Definition 11.8.2) but not regular, or even quasiregular (Definition 4.1.4(3, 5)). Consider the right-hand semitopology illustrated in Figure 6 and take $p = 0$. The reader can check that p is strongly compatible, but it is not quasiregular (i.e. $K(p) = \emptyset$) and thus also not regular.

Lemma 11.8.6 shows that the situation outlined in Proposition 11.7.3(2) cannot arise if we work with a strongly compatible point instead of an unconflicted one . . .

Lemma 11.8.6. *Suppose (P, Open) is a semitopology and $p, p' \in P$. Then the following are equivalent:*

- (1) p is hypertransitive in (P, Open) .
- (2) $\text{nbhd}(p)$ is hypertransitive in $\text{Soberify}(P, \text{Open})$ (Notation 10.1.3).

Proof. Note that from Lemma 11.8.3, p is hypertransitive in (P, Open) when

$$(\forall O \in \text{Open}. p \in O \implies O' \checkmark O \checkmark O'') \text{ implies } O' \checkmark O''$$

for every $O', O'' \in \text{Open}$. Also, from Definition 8.4.3(2) and Lemma 11.8.3, $\text{nbhd}(p)$ is hypertransitive in $\text{Soberify}(P, \text{Open})$ when

$$(\forall O \in \text{Open}. \text{nbhd}(p) \in \text{Op}(O) \implies \text{Op}(O') \checkmark \text{Op}(O) \checkmark \text{Op}(O'')) \text{ implies } \text{Op}(O') \checkmark \text{Op}(O'')$$

for every $\text{Op}(O'), \text{Op}(O'') \in \text{Opens}(\text{Soberify}(P, \text{Open}))$.

Now by Proposition 9.2.2(2), $\text{nbhd}(p) \in \text{Op}(O)$ if and only if $p \in O$, and by Corollary 9.2.3 $\text{Op}(O') \checkmark \text{Op}(O)$ if and only if $O' \checkmark O$, and $\text{Op}(O) \checkmark \text{Op}(O'')$ if and only if $O \checkmark O''$. The result follows. $\square$

... but, the situation outlined in Proposition 11.7.3(1) can arise, indeed we use the same counterexample:

Lemma 11.8.7. *It may be that every point in (P, Open) is hypertransitive, yet $\text{Soberify}(P, \text{Open})$ contains a point that is not hypertransitive.*

Proof. The same counterexample as used in Proposition 11.7.3(1) illustrates a space (P, Open) such that every point in (P, Open) is hypertransitive, but $\text{Soberify}(P, \text{Open})$ contains a point that is not hypertransitive. We note that $\bullet_A$ (the extra point in-between 3 and 0) is not hypertransitive, because both B and D intersect with every open neighbourhood of $\bullet_A$, but B does not intersect with D . $\square$

The development above suggests that we define:

Definition 11.8.8. Call a semitopology (P, Open) **strongly compatible** when $(\text{Open}, \subseteq, \checkmark)$ is strongly compatible in the sense of Definition 11.3.6(2).

The proof of Proposition 11.8.9 is then very easy:

Proposition 11.8.9. *Suppose (P, Open) is a semitopology. Then the following are equivalent:*

- (1) (P, Open) is strongly compatible in the sense of Definition 11.8.8.
- (2) $\text{Soberify}(P, \text{Open})$ is strongly compatible in the sense of Definition 11.8.8.
- (3) $\text{Soberify}(P, \text{Open})$ is strongly compatible in the sense of Definition 11.3.6(2).
- (4) $\text{Soberify}(P, \text{Open})$ is hypertransitive in the sense of Definition 6.2.3.

Proof. We unpack Definition 11.8.8 and note that strong compatibility of (P, Open) is expressed purely as a property of its semiframe of open sets $(\text{Open}, \subseteq, \checkmark)$. By Theorem 10.1.4(3), the semiframe of open sets of $\text{Soberify}(P, \text{Open})$ is isomorphic to $(\text{Open}, \subseteq, \checkmark)$, via nbhd^{-1} . Equivalence of parts 1 and 2 follows.

By Notation 10.1.3 and Remark 10.1.2(3), the points of $\text{Soberify}(\mathbf{P}, \text{Open})$ are just abstract points of $(\text{Open}, \subseteq, \emptyset)$. Equivalence of parts 2 and 3 follows.

Equivalence of part 4 with the other parts follows using Lemmas 11.8.3 and 11.8.6. $\square$

Recall from Definition 4.1.4(7) that $(\mathbf{P}, \text{Open})$ being (weakly) regular means that every point in $(\mathbf{P}, \text{Open})$ is (weakly) regular. Recall from Definition 11.8.8 that $(\mathbf{P}, \text{Open})$ being strongly compatible means that $(\text{Open}, \subseteq, \emptyset) = \text{Fr}(\mathbf{P}, \text{Open})$ is strongly compatible in the sense of Definition 11.3.6(2). We can now prove an analogue of Theorems 5.3.4 and 11.5.4:

Theorem 11.8.10. *Suppose $(\mathbf{P}, \text{Open})$ is a semitopology and $p \in \mathbf{P}$. Then the following are equivalent:*

- (1) $(\mathbf{P}, \text{Open})$ is regular.
- (2) $(\mathbf{P}, \text{Open})$ is weakly regular and strongly compatible.

Proof. Suppose $(\mathbf{P}, \text{Open})$ is regular, meaning that every $p \in \mathbf{P}$ is regular.

By Theorems 5.3.4 and 6.3.3 every $p \in \mathbf{P}$ is weakly regular and hypertransitive. (So by Lemma 11.8.3 every $\text{nbhd}(p)$ is strongly compatible, and by Lemma 11.8.6 also hypertransitive.) The definition of weak regularity for a space in Definition 4.1.4(7) is pointwise, so it follows immediately that $(\mathbf{P}, \text{Open})$ is weakly regular.

But, the definition of strong compatibility for a space in Definition 11.8.8 is on its semiframe of open sets, which may include abstract points not only of the form $\text{nbhd}(p)$. It therefore does not follow immediately that $(\mathbf{P}, \text{Open})$ is strongly compatible; Lemma 11.8.7 contains a counterexample.

We can still prove that $(\mathbf{P}, \text{Open})$ is strongly compatible – but we need to do a bit more work.

Unpacking Definition 11.8.8, we must show that $(\text{Open}, \subseteq, \emptyset)$ is strongly compatible. Unpacking Definition 11.3.6(2), we must show that every abstract point in $(\text{Open}, \subseteq, \emptyset)$ is strongly compatible.

So consider an abstract point $P \subseteq \text{Open}$. By Corollary 4.3.3 $\mathbf{P}$ has a topen partition $\mathcal{T}$, which means that: every $T \in \mathcal{T}$ is topen; the elements of $\mathcal{T}$ are disjoint; and $\bigcup \mathcal{T} = \mathbf{P}$.

Now $\bigcup \mathcal{T} = \mathbf{P} \in T$ by Definition 8.1.1(7) and Lemma 8.2.1(1), so by Definition 8.1.1(2) there exists at least one (and in fact precisely one) $T \in \mathcal{T}$ such that $T \in P$. Now T is a transitive element in Open , so by Proposition 11.3.13 $P \subseteq \text{Open}$ is strongly compatible as required. $\square$

Remark 11.8.11. We summarise what we have seen:

- (1) The notions of (quasi/weak)regularity match up nicely between a semitopology, and its semiframe soberification (Proposition 11.6.2).
- (2) We saw in Proposition 11.7.3 that the notions of (un)conflicted point and unconflicted space from Definition 5.3.1(2) are not robust under forming soberification (Notation 10.1.3). From the point of view of a pointless methodology in semitopologies – in which we seek to understand a semitopology $(\mathbf{P}, \text{Open})$ starting from its semiframe structure $(\text{Open}, \subseteq, \emptyset)$ – this is a defect.
- (3) A pointwise notion of strong compatibility exists; by Lemma 11.8.3 it is actually hypertransitivity from Definition 6.2.3. This is preserved pointwise by soberification (Lemma 11.8.6), but soberification can still introduce *extra* points, and it turns out that the property of a space being pointwise hypertransitive is still not robust under soberification because the extra points need not necessarily be hypertransitive; see Lemma 11.8.7.
- (4) This motivates the notion of a *strongly compatible* semitopology from Definition 11.8.8; and then Proposition 11.8.9 becomes easy.

Our larger point (no pun intended) is that the Definition and its corresponding Proposition are natural, *and also* that the other design decisions are less natural, as noted above. Perhaps somewhat unexpectedly, ‘regular = weakly regular + strongly compatible’ then works pointwise *and* for the entire space; see Theorem 11.8.10. Thus Definition 11.8.8 has good properties and is natural from a pointless/semiframe/open sets perspective.

12. Conclusions and Related and Future Work

12.1 Topology versus semitopology

We briefly compare and contrast topology/frames and semitopology/semiframes. This list is not exhaustive but we hope it will give a feel for how the two differ:

- (1) *Topology*: We are typically interested in spaces with separation axioms.³³
Semitopology: We are interested in guaranteeing agreement between participants of a distributed system, and this is all about *anti-separation* properties of their actionable coalitions. Semifilters have a compatibility condition (Definition 8.1.1(4)); and regularity, being intertwined, and being unconflicted or strongly compatible are anti-separation properties (see Definitions 3.5.1, 4.1.4, 5.3.1, and 11.3.6, and Remark 3.5.7).³⁴
- (2) *Topology*: If a minimal open neighbourhood of a point exists then it is least, because we can intersect two minimal neighbourhoods to get a smaller one which by minimality is equal to both. A finite filter has a least element.
Semitopology: A point may have multiple minimal open neighbourhoods – examples are very easy to generate, see for example the top-right example in Figure 2. A finite semifilter need not have a least element (see Remark 8.1.6).
- (3) *Topology*: Every finite T_0 topology is sober. A topology is sober if and only if every nonempty irreducible closed set is the closure of a unique point.
Semitopology: Neither property holds. See Lemma 9.3.8.
- (4) Semitopological questions such as ‘*is this a topen set*’ or ‘*are these two points intertwined*’ or ‘*does this point have a topen neighbourhood*’ – and many other definitions in this paper, such as our taxonomy of points into *regular*, *weakly regular*, *quasiregular*, *conflicted*, and *strongly compatible* are novel and/or play a larger role in the theory than they typically do in topology.

12.2 Related work

Dualities

We discussed duality results in detail in Remark 10.4.1. The reader may know that there are many such results, starting with Stone’s classic duality between Boolean algebras and compact Hausdorff spaces with a basis of clopen sets (Stone, 1936; Johnstone, 1986). The duality between frames and topologies is described in Mac Lane and Moerdijk (1992, p. 479, Corollary 4). See also the encyclopaedic treatment in Caramello (2011), with an overview in Example 2.9 on page 17. Our duality between semiframes and semitopologies fits into this canon.

Union sets, closure spaces, and minimal structures

There is a thread of research into *union-closed families*; these are subsets of a finite powerset closed under unions, so that a union-closed families is precisely just a finite semitopology. The motivation is to study the combinatorics of finite subsemilattices of a powerset. Some progress has been made in this Poonen (1992); the canonical reference for the relevant combinatorial conjectures is the ‘problem session’ on page 525 (conjectures 1.9, 1.9’, and 1.9’’) of Rival (1985). See also recent

progress in a conjecture about union-closed families.³⁵ There is no direct connection to semitopologies, and certainly no consideration of duality results. Perhaps the duality in this paper may be of some interest in that community.

A *closure space* is a subset of a powerset that is closed under intersections (Erné, 2009, p.173). Up to taking sets complements, a closure space is a semitopology, and likewise a finite closure space is, up to taking sets complements, a union-closed family. The motivation for closure spaces is to study closure operations in a topology-flavoured style, so closure spaces (unlike union-closed sets) share a topological flavour with semitopologies. However the applications are completely different, and algebra reveals how the underlying structures are also mathematically distinct, as is made clear by a lattice-based presentation of closure spaces and morphisms (Erné, 2009, Section 2.4), which uses a structure of ‘based lattices’ (essentially: lattices plus a generating basis): this differs from semitopologies with their semiframe presentation as compatible complete semilattices.³⁶

A *minimal structure* on a set X is a subset of $\text{pow}(X)$ that contains $\emptyset$ and X . Thus a semitopology is a minimal structure that is also closed under arbitrary unions. There is a thread of research into minimal structures, studying how notions familiar from topology (such as continuity) fare in weak (minimal) settings (Popa and Noiri, 2001) and how this changes as axioms (such as closure under unions) are added or removed. An accessible discussion is in Száz (2007), and see the brief but comprehensive references in Remark 3.7 of that paper. Of course our focus is on properties of semitopologies which are not considered in that literature; but we share an observation with minimal structures that it is useful to study topology-like constructs, in the absence of closure under intersections.

Algebraic topology as applied to distributed computing tasks

The reader may know that solvability results about distributed computing tasks have been obtained from algebraic topology, starting with the impossibility of k -set consensus and the Asynchronous Computability Theorem (Herlihy and Shavit, 1993; Borowsky and Gafni, 1993; Saks and Zaharoglou, 2000) in 1993. See Herlihy et al. (2013) for numerous such results.

The basic observation is that states of a distributed algorithm form a simplicial complex, called its *protocol complex*, and topological properties of this complex, like connectivity, are constrained by the underlying communication and fault model. These topological properties in turn can determine what tasks are solvable. For example: every algorithm in the wait-free model with atomic read-write registers has a connected protocol complex, and because the consensus task’s output complex is disconnected, consensus in this model is not solvable (Herlihy et al., 2013, Chapter 4).

This paper is also topological, but in a different way: we use (semi)topologies to study consensus in and of itself, rather than the solvability of consensus or other tasks in particular computation models. Put another way: the papers cited above use topology to study the solvability of distributed tasks, but this paper shows how the very idea of ‘distribution’ can be viewed as having a semitopological foundation.

Of course we can imagine that these might be combined – that in future work we may find interesting and useful things to say about the topologies of distributed algorithms when viewed as algorithms *on* and *in* a semitopology.

Fail-prone systems and quorum systems

Given a set of processes P , a *fail-prone system* (Malkhi and Reiter, 1998) (or *adversary structure* (Hirt and Maurer, 2000)) is a set of *fail-prone sets* $\mathcal{F} = \{F_1, \dots, F_n\}$ where, for every $1 \leq i \leq n$, $F_i \subseteq P$. $\mathcal{F}$ denotes the assumptions that the set of processes that will fail (potentially maliciously) is a subset of one of the fail-prone sets. A *dissemination quorum system* for $\mathcal{F}$ is a set $\{Q_1, \dots, Q_m\}$ of quorums where, for every $1 \leq i \leq m$, $Q_i \subseteq P$, and such that

- for every two quorums Q and Q' and for every fail-prone set F , $(Q \cap Q') \setminus F \neq \emptyset$ and

- for every fail-prone set F , there exists a quorum disjoint from F .

Several distributed algorithms, such as Bracha Broadcast (Bracha, 1987) and PBFT Castro and Liskov, (2002), rely on a quorum system for a fail-prone system $\mathcal{F}$ in order to solve problems such as reliable broadcast and consensus assuming (at least) that the assumptions denoted by $\mathcal{F}$ are satisfied.

Several recent works generalise the fail-prone system model to heterogeneous systems. Under the failure assumptions of a traditional fail-prone system, Bezerra et al. (2022) study reliable broadcast when participants each have their own set of quorums. Asymmetric Fail-Prone Systems (Alpos et al., 2024) generalise fail-prone systems to allow participants to make different failure assumption and have different quorums. In Permissionless Fail-Prone Systems (Cachin et al., 2023), participants not only make assumptions about failures, but also make assumptions about the assumptions of other processes; the resulting structure seems closely related to witness semitopologies, but the exact relationship still needs to be elucidated.

Federated Byzantine Agreement Systems (Mazières, 2015) are an instance of semitopologies. García-Pérez and Gotsman (2018) rigorously prove the correctness of broadcast abstractions in Stellar's Federated Byzantine Agreement model and investigate the model's relationship to dissemination quorum systems. The Personal Byzantine Quorum System model (Losa et al., 2019) is an abstraction of Stellar's Federated Byzantine Agreement System model and accounts for the existence of disjoint consensus clusters (in the terminology of the paper) which can each stay in agreement internally but may disagree between each other. Consensus clusters are closely related to the notion of *topen* in Definition 3.2.2(2).

Sheff et al., study heterogeneous consensus in a model called Learner Graphs (Sheff et al., 2021) and propose a consensus algorithm called Heterogeneous Paxos.

Cobalt, the Stellar Consensus Protocol, Heterogeneous Paxos, and the Ripple Consensus Algorithm (Schwartz et al., 2014; Mazières, 2015; MacBrough, 2018; Sheff et al., 2021) are consensus algorithms that rely on heterogeneous quorums or variants thereof. The Stellar network (Lokhava et al., 2019) and the XRP Ledger (Schwartz et al., 2014) are two global payment networks that use heterogeneous quorums to achieve consensus among an open set of participants; the Stellar network is an instance of a witness semitopology.

The literature on fail-prone systems and quorum systems is most interested in synchronisation algorithms for distributed systems and has been less concerned with their deeper mathematical structure. Some work by the second author and others (Losa et al., 2019) gets as far as proving an analogue to Proposition 3.4.4 (though we think it is fair to say that the presentation in this paper is simpler and clearer), but it fails to notice the connection with topology and the subsequent results which we present in this paper, and there is no consideration of algebra as used in this paper.

(Semi)lattices with extra structure

I am not aware of semiframes having been studied in the literature, but they are in excellent company, in the sense that things have been studied that are structurally similar. We mention two examples to give a flavour of this extensive literature:

- (1) A **quantale** is a complete lattice $(Q, \vee)$ with an associative *multiplication* operation $*$: $(Q \times Q) \rightarrow Q$ that distributes over $\vee$ in both arguments (Rosenthal, 1990). A commutative quantale whose multiplication is restricted to map to either the top or bottom element in Q is close being a semiframe.³⁷ For reference, a pleasingly simple representation result for quantales is given in Brown and Gurr (1993).
- (2) An **overlap algebra** is a complete Heyting algebra X with an *overlap relation* $\bowtie \subseteq X \times X$ whose intuition is that $x \bowtie y$ when $x \wedge y$ is *inhabited*. The motivation for this comes from constructive logic, in which $\exists p.(p \in x \wedge p \in y)$ is a different and stronger statement than

$\neg\forall p.\neg(p \in x \wedge p \in y)$. Accordingly, overlap algebras are described as ‘a constructive look at Boolean algebras’ (Ciraulo and Contente, 2020). Overlap algebras are not semiframes, but they share an idea with semiframes in making a structural distinction between ‘intersect’ and ‘have a non-empty join’.

12.3 Future work

The list of future work below is not exhaustive, of course. If this list inspires, through some omission that is obvious to the reader, a new idea, then that is best of all: finding good questions is the first and arguably most important step in research.

Remark 12.3.1 (Other notions of morphism). In Definition 10.1.1(1) we take a morphism of semitopologies $f : (P, \text{Open}) \rightarrow (P', \text{Open}')$ to be a continuous function $f : P \rightarrow P'$. Correspondingly, in Definition 10.2.1(1) we take a morphism of semiframes $g : (X', \leq', *) \rightarrow (X, \leq, *)$ to be a compatible morphism of complete semilattices.

The reader may be familiar with conditions on maps between topologies other than continuity, such as being open (f maps open sets to open sets) and closed (f maps closed sets to closed sets). These also make sense in semitopologies.

A further natural design space in the semitopological case is to include conditions on set intersections and strict inclusions. We briefly list some conditions that we could impose on $f : P \rightarrow P'$:

- (1) If $O \tilde{\cap} O'$ then $f^{-1}(O) \tilde{\cap} f^{-1}(O')$ (it is automatic that if $f^{-1}(O) \tilde{\cap} f^{-1}(O')$ then $O \tilde{\cap} O'$, but the reverse implication is a distinct condition).
- (2) If $O \subsetneq O'$ then $f^{-1}(O) \subsetneq f^{-1}(O')$.
- (3) $O \tilde{\cap} O'$ implies $f^{-1}(O) \subseteq f^{-1}(O')$ implies $O \subseteq O'$, or an equivalent contrapositive: $O \tilde{\cap} O'$ implies $O \not\subseteq O'$ implies $f^{-1}(O) \not\subseteq f^{-1}(O')$.³⁸

Remark 12.3.2 (Representations). The Sierpiński space $Sk = (P, \text{Open})$ sets $P = \{0, 1\}$ and $\text{Open} = \{\emptyset, \{1\}, \{0, 1\}\}$. This is both a topology and a semitopology, and it is a classifying space for open sets, in the sense that $\text{Hom}(-, Sk) : \text{SemiTop} \rightarrow \text{Set}$ is naturally isomorphic to $\text{Opns} : \text{SemiTop} \rightarrow \text{Set}$ which maps (P, Open) to Open .

Sk does not classify semitopologies, because semiframes suggest that we should view the set of open sets Open of a semitopology as a semiframe structure, having a subset inclusion (of course) and also a generalised intersection $\tilde{\cap}$. A likely classifying space for this is the top-left example in Figure 2, such that

- $P = \{0, 1, 2\}$ and
- $\text{Open} = \{\emptyset, \{0\}, \{2\}, \{0, 2\}, \{0, 1, 2\}\}$.

With Sk in mind, this space looks like two copies of Sk glued end-to-end – that is like two open sets – where the $1 \in P$ represents where they might intersect. Call this space **3**.

3 suggests that a logic for semiframes might be naturally three-valued, with values **t**, **f**, and also **b** for ‘is in the intersection’, and that the spaces $\text{Hom}(-, \mathbf{3})$ of continuous mappings to **3** might play a useful role. This might also have some impact on suitable notions of exponential space (see Remark 12.3.3). Current research (Gabbay and Zanolini, 2025) considers applying three-valued modal logics based on semitopologies to reason explicitly about decentralised algorithms, like Paxos (Lamport, 2001).

Remark 12.3.3 (Exponential spaces). It remains to check whether the category *SemiFrame* of semiframes is closed (Mac Lane, 1971, p. 180, Section VII.7), or Cartesian.³⁹

It remains to look into the Vietoris (also called the exponential) semitopologies (Engelking, 1989, Exercise 2.7.20, p. 120), though in view of Remark 12.3.2 it is also an open question what the most suitable notion of exponential would be – we speculate that exponentials over **3** would be suitable.

More generally, it remains to study functors of the form $Hom(-, B)$ and $Hom(A, -)$, for different values of A and B .

Remark 12.3.4 (Finiteness and compactness). The relation of semitopologies to finiteness is interesting. On the one hand, our motivating examples – distributed networks – are finite because they exist in the real world. On the other hand, in distributed networks, precisely because they are distributed, participants may not be able to depend on an exhaustive search of the full network being practical (or even permitted – this could be interpreted as a waste of resources or even as hostile or dangerous).

This requires mathematical models and algorithms that make sense on at least countably infinitely many points.⁴⁰

In fact, arguably even ‘countably large’ is not quite right. The natural cardinality for semitopologies may be uncountable, since network latency means that we cannot even enumerate the network: no matter how carefully we count, we could always in principle discover new participants who have joined in the past (but we just had not heard of them yet).

This motivates future work in which we consider algebraic conditions on a semiframe $(X, \leq, *)$ that mimic some of the properties of open sets of finite semitopologies (without necessarily insisting on finiteness itself). For instance:

- (1) We could insist that a $\leq$ -descending chain of non- $\perp_X$ elements in X have a non- $\perp_X$ greatest lower bound in X .
- (2) We could insist that a $\leq$ -descending chain of elements strictly $\leq$ -greater than some $x \in X$ have a greatest lower bound that is strictly $\leq$ -greater than x .
- (3) We could insist that if $(x_i \mid i \geq 0)$ and $(y_i \mid i \geq 0)$ are two $\leq$ -descending chains of elements, and $x_i * y_i$ for every $i \geq 0$ – in words: x_i is compatible with y_i – then the greatest lower bounds of the two chains are compatible.

The reader may notice how these conditions are reminiscent of compactness conditions from topology: for example a metric space is compact if and only if every descending chain of open sets has a nonempty intersection. This is no coincidence, since one of the uses of compactness in topology is precisely to recover some of the characteristics of finite topologies.

Considering semiframes (and indeed semitopologies) with compactness/finiteness flavoured conditions, is future work.

Remark 12.3.5 (Computational/logical behaviour). The original motivation for semiframes comes from semitopologies, which are motivated by blockchains and other decentralised systems, which are often real software artefacts that do real computation. It might therefore be useful to think about ‘computable’ semiframes, whatever this should mean. Gabbay (2024) considers *witness semitopologies* as a model of computably tractable semitopologies (being a witness semitopology is a concrete condition on sets), and we propose an algebraic abstraction of these as (strongly) *chain-complete* semitopologies; a semitopology is (strongly) chain-complete when any infinite descending chain of (nonempty) open sets has a (nonempty) open intersection. we argue that being chain-complete is an appropriate generalisation of the familiar Alexandrov condition on topologies.

This observation, along with Remark 12.3.4, suggests that it would be interesting to look at semiframes with additional conditions having to do with limits of possibly infinite descending chains.

Remark 12.3.6 (Generalising $*$). In Remark 7.2.2 we mentioned that we can think of semitopologies not as ‘*topologies without intersections*’ so much as ‘*topologies with a generalised intersection*’. In this paper we have studied a relation called $\tilde{\cap}$ (for point-set semitopologies) and $*$ (for semiframes), which intuitively measure whether two elements intersect. But really, this is just a notion of generalised meet, which suggests:

- We would take $(X, \leq)$ and $(X', \leq')$ to be complete join-semilattices and the generalised meet $*$: $(X \times X) \rightarrow X'$ is any commutative distributive map.
- We could generalise in a different direction and consider (e.g.,) cocomplete symmetric monoidal categories: $*$ becomes the (symmetric) monoid action.
- We could generalise from binary intersections to n -ary intersections. This generalisation deserves its own brief discussion:

The literature on decentralised systems is rich in intersection conditions. A Q^2 property from (Damgård et al., 2007, Theorem 1) asserts (in semitopological language) that all pairs of nonempty open set intersect – but there is more, for example, a Q^3 property (Alpos et al., 2024, Definition 2) asserts that all *triples* of nonempty open set intersect. Other unpublished work considers situations where up to *five* nonempty open sets should intersect. From the point of view of this paper, it suggests a sequence of n -ary generalisations of $\tilde{\cap}$ and $*$.

Furthermore, a B^3 condition (Damgård et al., 2007, Theorem 2) (Alpos et al., 2024, Definition 5), considers a situation of even greater generality (and worse behaviour!), where open neighbourhoods are local to points in the sense that if $O \in nbhd(p)$ and $p' \in O$ then it does not necessarily follow that $O \in nbhd(p')$. This suggests ‘heterogeneous’ generalisations of semitopology and semiframe whose mathematical behaviour is at time of writing unexplored.

Remark 12.3.7 (Homotopy and convergence). We have not looked in any detail at notions of path and convergence in semitopologies and semiframes. Preliminary considerations suggest that, in the finite case, a path converging to a point can be identified with a minimal open set containing that point. In the topological case this is trivial, since minimal open sets are least; in the semitopological case it is not, since there may be many minimal open sets (Lemma 2.1.2). Thus, even in a semitopology with no ‘holes’, it can be possible to approach a single point from multiple directions. Developing this part of the theory is future work.

Remark 12.3.8 (Constructive mathematics). We have not considered what semiframes would look like in a constructive setting. Much of the interest in frames and locales (vs. point-set topologies) comes from working in a constructive setting; for example, in the topos of sheaves over a base space, locales give a good fibrewise topology of bundles. To what extent similar structures might be built using semiframes, or what other structures might emerge instead, are currently entirely open questions.

Remark 12.3.9 (Semiframes, logic and analysis of real systems). A natural application of semiframes is to help build logics for consensus, and in particular logics for consensus *algorithms* (like Paxos Lamport (2001)). A declarative approach to specifying consensus algorithms, based on the semitopological ideas in this paper, is current research recently submitted for publication (Gabbay and Zanolini, 2025). In separate work current at the time of writing, these techniques have helped find errors in a real consensus algorithm; details are to be published in due course.

That work illustrates how the ideas in this paper can be made actionable to analyse properties of real consensus algorithms.

12.4 Final comments

Innovation in decentralised systems design is currently very active, and in particular there is much work on notions of consensus and quorum (García-Pérez and Gotsman, 2018; Lokhava et al., 2019; Losa et al., 2019; Sheff et al., 2021; Bezerra et al., 2022; Florian et al., 2022; Cachin et al., 2023; Li et al., 2023; Li and Lesani, 2023; Alpos et al., 2024), and many new systems (Schwartz et al., 2014; Mazières, 2015; MacBrough, 2018; Sheff et al., 2021) (and this is just a sample of an active literature). This explosion of practical design is also an explosion of suggestions to the theory community, of new structures to consider. One way to do this is to apply a classic technique which has been usefully applied many times before: *topologise, then dualise*.

Of course, practitioners are also invited to learn more topology and algebra and consider phrasing some aspects of their systems in these terms.⁴¹ If there is one thing holding back practical progress in decentralised algorithm design, it is that it is exceptionally hard. If (semi)topology, algebra, and logic can help, then that will be a useful contribution.

Acknowledgements. I dedicate this paper to the memory of Phil Scott: colleague, mentor, gentleman, friend, and inspiration to us all. I am extremely grateful to the anonymous referees for feedback on this work. Thanks to Giuliano Losa and the Stellar Development Foundation for their generous support and funding in pursuing this research.

Notes

1 It is actually much worse than this: we may also have to worry about messages being lost or delayed and about some participants being faulty and not following algorithms and protocols correctly. But even in the ideal case where communication is reliable and all participants are correct, decentralised algorithms are *still* hard.

2 Pedant point: some blockchains have a central control, but this is a special case; most are fully decentralised.

3 The empty set has no resources but is still an actionable coalition because it has no members, so all of those members can act as they like.

4 How to design an algorithm that can run on a decentralised system and get coherent and sensible behaviour from the overall system, even though any actionable coalition can (within the rules of the algorithm) potentially agree on some action and carry it out, without waiting for participants not in that coalition to participate, approve, or even necessarily know that an action has been taken – is *the* challenge of designing decentralised algorithms. It is not easy! Explaining the magic of practical decentralised algorithms is out of scope for this paper, but it *can be done*, and notions of actionable coalition are often key to such algorithms. The interested reader can find more information in Lamport (2001) and Cachin et al. (2011).

5 Our ants, being mathematical idealisations, are massless and occupy zero volume. However, even if the ants had mass and volume, some of them could just stand around drinking the ant equivalent of tea and encouraging the ones who are actually doing the heavy lifting; the mathematics of actionable coalitions makes no practical judgments about which participants are doing useful work in the actual implementation.

6 There is a little overlap between this clause and the first one: if $X = \emptyset$, then by convention $\bigcup X = \emptyset$. Thus, $\emptyset \in \text{Open}(P)$ follows from both clause 1 and clause 2. If desired, the reader can just remove the condition $\emptyset \in \text{Open}(P)$ from clause 1, and no harm would come of it.

7 **Affirmation:** Something declared to be true; a positive statement or judgment (<https://www.wordnik.com/words/affirmation>, permalink: <https://web.archive.org/web/20230608073651/https://www.wordnik.com/words/affirmation>).

8 The other main classification is **conflicted** points, in Definition 5.3.1. These properties are connected by an equation: regular = weakly regular + conflicted; see Theorem 5.3.4.

9 According to Wikipedia, this space is also called indiscrete, anti-discrete, concrete, and codiscrete (https://en.wikipedia.org/wiki/Trivial_topology).

10 The empty set is trivially transitive and open, so it would make sense to admit it as a (degenerate) topen. However, it turns out that we mostly need the notion of ‘topen’ to refer to certain kinds of neighbourhoods of points (we will call them communities; see Definition 4.1.4). It is therefore convenient to exclude the empty set from being topen, because while it is the neighbourhood of every point that it contains, it is not a neighbourhood of any point.

11 ‘Transitive open’ $\rightarrow$ ‘topen’, like ‘closed and open’ $\rightarrow$ ‘clopen’.

- 12 Following Definition 2.1.3 and Example 2.1.4(3), this is just $\{0, 1\}$ with the discrete semitopology.
- 13 See also Theorem 11.5.4, which does something similar for semiframes.
- 14 Lemma 6.2.2 packages this argument up nicely with some slick notation, which we have not yet set up.
- 15 ‘Properly reflexive’ is a loose riff on terminologies like ‘proper subset of’ or ‘proper ideal of a ring’. We might also call this ‘non- $\perp$ reflexive’, which is descriptive but perhaps a bit of a mouthful.
- 16 Just being a complete lattice is not enough; it has to be distributive as well. Consider $\omega+1 = \mathbb{N} \cup \{\omega\}$ with its usual ordering, augmented with an element d such that $0 \leq d \leq \omega$. Then $\omega = \bigvee \mathbb{N}$ and $d * \omega$, but $\neg(d * n)$ for every $n \in \mathbb{N}$.
- 17 Recall that a proper filter is a filter that does not contain $\perp$.
- 18 This is carefully worded. We care about *abstracting* properties of finite semitopologies, but we should not restrict to considering *only* semitopologies and semiframes that are actually finite! See Remark 12.3.4.
- 19 A proper filter is a filter that does not contain $\perp$. A maximal filter is a filter that is maximal amongst proper filters.
- 20 A succinct proof is in Wikipedia (2024b).
- 21 Unpacking what that means, we obtain this: $x \neq \perp \wedge x = x'$ or $x \neq \perp \wedge x' = \top$ or $x' \neq \perp \wedge x = \top$. This definition for $*$ is what we need for our counterexample, but other choices for $*$ also yield valid semiframes. For example, we can set $x * x'$ when $x, x' \neq \perp$.
- 22 See also a discussion of the design of the notion of semifilter in Remarks 8.1.6 and 10.3.4.
- 23 There are no guarantees in general about how many abstract points exist; e.g., Lemma 8.2.7 gives an example of a semiframe that has no abstract points at all and so maps to the empty semitopology. Later on in Definition 9.1.2 we consider conditions to ensure the existence of abstract points.
- 24 More on this topic later on in Definition 10.2.1, when we build the category of semiframes.
- 25 ‘Unnecessary’ and ‘enough’ are in scare quotes here because these are subjective terms. For example, if points represent computer servers on a network, then we might consider it a *feature* to not be T_0 by having multiple points that are topologically indistinguishable – e.g., for backup or to reduce latency – and likewise, we might consider it a feature to not have one concrete point for every abstract point if this avoids redundancies. There is no contradiction here: a computer network based on a small non-sober space with multiple backups of what it has may be a more efficient and reliable system than one based on a larger non-sober space that does not back up its servers but is full of redundant points. And this smaller, non-sober space may present itself to the user abstractly as the larger, sober space.
- Users may even forget about the computation that goes on under the hood of this abstraction, as illustrated by the following *true story*: I had a paper presenting an efficient algorithm rejected because it ‘lacked motivation’. Why? Because the algorithm was unnecessary: the reviewer claimed, apparently with a straight face, that guessing the answer until you got it right was computationally equivalent.
- 26 A discussion of possible alternatives for future work is in Remark 12.3.1. See also Remarks 8.1.3 and 8.1.6.
- 27 The Wikipedia page is also exceptionally clear (Wikipedia, 2024a).
- 28 This is not a necessary condition; *non-adjoint equivalences* are possible. But the duality in this paper comes from an adjoint equivalence.
- 29 *Confusing terminology alert*: Definition 3.2.2(1) also has a notion of *transitive set*. The notion of transitive set is well-defined for a set that may not be open. In the world of semiframes, we just have elements of the semiframe (which correspond, intuitively, to open sets). Thus, *transitive* semiframe elements correspond to (nonempty) transitive open sets of a semitopology, which are called *topens*.
- 30 1 is also a *conflicted* point; see Example 5.3.2(1). This is no accident: by Lemma 6.3.1(2), if p is conflicted, then it is not hypertransitive, and by Lemma 11.3.8, it follows that $nbhd(p)^*$ is not compatible.
- 31 If we stretch the English language, we might say that soberifying a space can conflictify one of its points.
- 32 ... meaning that every abstract point in $(\text{Open}, \subseteq, \check{\ })$ is the neighbourhood semifilter of a unique concrete point in P .
- 33 The Wikipedia page on separation axioms (Wikipedia, 2024c) includes an excellent overview with over a dozen separation axioms. No anti-separation axioms are discussed.
- 34 An extra word on this: our theory of semitopologies admits spaces whose points partition into distinct communities. Surely it *must be bad* if not all points need to be in consensus in a final state?
- Not at all: for example, most blockchains have a *mainnet* and several *testnets*, and it is understood that each should be coherent within itself, but different nets *need not* be in consensus with one another – indeed, if the mainnet had to agree with a testnet, then this would likely be a bug, not a feature. So the idea of a single space with multiple partitions of consensus is not a new idea; it is an old idea, which we frame in a new, fruitful, and more general way.
- 35 https://web.archive.org/web/20230330170701/https://en.wikipedia.org/wiki/Union-closed_sets_conjecture#Partial_results.
- 36 There are other differences too. For example, in a semitopology, the closure of the empty set is empty; in a closure space, it need not be. This is because the ‘closure’ in a closure space is intended in a subtly different sense, which is intended to model things that include ‘deductive closure of’, and the deductive closure of the empty set is the set of tautologies, which might be nonempty.
- 37 But not quite! We also need proper reflexivity (Definition 7.2.1(2)), and quantale morphisms do not necessarily map the top element to the top element like semiframe morphisms should (Definitions 7.1.2 and 10.2.1(1)).

38 This condition is motivated by a study of graph representations of semitopologies. See Gabbay (2024, Chapter 17).

39 It would be surprising if it were not Cartesian. The category of semitopologies is Cartesian.

40 This is no different than a programming language including a data type of arbitrary precision integers: the programme must eventually terminate, but because we do not know when, we need the *idea* of an infinity in the language.

41 A start on this is in Gabbay and Zanolini (2025). True story: I attended a talk on a novel distributed consensus algorithm which featured an interesting five-way quorum intersection condition. I said to the speaker after the talk, ‘So consider each quorum as a proposition: what are the five properties that your algorithm needs to guarantee, in the sense that they are always true at some point?’ – a natural question, given my background in logic. The speaker, who is no fool and is an expert in practical systems, had not thought of things in this way and did not know how to answer. I suggested that until this question was answered, he didn’t really understand his algorithm. Of course I tried to be polite, tactful and constructive when I said this, but it’s true.

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