$L^p$ BOUNDS FOR NONISOTROPIC MARCINKIEWICZ INTEGRALS ASSOCIATED TO SURFACES

FENG LIU and SUZHEN MAO

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Abstract

In an extrapolation argument, we prove certain $L^p$ $(1 < p < \infty)$ estimates for nonisotropic Marcinkiewicz operators associated to surfaces under the integral kernels given by the elliptic sphere functions $\Omega \in L(\log^+ L)^{\alpha}(\Sigma)$ and the radial function $h \in N_0(\mathbb{R}^n)$. As applications, the corresponding results for parametric Marcinkiewicz integral operators related to area integrals and Littlewood–Paley $g^*_\lambda$-functions are given.

Keywords and phrases: nonisotropic dilations, Marcinkiewicz integrals, rough kernels, extrapolation.

1. Introduction

As is well known, Marcinkiewicz integral operators belong to a broad class of Littlewood–Paley $g$-functions and $L^p$ bounds regarding them are useful in the study of smoothness properties of functions and behavior of integral transformations, such as Poisson integrals, singular integrals and, more generally, singular Radon transforms. In this paper we focus on the $L^p$ mapping properties for a class of nonisotropic Marcinkiewicz integral operators associated to surfaces.

Before establishing our main results, let us recall and introduce some notation. Let $n \geq 2$ and $\mathbb{R}^n$ be the $n$-dimensional Euclidean space with a nonisotropic dilation. Precisely, let $P$ be an $n \times n$ real matrix whose eigenvalues have positive real parts and let $\alpha = \text{tr} P$. Define a dilation group $\{A_t\}_{t>0}$ on $\mathbb{R}^n$ by $A_t = t^\alpha \exp((\log t)P)$. There is a nonnegative function $r$ on $\mathbb{R}^n$ associated with $\{A_t\}_{t>0}$. The function $r$ is continuous on $\mathbb{R}^n$ and infinitely differentiable in $\mathbb{R}^n \setminus \{0\}$; furthermore, it satisfies:

(i) $r(A_t x) = tr(x)$ for all $t > 0$ and $x \in \mathbb{R}^n$;

(ii) $r(x + y) \leq C(r(x) + r(y))$ for some $C > 0$;

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(iii) if \( \Sigma = \{ x \in \mathbb{R}^n | r(x) = 1 \} \), then \( \Sigma = \{ \theta \in \mathbb{R}^n | \langle B\theta, \theta \rangle = 1 \} \) for a positive symmetric matrix \( B \), where \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( \mathbb{R}^n \): then, the Lebesgue measure can be written as \( dx = r^{n-1} d\sigma \, dt \), that is,

\[
\int_{\mathbb{R}^n} f(x) \, dx = \int_0^\infty \int_\Sigma f(A_\theta x) r^{n-1} \, d\sigma(\theta) \, dt
\]

for appropriate functions \( f \), where \( d\sigma \) is a \( C^\infty \) measure on \( \Sigma \);

(iv) there are positive constants \( c_1, c_2, c_3, c_4, \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \) such that

\[
c_1|x|^{\alpha_1} \leq r(x) \leq c_2|x|^{\alpha_2} \quad \text{if} \quad r(x) \geq 1,
\]

\[
c_3|x|^{\beta_1} \leq r(x) \leq c_4|x|^{\beta_2} \quad \text{if} \quad r(x) \leq 1.
\]

See [5, 16, 19] for more details.

Let \( \Omega \) be a locally integrable function and homogeneous of degree 0 with respect to the dilation group \( \{ A_\theta \} \), that is, \( \Omega(A_\theta x) = \Omega(x) \) for \( x \neq 0 \). We assume that

\[
\int_\Sigma \Omega(\theta) \, d\sigma(\theta) = 0. \tag{1.1}
\]

For a suitable mapping \( \Phi : (0, \infty) \rightarrow (0, \infty) \), we define the parametric Marcinkiewicz integral operator along the surfaces \( \{ A_{\Phi(\tau y)} y' \} ; y \in \mathbb{R}^n \} \) by

\[
\mathcal{M}_{h, \Omega, \Phi, \theta}(f)(x) := \left( \int_0^\infty \left| \frac{1}{|\rho|^n} \int_{r(y) \leq \rho} \frac{h(r(y) \Omega(y) \phi(y)}{r(y)^{\alpha - \theta}} f(x - A_{\Phi(\tau y)} y') \, dy \right|^2 \frac{dt}{t} \right)^{1/2}, \quad x \in \mathbb{R}^n,
\]

where \( \phi = \sigma + i\tau \) (\( \sigma, \tau \in \mathbb{R} \) with \( \sigma > 0 \), \( \gamma' = A_{\tau y}^{-1} y' \), \( f \in \mathcal{S}(\mathbb{R}^n) \) (the Schwartz class) and \( h \in \Delta_1(\mathbb{R}^+) \). Here \( \Delta_\gamma(\mathbb{R}^+)(\gamma \geq 1) \) denotes the collection of measurable functions \( h \) on \( \mathbb{R}^+ := (0, \infty) \) satisfying

\[
\| h \|_{\Delta_\gamma(\mathbb{R}^+)} = \sup_{j \in \mathbb{Z}} \left( \int_2^{2^{j+1}} |h(t)|^\gamma \frac{dt}{t} \right)^{1/\gamma} < \infty.
\]

It is easy to check that \( L^\infty(\mathbb{R}^+) = \Delta_0(\mathbb{R}^+) \subseteq \Delta_{\gamma_1}(\mathbb{R}^+) \subseteq \Delta_{\gamma_2}(\mathbb{R}^+) \) for any \( 1 \leq \gamma_2 < \gamma_1 < \infty \). Let \( \mathcal{N}_\delta(\mathbb{R}^+)(\delta > 0) \) be the set of all measurable functions \( h \) on \( \mathbb{R}^+ \) satisfying

\[
\mathcal{N}_\delta(h) = \sum_{m=1}^\infty m^{\delta} 2^m d_m(h) < \infty \quad \text{with} \quad d_m(h) = \sup_{k \in \mathbb{Z}} 2^{-k} |E(k, m)|,
\]

where \( E(k, 1) = \{ t \in (2^k, 2^{k+1}) : |h(t)| \leq 2 \} \) and

\[
E(k, m) = \{ t \in (2^k, 2^{k+1}) : 2^{m-1} \leq |h(t)| \leq 2^m \} \quad \text{for} \; m \geq 2.
\]

It follows from [18] that

\[
\Delta_\gamma(\mathbb{R}^+) \subseteq \mathcal{N}_{\delta_1}(\mathbb{R}^+) \subseteq \mathcal{N}_{\delta_2}(\mathbb{R}^+), \quad \forall \delta_1 > \delta_2 > 0 \quad \text{and} \; 1 < \gamma < \infty. \tag{1.3}
\]

We denote by \( L(\log^+ L)^{\beta}(\Sigma)(\beta > 0) \) the space of all those functions \( \Omega \) on \( \Sigma \) which satisfy

\[
\int_\Sigma |\Omega(\theta)| \log^2(2 + |\Omega(\theta)|) \, d\sigma(\theta) < \infty.
\]
Also, we consider the $L^q(\Sigma)$ spaces and write $\|\Omega\|_q = (\int_\Sigma |\Omega(\theta)|^q \, d\sigma(\theta))^{1/q}$ for $\Omega \in L^q(\Sigma)$. Note that

$$L^q(\Sigma) \subseteq L(\log^+ L)^{\beta_1}(\Sigma) \subseteq L(\log^+ L)^{\beta_2}(\Sigma), \quad q > 1 \text{ and } \beta_2 < \beta_1. \quad (1.4)$$

When $\Phi(t) = t$, we denote $\mathcal{M}_{h,\Omega,\varphi,\varrho}$ by $\mathcal{M}_{h,\Omega,\varrho}$. When $A_t = tE$ with $E$ being the identity matrix and $r(x) = |x|$ (the Euclidean norm), $\Sigma$ recovers the unit sphere in $\mathbb{R}^n$ denoted by $S^{n-1}$, and the operator $\mathcal{M}_{h,\Omega,\varrho}$ reduces to the classical parametric Marcinkiewicz integral operator, which has been studied by many authors. For example, see [4, 20] for the case $h(t) = \varrho = 1$ and $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$, [8, 9] for the case $\varrho = 1$, $h(t) \in \Delta_\infty(\mathbb{R}^+)$ and $\Omega \in H^1(S^{n-1})$, [1] for the case $h(t) \in \Delta_\chi(\mathbb{R}^+)$ and $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$ and [1, 13] for the case $h \in \mathcal{N}_{1/2}(\mathbb{R}^+)$ and $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$. When $A_t x = (t^{\alpha_1} x_1, t^{\alpha_2} x_2, \ldots, t^{\alpha_n} x_n)$ with $\alpha_1, \ldots, \alpha_n$ being integers greater than one and $r(x) = \rho(x)$ with $\rho(x)$ being the solution to the equation $\sum_{j=1}^n x_j^2 \rho(x)^{-2\alpha_j} = 1$, the operator $\mathcal{M}_{h,\Omega,\varphi}$ recovers the parabolic parametric Marcinkiewicz integral operators denoted by $\mu_{h,\Omega,\varphi}$, and then $\Sigma$ recovers $S^{n-1}$. The $L^p$ mapping properties of $\mu_{h,\Omega,\varphi}$ have been discussed extensively by many authors. Xue et al. [21] proved that $\mu_{h,\Omega,\varphi}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, provided that $h(t) = \varrho = 1$ and $\Omega \in L^q(S^{n-1})$ for some $q > 1$. Chen and Ding [6] (respectively, [7]) extended the above result to the case $\Omega \in L(\log^+ L)^{1/2}(S^{n-1})$ (respectively, $\Omega \in H^1(S^{n-1})$). The investigation of the parabolic parametric Marcinkiewicz integral operators $\mu_{h,\Omega,\varphi}$ with additional roughness in the radial direction has also received a large amount of attention by many authors (see [14, 15] for example).

On the other hand, to study further the singular integral operator with rough kernel both on the unit sphere and in the radial direction, Sato [17] first introduced the radial condition $\mathcal{N}_P(\mathbb{R}^+)$ and proved the following result.

**Theorem A.** Let $\Omega \in L(\log^+ L)(\Sigma)$ satisfy (1.1) and $h \in \mathcal{N}_1(\mathbb{R}^+)$; then the nonisotropic singular integral operator $T_{h,\Omega}$ defined by

$$T_{h,\Omega}(f)(x) = p.v. \int_{\mathbb{R}^n} \frac{h(r(y))\Omega(y)}{r(y)^\alpha} f(x - y) \, dy, \quad x \in \mathbb{R}^n,$$

is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.

Based on the above, a natural question is the following.

**Question.** Is $\mathcal{M}_{h,\Omega,\varrho}$ bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ under the condition that $\Omega \in L(\log^+ L)^a(\Sigma)$ and $h \in \mathcal{N}_p(\mathbb{R}^+)$?

In this paper, we will give an affirmative answer to this question by considering a class of operators broader than $\mathcal{M}_{h,\Omega,\varrho}$. More precisely, we denote by $\mathcal{G}$ the set of all functions $\varphi$ satisfying the following conditions (a) or (b):

(a) $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is an increasing $C^1$ function such that $t\varphi'(t) \geq C_\varphi \varphi(t)$ and $\varphi(2t) \leq c_\varphi \varphi(t)$ for all $t > 0$, where $C_\varphi$ and $c_\varphi$ are independent of $t$. Moreover, $\varphi'$ is monotonic.
(b) \( \varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is a decreasing \( C^1 \) function such that \( t \varphi'(t) \leq -C_\varphi \varphi(t) \) and \( \varphi(t) \leq c_\varphi(2t) \) for all \( t > 0 \), where \( C_\varphi \) and \( c_\varphi \) are independent of \( t \). Moreover, \( \varphi' \) is monotonic.

**Remark 1.1.** There are some model examples on the class \( \mathcal{H} \) satisfying (a), such as \( t^\beta (\beta > 0), t^\beta (\ln(1 + t))^{\gamma} (\beta, \gamma > 0), t \ln(e + t), \) real-valued polynomials \( P \) on \( \mathbb{R} \) with positive coefficients and \( P(0) = 0 \) and so on. The model example of functions \( \phi \in \mathcal{H} \) which satisfy (b) are \( t^\delta (\delta < 0), t^{-1} \ln(1 + 1/t) \). It should be pointed out that there are two important facts, as follows.

(i) If \( \varphi(t) \in C^1(\mathbb{R}^+) \) is nonnegative and increasing (respectively, decreasing) on \( \mathbb{R}^+ \) and \( \varphi(t)/(t \varphi'(t)) \) is bounded on \( \mathbb{R}^+ \), then \( \lim_{t \rightarrow 0} \varphi(t) = 0 \) (respectively, \( \lim_{t \rightarrow 0} \varphi(t) = +\infty \)) and \( \lim_{t \rightarrow +\infty} \varphi(t) = +\infty \) (respectively, \( \lim_{t \rightarrow +\infty} \varphi(t) = 0 \) (see [11]).

(ii) If \( \varphi \in \mathcal{H} \) and satisfies (a), there exists a constant \( B_\varphi > 1 \) such that \( \varphi(2t) \geq B_\varphi \varphi(t) \) (see [2, 3] for example). Similarly, one can easily check that if \( \varphi \in \mathcal{H} \) and satisfies (b), then there exists a constant \( B_\varphi > 1 \) such that \( \varphi(t) \geq B_\varphi \varphi(2t) \).

Our main results can be stated as follows.

**Theorem 1.2.** Let \( \mathcal{M}_{h, \Omega, \phi, q} \) be as in (1.2) and \( \Phi \in \mathcal{H} \). Suppose that \( \Omega \in L^q(\Sigma) \) for some \( q \in (1, 2) \) satisfying (1.1) and \( h \in \Delta_h^q(\mathbb{R}^+) \) for some \( \gamma \in (1, 2) \). Then:

(i) for \( 2 \leq p < \infty \),

\[
\|\mathcal{M}_{h, \Omega, \phi, q}(f)\|_{L^p(\mathbb{R}^+)} \leq C_p(\gamma - 1)^{-1/2}(q - 1)^{-1/2}\|h\|_{\Delta_h^q(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}\|f\|_{L^p(\mathbb{R}^+)};
\]

(ii) for \( 1 < p < 2 \),

\[
\|\mathcal{M}_{h, \Omega, \phi, q}(f)\|_{L^p(\mathbb{R}^+)} \leq C_p(\gamma - 1)^{-1}(q - 1)^{-1}\|h\|_{\Delta_h^q(\mathbb{R}^+)}\|\Omega\|_{L^q(\Sigma)}\|f\|_{L^p(\mathbb{R}^+)}.
\]

The constants \( C_p > 0 \) are independent of \( h, \Omega, q \) and \( \gamma \), but depend on \( \Phi \).

**Theorem 1.3.** Let \( \mathcal{M}_{h, \Omega, \phi, q} \) be as in (1.2) and \( \Phi \in \mathcal{H} \). Suppose that \( \Omega \) satisfies (1.1).

(i) If \( \Omega \in L(\log^+ L)^{(1/2)}(\Sigma) \) and \( h \in \mathcal{N}_{1/2}(\mathbb{R}^+) \), then, for \( 2 \leq p < \infty \),

\[
\|\mathcal{M}_{h, \Omega, \phi, q}(f)\|_{L^p(\mathbb{R}^+)} \leq C(1 + \|\Omega\|_{L(\log^+ L)^{(1/2)}(\Sigma)}(1 + N_{1/2}(h))\|f\|_{L^p(\mathbb{R}^+)};
\]

(ii) If \( \Omega \in L(\log^+ L(\Sigma) \) and \( h \in \mathcal{N}_1(\mathbb{R}^+) \), then, for \( 1 < p < 2 \),

\[
\|\mathcal{M}_{h, \Omega, \phi, q}(f)\|_{L^p(\mathbb{R}^+)} \leq C(1 + \|\Omega\|_{L(\log^+ L(\Sigma)}(1 + N_1(h))\|f\|_{L^p(\mathbb{R}^+)}.
\]

The constants \( C_p > 0 \) depend on \( \Phi \).

**Remark 1.4.** When \( A_t = tE \) with \( E \) being the identity matrix and \( r(x) = |x| \) (the Euclidean norm), Theorem 1.3 was shown by Liu and Wu in more general form (see [13, Theorem 1.6]) (also see [1] for the case \( \Phi(t) = t \)). When \( A_t x = (t^{\alpha_1} x_1, t^{\alpha_2} x_2, \ldots, t^{\alpha_n} x_n) \) with \( \alpha_1, \ldots, \alpha_n \) being integers greater than one and \( r(x) = \rho(x) \) with \( \rho(x) \) being the solution to the equation \( \sum_{j=1}^n \lambda_j \rho(x)^{\alpha_j} = 1 \), Theorem 1.3 was proved by Liu and Zhang in more general form (see [15, Theorem 1]). It should be pointed out that our main results are also new, even in the special case \( \Phi(t) = t \) and \( q = 1 \).
The rest of this paper is organized as follows. In Section 2 we present some preliminary lemmas. The proofs of main results will be given in Section 3. Finally, we consider the $L^p$ bounds of the corresponding parametric Marcinkiewicz integral operators related to area integrals and Littlewood–Paley $g^*_a$-functions in Section 4. We remark that the proof of Theorem 1.2 is based on the method of [1], but we add some new techniques. The main ingredients of our proofs in Theorem 1.2 are to give two sharp estimates for two maximal operators (see Lemma 2.3). As a consequence of Theorem 1.2, we can prove Theorem 1.3 via an extrapolation method which was originally by Yano (see [22]) and developed by Sato (see [17]).

Throughout the paper, we let $p'$ denote the conjugate index of $p$ which satisfies $1/p + 1/p' = 1$. For $x \in \mathbb{R}$, we set $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$. The letter $C$ will stand for positive constants that are not necessarily the same at each occurrence but that are independent of the essential variables.

2. Preliminary lemmas

Following the notation in [17], let $P^*$ denote the adjoint of the matrix $P$. Then $A^*_t = \exp((\log t)P^*)$. We can define a nonnegative function $s$ from $\{A^*_t\}$ in exactly the same way as we define $r$ from $\{A_t\}$.

We will use the following estimates (see [19]):

\begin{align}
   d_1|\xi|^{a_1} < s(\xi) < d_2|\xi|^{a_2} & \quad \text{ if } s(\xi) \geq 1, \tag{2.1} \\
   d_3|\xi|^{b_1} < s(\xi) < d_4|\xi|^{b_2} & \quad \text{ if } 0 < s(\xi) \leq 1, \tag{2.2}
\end{align}

where $d_j$ ($j = 1, 2, 3, 4$), $a_k, b_k$ ($k = 1, 2$) are positive constants. It follows from (2.1)–(2.2) that

\begin{align}
   |\xi| \leq C_1(s(\xi)^{1/a_1} + s(\xi)^{1/b_1}), \tag{2.3} \\
   |\xi|^{-1} \leq C_2(s(\xi)^{-1/a_2} + s(\xi)^{-1/b_2}). \tag{2.4}
\end{align}

First we give the following estimate, which follows from [17, Corollary 4.2] via an integration by parts argument.

**Lemma 2.1.** Let $L$ be the degree of the minimal polynomial of $P$ and $\Psi \in C^1([a, b])$ with $0 < a < b$. Then, for $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$,

$$
\left| \int_a^b \exp(i\eta \cdot A_t \xi)\Psi(t) \, dt \right| \leq C|\eta| \cdot P\xi|^{-1/L} \left( \sup_{t \in [a, b]} |\Psi(t)| + \int_a^b |\Psi'(t)| \, dt \right)
$$

for some positive constant $C$ independent of $\xi, \eta$ and $\Psi$. Applying Lemma 2.1, we shall establish the following result.

**Lemma 2.2.** Let $L$ be as in Lemma 2.1 and $\Phi \in \mathcal{S}$. Then, for $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$ and $t > 0$, there exists a constant $C > 0$ such that

$$
\left| \int_{t/2}^t \exp(i\eta \cdot A_{\Phi(u)} \xi) \frac{du}{u} \right| \leq C|\eta| \cdot PA_{\Phi(t)}\xi|^{-1/L}.
$$

The constant $C$ is independent of $\xi, \eta$, but depends on $\Phi$. 

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This proves Lemma 2.2. □

In what follows, we will establish some lemmas, which will play key roles in the proofs of our main results.

For \( q, \gamma \in (1, \infty) \) and \( t > 0 \), we define the family of measures \( \{\sigma_{h,t}\}_{t > 0} \) and the related maximal operators \( \sigma^*_h \) and \( M_{h,q,\gamma} \) on \( \mathbb{R}^n \) by

\[
\overline{\sigma}_{h,t}(\xi) = \frac{1}{t^n} \int_{t/2 < |y| \leq t} \exp(-2\pi i \xi \cdot A_{\Phi(\xi)} y) \frac{h(r(y))\Omega(y')}{r(y)^{n-\gamma}} dy,
\]

\[
\sigma^*_h(f)(x) = \sup_{r \in \mathbb{R}^n} \|\sigma_{h,r}\| f(x),
\]

\[
M_{h,q,\gamma}(f)(x) = \sup_{k \in \mathbb{Z}} \int_{2^k r_{q,\gamma} < t \leq 2^{k+1} r_{q,\gamma}} \|\sigma_{h,r}\| f(x) \frac{dt}{t},
\]

where \( |\sigma_{h,t}| \) is defined in the same way as \( \sigma_{h,t} \), but with \( \Omega \) replaced by \( |\Omega| \) and \( h \) replaced by \( |h| \).

In what follows, we will establish some lemmas, which will play key roles in the proofs of our main results.
Lemma 2.3. Let $\Omega \in L^q(\Sigma)$ for some $1 < q < \infty$ and satisfy (1.1). Suppose that $h \in \Delta_\gamma(\mathbb{R}^+) \text{ for some } \gamma > 1$ and $\Phi \in \mathfrak{F}_+$. Then, for any $t > 0$ and $\xi \in \mathbb{R}^n$, there exists $C > 0$ such that

$$\max\{||\hat{\sigma}_{h,t}(\xi)||, ||\hat{\sigma}_{h,t}(\xi)|| \leq C||h||_{\Delta_\gamma(\mathbb{R}^+)}||\Omega||_{L^q(\Sigma)} \max\{1, |A^*_{\Phi(t)}\xi|^{-1/(4q'\gamma L)}\},$$

$$\max\{||\hat{\sigma}_{h,t}(\xi)||, ||\hat{\sigma}_{h,t}(\xi) - ||\hat{\sigma}_{h,t}(0)|| \leq C||h||_{\Delta_\gamma(\mathbb{R}^+)}||\Omega||_{L^q(\Sigma)}|A^*_{\Phi(t)}\xi|^{1/(4q'\gamma L)}.$$

The constant $C$ is independent of $h$, $\Omega$, $q$, $\gamma$, but depends on $\Phi$.

Proof. We only consider the case $\Phi \in \mathfrak{F}_+$ satisfying the condition (a), since the other case can be proved similarly. By a change of variable and Hölder’s inequality,

$$\max\{||\hat{\sigma}_{h,t}(\xi)||, ||\hat{\sigma}_{h,t}(\xi)|| \leq C||h||_{\Delta_\gamma(\mathbb{R}^+)}||\Omega||_{L^q(\Sigma)},$$

Similarly,

$$\max\{||\hat{\sigma}_{h,t}(\xi)||, ||\hat{\sigma}_{h,t}(\xi) - ||\hat{\sigma}_{h,t}(0)|| \leq C||h||_{\Delta_\gamma(\mathbb{R}^+)}||\Omega||_{L^q(\Sigma)}.$$ 

On the other hand, by a change of variable and Hölder’s inequality,

$$||\hat{\sigma}_{h,t}(\xi)|| = \left|\frac{1}{|\mathbb{R}^n|} \int_{1/2}^{t} \int_{\Sigma} \exp(-2\pi i\xi \cdot A_{\Phi(t)}(\theta)) \Omega(\theta) d\sigma(\theta) \frac{h(u)}{u^{1-\epsilon}} du \right|$$

$$\leq \int_{1/2}^{t} \left|\int_{\Sigma} \exp(-2\pi i\xi \cdot A_{\Phi(t)}(\theta)) \Omega(\theta) d\sigma(\theta) \right| \left|\frac{h(u)}{u^{1-\epsilon}} du \right|^{1/\gamma'}$$

$$\leq C||h||_{\Delta_\gamma(\mathbb{R}^+)} \left(\int_{1/2}^{t} \left|\int_{\Sigma} \exp(-2\pi i\xi \cdot A_{\Phi(t)}(\theta)) \Omega(\theta) d\sigma(\theta) \right|^{1/\gamma'} du \right)^{\max\{0,1-2/\gamma'\}}$$

$$\times \left(\int_{1/2}^{t} \left|\int_{\Sigma} \exp(-2\pi i\xi \cdot A_{\Phi(t)}(\theta)) \Omega(\theta) d\sigma(\theta) \right|^{2 du \left/ u \right.} \right)^{1/\max\{2,\gamma'\}}.$$ 

By Lemma 2.1 and Hölder’s inequality, for any $0 < \epsilon < \min\{1/(2q'), 1/L\},$

$$\int_{1/2}^{t} \left|\int_{\Sigma} \exp(-2\pi i\xi \cdot A_{\Phi(t)}(\theta)) \Omega(\theta) d\sigma(\theta) \right|^{2 du \left/ u \right.}$$

$$= \int_{1/2}^{t} \int_{\Sigma} \exp(-2\pi i A^*_{\Phi(t)}\xi \cdot (\theta - w))\Omega(\theta)\overline{\Omega(w)} d\sigma(\theta) d\sigma(w) \frac{du}{u}$$

$$\leq \int_{\Sigma} \int_{1/2}^{t} \exp(-2\pi i\xi \cdot A_{\Phi(t)}(\theta - w)) \frac{du}{u} \left|\Omega(\theta)\overline{\Omega(w)}\right| d\sigma(\theta) d\sigma(w)$$

$$\leq C \int_{\Sigma} \int_{1/2}^{t} \left|\xi \cdot (A_{\Phi(t)} P(\theta - w))^{-\epsilon} \Omega(\theta)\overline{\Omega(w)}\right| d\sigma(\theta) d\sigma(w)$$

$$\leq C||\Omega||_{L^q(\Sigma)} \left(\int_{\Sigma} \int_{1/2}^{t} \left|P^* A^*_{\Phi(t)}\xi \cdot (\theta - w)\right|^{-\epsilon q'} d\sigma(\theta) d\sigma(w) \right)^{1/q'}$$

$$\leq C||\Omega||_{L^q(\Sigma)} \left| A^*_{\Phi(t)}\xi \right|^{\epsilon - \epsilon q'}.$$ 

(2.10)
where the last inequality follows from \([12, \text{ page 533}]\) (also see \([17, \text{ proof of Lemma 1}]\)). If follows from (2.9) and (2.10) that

\[
|\sigma_{h,t}(\xi)| \leq C \|h\|_{L^2(\mathbb{R}^n)} \|\Omega\|_{L^q(\mathcal{S})} |A_{\Phi(t)}^*| \xi|^{1/(2q' \max\{2,\gamma\}|L|}, \tag{2.11}
\]

where we take \(\epsilon = 1/(2q' L)\). Similarly,

\[
|\sigma_{h,t}(\xi)| \leq C \|h\|_{L^2(\mathbb{R}^n)} \|\Omega\|_{L^q(\mathcal{S})} |A_{\Phi(t)}^*| \xi|^{1/(2q' \max\{2,\gamma\}|L|}.
\]

This, together with (2.7), (2.8) and (2.11), implies (2.5). On the other hand, by a change of variables, (1.1) and Hölder’s inequality,

\[
|\sigma_{h,t}(\xi)| = \left| \frac{1}{t^\epsilon} \int_{1/2}^t \int_{\mathcal{S}} (\exp(-2\pi i \xi \cdot A_{\Phi(u)} \theta) - 1) \Omega(\theta) d\sigma(\theta) \frac{h(u)}{u^{1+\epsilon}} du \right|
\]

\[
\leq C \int_{1/2}^t \int_{\mathcal{S}} \Omega(\theta) |\xi \cdot A_{\Phi(u)} \theta| d\sigma(\theta) \left| \frac{h(u)}{u} \right|^{1/\gamma'} \frac{du}{u}.
\]

\[
\leq C \|h\|_{L^2(\mathbb{R}^n)} \left( \int_{1/2}^t \int_{\mathcal{S}} \Omega(\theta) |\xi \cdot A_{\Phi(u)} \theta| d\sigma(\theta) \right)^{1/\gamma'} \frac{du}{u}.
\]

\[
\leq C_\Phi^{-1/\gamma'} \|h\|_{L^2(\mathbb{R}^n)} \left( \int_{1/2}^t \int_{\mathcal{S}} \Omega(\theta) |\xi \cdot A_{\Phi(u)} \theta| d\sigma(\theta) \right)^{1/\gamma'} \frac{du}{u}.
\]

where \(\zeta\) is as in Lemma 2.2. Note that \(\zeta \geq c_\Phi^{-1}\) and \(|A_{\Phi(u)} \theta| \leq C\) for \(u \in [\zeta, 1]\) and \(\theta \in \Sigma\). Thus,

\[
|\sigma_{h,t}(\xi)| \leq C \|h\|_{L^2(\mathbb{R}^n)} \|\Omega\|_{L^q(\mathcal{S})} |A_{\Phi(t)}^*| \xi|^{1/(4q' \gamma'|L|)}. \tag{2.12}
\]

It follows from (2.7) and (2.12) that

\[
|\sigma_{h,t}(\xi)| \leq C \|h\|_{L^2(\mathbb{R}^n)} \|\Omega\|_{L^q(\mathcal{S})} |A_{\Phi(t)}^*| \xi|^{1/(4q' \gamma'|L|)}.
\]

Similarly, we can prove that

\[
|\sigma_{h,t}(\xi) - \sigma_{h,0}(\xi)| \leq C \|h\|_{L^2(\mathbb{R}^n)} \|\Omega\|_{L^q(\mathcal{S})} |A_{\Phi(t)}^*| \xi|^{1/(4q' \gamma'|L|),}
\]

which, combined with (2.13), implies (2.6). This proves Lemma 2.3. \(\square\)

**Lemma 2.4.** Let \(h, \Omega, \Phi\) be as in Lemma 2.3. Then, for any \(1 < p < \infty\), there exists a constant \(C > 0\) such that

\[
\|\sigma_h^p(f)\|_{L^p(\mathbb{R}^n)} \leq C q' \gamma' \|h\|_{L^q(\mathbb{R}^n)} \|\Omega\|_{L^q(\mathcal{S})} \|f\|_{L^p(\mathbb{R}^n)}, \tag{2.14}
\]

\[
|M_{h,q,p}(f)\|_{L^p(\mathbb{R}^n)} \leq C q' \gamma' \|h\|_{L^q(\mathbb{R}^n)} \|\Omega\|_{L^q(\mathcal{S})} \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.15}
\]

The constant \(C\) is independent of \(h, \Omega, q, \gamma\), but depends on \(\Phi\).
Interpolating between (2.17) and (2.18) leads to

\[ \| \mu^*(f) \|_{L^p(\mathbb{R}^n)} \leq C_p N_{q, \gamma} \| f \|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty. \]  

(2.16)

Below we estimate \( |\tilde{\mu}_k(\xi)| \). By a change of variable, (2.3) and the same argument as in getting (2.12),

\[
|\tilde{\mu}_k(\xi)| - |\tilde{\mu}_k(0)| \leq \sum_{i=0}^{[\gamma'] + 1} \left( \int_{2^{i} \gamma' + 1}^{2^{i+1} \gamma'} \left| \sum_{\gamma \in \mathbb{Z}^n} \int_{\mathbb{R}^n} \frac{h(r(y))\Omega(y)}{\rho(y)^\gamma} f(A_{\Phi(r(y))}y') dy \right| \right) \left( \int \left| A_{\Phi(r(y)(\Sigma_k) - \gamma \cdot \xi)}(\mathbb{R}^n) \right| \right)
\]

(2.17)

One can easily check that

\[ |\tilde{\mu}_k(\xi)| \leq C_{N, q, \gamma}, \quad \forall \xi \in \mathbb{R}^n. \]  

(2.18)

Interpolating between (2.17) and (2.18) leads to

\[
|\tilde{\mu}_k(\xi)| - |\tilde{\mu}_k(0)| \leq C_{N, q, \gamma} \left( (\Phi(2^{q' \gamma'}) s(\xi))^{1/(4q' \gamma' a_1 L)} + (\Phi(2^{q' \gamma'} s(\xi))^{1/(4q' \gamma' b_1 L)} \right). \]  

(2.19)
On the other hand, by a change of variable, Hölder’s inequality and (2.10), for any \(0 < \epsilon < \min\{1/(2q'), 1/L\},\)
\[
|\hat{\mu}_k(\xi)| = \left|\int_{2^{q'\gamma'k+1}}^{2^{q'\gamma'(k+1)}} \exp(-2\pi i \xi \cdot A_{\Phi(\theta)}(\theta)) |\Omega(\theta)| d\sigma(\theta) |h(u)\frac{du}{u}| \right|
\leq \left( \int_{2^{q'\gamma'k+1}}^{2^{q'\gamma(k+1)}} |h(u)|^\gamma \frac{du}{u} \right)^{1/\gamma}
\times \left( \int_{2^{q'\gamma'k+1}}^{2^{q'\gamma(k+1)}} \left|\int_\Sigma \exp(-2\pi i \xi \cdot A_{\Phi(\theta)}(\theta)) |\Omega(\theta)| d\sigma(\theta) \right|^\gamma \frac{du}{u} \right)^{1/\gamma'}
\leq C(q'\gamma')^{1/\gamma} |h|_{\Delta_{q'}(\mathbb{R}^n)}^{1/\gamma'} |\Omega|_{L^1(\Sigma)}^{\max(0,1-2/\gamma')}
\times \left( \sum_{l=0}^{[q'\gamma']} \left( \int_{2^{q'\gamma'k+1}}^{2^{q'\gamma'(k+1)}} \left|\int_\Sigma \exp(-2\pi i \xi \cdot A_{\Phi(\theta)}(\theta)) |\Omega(\theta)| d\sigma(\theta) \right|^\gamma \frac{du}{u} \right)^{1/\gamma'} \right)
\leq C(q'\gamma')^{1/\gamma} |h|_{\Delta_{q'}(\mathbb{R}^n)}^{1/\gamma'} |\Omega|_{L^1(\Sigma)}^{\max(0,1-2/\gamma')}
\times \left( \sum_{l=0}^{[q'\gamma']} |\Omega|_{L^1(\Sigma)}^{2} |A_{\Phi(2^{q'\gamma'(k+1)})}^{*} \xi^{1-\epsilon} |^{\max(2,\gamma')} \right)^{1/\gamma'}
\leq C(q'\gamma')^{1/\gamma} |h|_{\Delta_{q'}(\mathbb{R}^n)}^{1/\gamma'} |\Omega|_{L^1(\Sigma)}^{\max(0,1-2/\gamma')}
\times \left( \sum_{l=0}^{[q'\gamma']} |A_{\Phi(2^{q'\gamma'(k+1)})}^{*} \xi^{1-\epsilon} |^{\max(2,\gamma')} \right)^{1/\gamma'}.
\]
This, together with (2.4) and (2.18), leads to
\[
|\hat{\mu}_k(\xi)| \leq C N_{q',\gamma'}((\Phi(2^{q'\gamma'k} s(\xi)))^{-1/(4q'\gamma'a_2 L)} + (\Phi(2^{q'\gamma'k} s(\xi)))^{-1/(4q'\gamma''b_2 L))}. \tag{2.20}
\]
We can choose a nonnegative \(C_0^\infty(\mathbb{R}^n)\) function \(\hat{\psi}(0) = 1\) and \(\text{supp}(\psi) \subset \{x \in \mathbb{R}^n : r(x) \leq 1\}\). Define the family of measures \(\{\nu_k\}_{k \in \mathbb{Z}}\) on \(\mathbb{R}^n\) by
\[
\nu_k(\xi) = |\mu_k(\xi) - \psi_k(\xi)|\mu_k(0), \tag{2.21}
\]
where \(\psi_k(x) = \Phi(2^{q'\gamma'k})^{-a} \psi(\Phi(2^{q'\gamma'k} x)).\) Let \(\Psi_k = |\mu_k(0)\psi_k\). One can easily check that
\[
\mu^*(f) \leq G(f) + \Psi^*(|f|), \tag{2.22}
\]
\[
\nu^*(f) \leq \mu^*(f) + \Psi^*(|f|), \tag{2.23}
\]
where \(\nu^*(f) = \sup_{k \in \mathbb{Z}} |\nu_k| * |f|, \Psi^*(f) = \sup_{k \in \mathbb{Z}} |\Psi_k| * |f| \) and \(G(f) = (\sum_{k \in \mathbb{Z}} |\nu_k| * |f|^2)^{1/2}\). By the \(L^p\) boundedness of the Hardy–Littlewood maximal function on \(\mathbb{R}^n\) with respect to the function \(r(\cdot),\)
\[
\left\|\sup_{k \in \mathbb{Z}} |\psi_k| * |f| \right\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty, \tag{2.24}
\]
where positive $C$ is independent of $\gamma$ and $q$. Thus, by (2.18),
\[
\|\Psi'(f)\|_{L^p(\mathbb{R}^n)} \leq C N_{q,\gamma} \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty,
\]
(2.25)
where $C$ is independent of $\gamma$ and $q$. By (2.22) and (2.25), to prove (2.16), it suffices to prove that
\[
\|G(f)\|_{L^p(\mathbb{R}^n)} \leq C N_{q,\gamma} \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty,
\]
(2.26)
where $C$ is independent of $\gamma$ and $q$. By a well-known property of Rademacher’s function, (2.26) follows from
\[
\|\tau_\epsilon(f)\|_{L^p(\mathbb{R}^n)} \leq C N_{q,\gamma} \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty,
\]
where $\tau_\epsilon(f) = \sum_{k \in \mathbb{Z}} \epsilon_k \nu_k \ast f$ with $\epsilon = \{\epsilon_k\}$, $\epsilon_k = 1$ or $-1$ (the inequality is uniform in $\epsilon$) and $C$ is independent of $\gamma$ and $q$. It follows from (2.3)–(2.4) and (2.18)–(2.20) that
\[
|\hat{\nu}_k(\xi)| \leq C N_{q,\gamma} \min\{1, (\Phi(2^q \gamma^k)s(\xi))^{-1/(4q'\gamma^k a_1 b_1 L)}, (\Phi(2^q \gamma^k s(\xi))^{-1/(4q'\gamma^k a_2 b_2 L)}\},
\]
(2.27)
\[
|\hat{\nu}_k(\xi)| \leq C N_{q,\gamma} (\Phi(2^q \gamma^k s(\xi))^{-1/(4q'\gamma^k a_1 L)} + (\Phi(2^q \gamma^k) s(\xi))^{1/(4q'\gamma^k b_1 L)}).
\]
(2.28)

Let $\{\Gamma_k\}_{k \in \mathbb{Z}}$ be a sequence of nonnegative functions in $C_0^\infty((0, \infty))$ such that
\[
\text{supp}(\Gamma_k) \subset [\Phi(2^j \gamma^k(L+1))^{-1}, \Phi(2^j \gamma^k(L+1)-1)], \quad \sum_{k \in \mathbb{Z}} \Gamma_k^2(t) = 1,
\]
where $C_j$ $(j = 1, 2, \ldots)$ are independent of $q$ and $\gamma$. Define the Fourier multiplier operators $S_k$ by
\[
\hat{S_k(f)}(\xi) = \Gamma_k(s(\xi)) \hat{f}(\xi).
\]
(2.29)

By Littlewood–Paley theory, for any $1 < p < \infty$, $\{g_k\} \in L^p(\mathbb{R}^n, \ell^2)$ and $f \in L^p(\mathbb{R}^n)$, there exists $C_p > 0$ which is independent of $q$ and $\gamma$ such that
\[
\left\| \sum_{k \in \mathbb{Z}} S_k(g_k) \right\|_{L^p(\mathbb{R}^n)} \leq C_p \left\| \left( \sum_{k \in \mathbb{Z}} |g_k|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)},
\]
(2.30)
\[
\left\| \left( \sum_{k \in \mathbb{Z}} |S_k(f)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}.
\]
(2.31)

By the definition of $S_k$, we can write
\[
\tau_\epsilon(f) = \sum_{k \in \mathbb{Z}} \epsilon_k \nu_k \ast S_j \nu_k \ast f = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \epsilon_k S_j \nu_k \ast (\nu_k 
\ast S_j \nu_k \ast f) := \sum_{j \in \mathbb{Z}} \tau_j(f).
\]
(2.32)

Then, by Plancherel’s theorem, (2.27)–(2.28) and (2.30)–(2.31),
\[
\left\| \tau_\epsilon(f) \right\|^2_{L^2(\mathbb{R}^n)} \leq C \sum_{k \in \mathbb{Z}} \int_{s(\xi) \in \Phi(2^q \gamma^k(L+1))^{-1} \leq \xi \leq \Phi(2^q \gamma^k(L+1)-1)} |\hat{f}(\xi)|^2 |\hat{\nu}_k(\xi)|^2 d\xi
\]
\[
\leq C (N_{q,\gamma} D_j)^2 \|f\|^2_{L^2(\mathbb{R}^n)}.
\]
where \( D_j = (B_\Phi^{-(j-1)/(8a_1L)} + B_\Phi^{-(j-1)/(8b_1L)}) \chi_{(j\geq 1)}(j) \). Then
\[
\| \tau_j f \|_{L^2(\mathbb{R}^n)} \leq CN_{q,\gamma} B_\Phi^{-\gamma} \| f \|_{L^2(\mathbb{R}^n)},
\] (2.33)
where \( C \) and \( c \) are independent of \( \gamma \) and \( q \). This, together with the trivial estimate \( \sup_{k \in \mathbb{Z}} \| \psi_k \| \leq CN_{q,\gamma} \) and the proof of [12, Lemma, page 544], implies that
\[
\| \tau_{\epsilon} f \|_{L^2(\mathbb{R}^n)} \leq CN_{q,\gamma} \| f \|_{L^2(\mathbb{R}^n)}.
\]
We also obtain that
\[
\| G(f) \|_{L^2(\mathbb{R}^n)} \leq CN_{q,\gamma} \| f \|_{L^2(\mathbb{R}^n)},
\]
which, by combining (2.22) and (2.23) with (2.25), yields
\[
\| \psi'(f) \|_{L^2(\mathbb{R}^n)} \leq CN_{q,\gamma} \| f \|_{L^2(\mathbb{R}^n)}.
\]
This, together with the trivial estimate \( \sup_{k \in \mathbb{Z}} \| \psi_k \| \leq CN_{q,\gamma} \) and the proof of [12, Lemma, page 544], implies that
\[
\left\| \left( \sum_{k \in \mathbb{Z}} | \psi_k * g_k |^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \leq CN_{q,\gamma} \left\| \left( \sum_{k \in \mathbb{Z}} | g_k |^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)}
\]
holds for arbitrary functions \( \{ g_k \} \in L^p(\mathbb{R}^n, \ell^2) \) with \( p = 4 \) or \( p = 4/3 \). This, combining (2.30) with (2.31), implies that
\[
\| \tau_{\epsilon} f \|_{L^p(\mathbb{R}^n)} \leq CN_{q,\gamma} \| f \|_{L^p(\mathbb{R}^n)}
\] (2.34)
for \( p = 4 \) or \( p = 4/3 \). By (2.32) and the interpolation between (2.33) and (2.34),
\[
\| \tau_{\epsilon} f \|_{L^p(\mathbb{R}^n)} \leq CN_{q,\gamma} \| f \|_{L^p(\mathbb{R}^n)}, \quad 4/3 < p < 4.
\]
Consequently,
\[
\| G(f) \|_{L^p(\mathbb{R}^n)} \leq CN_{q,\gamma} \| f \|_{L^p(\mathbb{R}^n)}, \quad 4/3 < p < 4.
\]
Reasoning as above, (2.22)–(2.23), (2.25), (2.30)–(2.33), the trivial estimate \( \sup_{k \in \mathbb{Z}} \| \psi_k \| \leq CN_{q,\gamma} \), the proof of [12, Lemma, page 544] and an interpolation argument yield
\[
\| G(f) \|_{L^p(\mathbb{R}^n)} \leq CN_{q,\gamma} \| f \|_{L^p(\mathbb{R}^n)}, \quad 8/7 < p < 8.
\]
By using this argument repeatedly, we can obtain ultimately (2.26). Equation (2.14) is proved.

It remains to prove (2.15). Let \( \psi_k \) be as in (2.21). Define the family of measures \( \{ \omega_k \} \) on \( \mathbb{R}^n \) by
\[
\omega_k(\xi) = \int_{2^{-\gamma/2} \xi}^{2^{\gamma/2} \xi} | \sigma_{h,t}(\xi) | \frac{dt}{t} - \int_{2^{-\gamma/2} \xi} \frac{dt}{t} \psi_k(\xi).
\] (2.35)

By Lemma 2.3, one can easily check that
\[
| | \sigma_{h,t}(\xi) | - | \sigma_{h,t}(0) \psi_k(\xi) | |
\leq C \| h \|_{L^p(\mathbb{R}^n)} \| \Omega \|_{L^p(\mathbb{R}^n)} \min \{ 1, | A_{\Phi(0)}\epsilon \|^1/(4q' L) + | A_{\Phi(2q') \epsilon} \|^1/(4q' L) \},
\]
(2.36)
\[
| | \sigma_{h,t}(\xi) | - | \sigma_{h,t}(0) \psi_k(\xi) | |
\leq C \| h \|_{L^p(\mathbb{R}^n)} \| \Omega \|_{L^p(\mathbb{R}^n)} (| A_{\Phi(0)}\epsilon |^{-1/(4q' L)} + | A_{\Phi(2q') \epsilon} \|^1/(4q' L) )
\]
(2.37)
It follows from (2.3)–(2.4) and (2.36)–(2.37) that
\[
|\omega_k(\xi)| \leq CN_{q,\gamma} \min\{1, (\Phi(2^dq'k)s(\xi))^{1/(4dq'2L)} + (\Phi(2^dq'k)s(\xi))^{-1/(4dq'2L)}\},
\]
\[
|\omega_k(\xi)| \leq CN_{q,\gamma}((\Phi(2^dq'k)s(\xi))^{-1/(4dq'2L)} + (\Phi(2^dq'k)s(\xi))^{1/(4dq'2L)}).
\]

We get from (2.35) that
\[
M_{h,q,\gamma}(f) \leq g(f) + \Theta^*(|f|),
\]
\[
\omega^*(f) \leq M_{h,q,\gamma}(f) + \Theta^*(|f|),
\]
where \(\omega^*(f) = \sup_{k \in \mathbb{Z}} |\omega_k| * |f|, g(f) = (\sum_{k \in \mathbb{Z}} |\omega_k * f|^2)^{1/2}\) and \(\Theta^*(f) = \sup_{k \in \mathbb{Z}} |\Theta_k| * |f|\) with \(\Theta_k = \int_{2^dq'k} |\sigma_{h,t}|(0)(dt/t)\psi_k\). It follows from (2.5) and (2.24) that
\[
\|\Theta^*(f)\|_{L^p(\mathbb{R}^n)} \leq CN_{q,\gamma} \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty,
\]
where \(C\) is independent of \(q\) and \(\gamma\). By (2.38)–(2.42), the trivial estimate \(\sup_{k \in \mathbb{Z}} \|\omega_k\| \leq CN_{q,\gamma}\) and the same arguments as in getting (2.14), we obtain (2.15). This completes the proof of Lemma 2.4.

Applying Lemma 2.4, we obtain the following result.

**Lemma 2.5.** Let \(\Omega, \Phi\) be as in Lemma 2.3 and \(h \in \Delta_{\gamma}(\mathbb{R}^+)\) for some \(\gamma \in (1, 2]\). Then there exists \(C > 0\) such that
\[
\left\|\left(\sum_{k \in \mathbb{Z}} \int_{2^dq'k} \left|\sigma_{h,t} * g_k\right|^2 \frac{dt}{t}\right)^{1/2}\right\|_{L^p(\mathbb{R}^n)} 
\leq C(q', \gamma')^{-1/2} \|h\|_{\Delta_{\gamma}(\mathbb{R}^+)} \|\Omega\|_{L^p(\mathbb{S})} \left\|\left(\sum_{k \in \mathbb{Z}} |g_k|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^n)}, \quad 2 \leq p < \infty, \quad (2.43)
\]
\[
\left\|\left(\sum_{k \in \mathbb{Z}} \int_{2^dq'k} \left|\sigma_{h,t} * g_k\right|^2 \frac{dt}{t}\right)^{1/2}\right\|_{L^p(\mathbb{R}^n)} 
\leq Cq' \gamma' \|h\|_{\Delta_{\gamma}(\mathbb{R}^+)} \|\Omega\|_{L^p(\mathbb{S})} \left\|\left(\sum_{k \in \mathbb{Z}} |g_k|^2\right)^{1/2}\right\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < 2. \quad (2.44)
\]

The constant \(C\) is independent of \(h, \Omega, q, \gamma,\) but depends on \(\Phi\).

**Proof.** The idea of the proof is similar to the one appearing in the proof of [1, Lemma 3.7]. First we prove (2.43). For fixed \(2 \leq p < \infty\), by duality, there exists a nonnegative function \(f \in L^{p/(2')}((\mathbb{R}^n))\) with \(\|f\|_{L^{p/(2')}((\mathbb{R}^n))} \leq 1\) such that
\[
\left\|\left(\sum_{k \in \mathbb{Z}} \int_{2^dq'k} \left|\sigma_{h,t} * g_k\right|^2 \frac{dt}{t}\right)^{1/2}\right\|_{L^p(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} \sum_{k \in \mathbb{Z}} \int_{2^dq'k} \left|\sigma_{h,t} * g_k\right|^2 \frac{dt}{t} f(x) \ dx. \quad (2.45)
\]
By a change of variable and Hölder’s inequality, we obtain
\[
|σ_{h,t} * g_k(x)|^2 \leq \left( \int_{t/2 < r(y) \leq t} \frac{|h(r(y))Ω(y)|}{r(y)^a} |g_k(x - A_Φ(r(y))y')| \, dy \right)^2 \leq \left( \int_{t/2}^t \int_\Sigma |g_k(x - A_Φ(u)y')| |Ω(y')| \, dσ(y') \, |h(u)| \, \frac{du}{u} \right)^2 \leq C |h|^\gamma_{Δ_r}\|Ω\|_{L^p(Σ)} \left( \int_{t/2}^t \int_Σ |g_k(x - A_Φ(u)y')|^2 |Ω(y')| \, dσ(y') \, |h(u)|^{2-γ} \frac{du}{u} \right). \tag{2.46}
\]

Thus, by (2.45), (2.46) and Hölder’s inequality, one can check that
\[
\left\| \left( \sum_{k ∈ ℤ} \int_{2^q r_k}^{2^{q'} r_k} |σ_{h,t} * g_k|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(ℝ^n)} \leq C |h|^\gamma_{Δ_r} \|Ω\|_{L^p(Σ)} \int_{ℝ^n} \sum_{k ∈ ℤ} |g_k(x)|^2 \tilde{M}_{[|h|^{2-γ},q,γ]}(\tilde{f})(-x) \, dx \leq C |h|^\gamma_{Δ_r} \|Ω\|_{L^p(Σ)} \left\| \left( \sum_{k ∈ ℤ} |g_k|^2 \right)^{1/2} \right\|_{L^p(ℝ^n)} \|\tilde{M}_{[|h|^{2-γ},q,γ]}(\tilde{f})\|_{L^{p/2'}(ℝ^n)}, \tag{2.47}
\]

where \( \tilde{f}(x) = f(-x) \) and \( \tilde{M}_{[|h|^{2-γ},q,γ]} \) denotes \( M_{[|h|^{2-γ},q,γ]} \) with \( q = 1 \). It is easy to check that \( |h(·)|^{2-γ} \in Δ_{γ/(2-γ)}(ℝ^n) \); thus, by (2.15),
\[
\|\tilde{M}_{[|h|^{2-γ},q,γ]}(\tilde{f})\|_{L^{p/2'}(ℝ^n)} \leq C q' \left( \frac{γ}{2 - γ} \right)' |h|^{2-γ} \|Ω\|_{L^p(Σ)} \|f\|_{L^{p/2}(ℝ^n)} \leq C q' γ' |h|^{2-γ} \|Ω\|_{L^p(Σ)} ,
\]

which, combined with (2.47), implies (2.43).

Next, we prove (2.44). Assume that \( 1 < p < 2 \); by duality, there exist functions \( \{f_k(x,t)\} \) defined on \( ℝ^n × ℤ^+ \) with \( \|f_k(·,·)\|_{L^{p',p}(L^{2,2})(2^{q' r_k},2^{(q'+1) r_k},dt)(t))} \leq 1 \) such that
\[
\left\| \left( \sum_{k ∈ ℤ} \int_{2^{q' r_k}}^{2^{q'} (r_k+1)} |σ_{h,t} * g_k|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^p(ℝ^n)} \leq \int_{ℝ^n} \sum_{k ∈ ℤ} \int_{2^{q' r_k}}^{2^{q'} (r_k+1)} |σ_{h,t} * g_k|^2 \frac{dt}{t} \, dx \leq C (q' γ')^{1/2} \left\| \left( \sum_{k ∈ ℤ} |g_k|^2 \right)^{1/2} \right\|_{L^{p'}(ℝ^n)} \|H\|_{L^{p/2}(ℝ^n)}^{1/2}, \tag{2.48}
\]

where
\[
H(x) = \sum_{k ∈ ℤ} \int_{2^{q' r_k}}^{2^{q'} (r_k+1)} |σ_{h,t} * \tilde{f}_k(x,t)|^2 \frac{dt}{t} \quad \text{and} \quad \tilde{f}_k(x,t) = f(-x,t).
\]
Since \( p' > 2 \), there exists a nonnegative function \( u \in L^{(p' / 2)'}(\mathbb{R}^n) \) such that
\[
\|H\|_{L^{p'/2}(\mathbb{R}^n)} = \sum_{k \in \mathbb{Z}^n} \int_{\mathbb{R}^n} |\sigma_{h,t} \ast \tilde{f}_k(x,t)|^2 \frac{dt}{t} u(x) \, dx.
\]
By (2.14), Hölder’s inequality and the fact that \( |h(\cdot)|^{2-\gamma} \in \Delta_{\gamma/(2-\gamma)}(\mathbb{R}^+) \),
\[
\|H\|_{L^{p'/2}(\mathbb{R}^n)} \leq C \|h\|_{\Delta_{\gamma}(\mathbb{R}^+)}^2 \|\Omega\|_{L^\infty(\Sigma)} \int_{\mathbb{R}^n} \tilde{\sigma}^{\ast}_{|h|^{2-\gamma}}(\tilde{u})(x) \left( \sum_{k \in \mathbb{Z}^n} \int_{2^{k-1} < r(y) \leq 2^k} \|\tilde{f}_k(x,t)\|^2 \, dx \right) \frac{dt}{t}
\]
\[
\leq C \|h\|_{\Delta_{\gamma}(\mathbb{R}^+)} \|\Omega\|_{L^\infty(\Sigma)} \left( \sum_{k \in \mathbb{Z}^n} \int_{2^{k-1} < r(y) \leq 2^k} \Omega(y) h(r(y)) \frac{1}{r(y)^{n-p}} f(x-A_{\Phi(r(y))}y') \, dy \right) \left( \sum_{k \in \mathbb{Z}^n} \int_{2^{k-1} < r(y) \leq 2^k} \|\tilde{f}_k(x,t)\|^2 \, dx \right) \frac{dt}{t}
\]
\[
\leq C q' \left( \frac{\gamma}{2-\gamma} \right) \|\Omega\|_{L^\infty(\Sigma)} \|\Omega\|^2 \|\Omega\|^2 \|\Omega\|^2 \|\Omega\|^2 \|\Omega\|^2.
\]
where \( \tilde{u}(x) = u(-x) \) and \( \tilde{\sigma}^{\ast}_{|h|^{2-\gamma}}(\tilde{u}) \) denotes \( \sigma^{\ast}_{|h|^{2-\gamma}}(\tilde{u}) \) with \( \varphi = 1 \). Equation (2.44) follows from (2.48) and (2.49). This proves Lemma 2.5. \( \square \)

3. Proof of main results

This section is devoted to the proofs of the main results.

**Proof of Theorem 1.1.** Let \( h, \Omega, \Phi \) be as in Theorem 1.2. By Minkowski’s inequality, we can write
\[
M_{h,\Omega,\varphi}(f)(x) = \left( \int_0^\infty \sum_{k=-\infty}^\infty \frac{1}{|t|} \int_{2^{k-1} < r(y) \leq 2^k} \Omega(y) h(r(y)) \frac{1}{r(y)^{n-p}} f(x-A_{\Phi(r(y))}y') \, dy \right) \left( \sum_{k=-\infty}^\infty \int_{2^{k-1} < r(y) \leq 2^k} \|\tilde{f}_k(x,t)\|^2 \, dx \right) \frac{dt}{t}.
\]
\[
\leq \sum_{k=-\infty}^\infty \left( \int_0^\infty \frac{1}{|t|} \int_{2^{k-1} < r(y) \leq 2^k} \Omega(y) h(r(y)) \frac{1}{r(y)^{n-p}} f(x-A_{\Phi(r(y))}y') \, dy \right) \left( \sum_{k=-\infty}^\infty \int_{2^{k-1} < r(y) \leq 2^k} \|\tilde{f}_k(x,t)\|^2 \, dx \right) \frac{dt}{t}.
\]
\[
\leq (1-\varphi^{-1})^{-1} \left( \int_0^\infty \|\sigma_{h,t} \ast f(x)\|^2 \frac{dt}{t} \right)^{1/2}.
\]
Let
\[
S_{h,\Omega,\varphi}(f)(x) := \left( \int_0^\infty \|\sigma_{h,t} \ast f(x)\|^2 \frac{dt}{t} \right)^{1/2}.
\]
By Lemma 2.3 and (2.3)–(2.4), one can verify that
\[
\left( \int_{2^{k-1} < r(y) \leq 2^k} \|\sigma_{h,t}(\xi)\|^2 \frac{dt}{t} \right)^{1/2}
\]
\[
\leq \frac{C(\gamma-1)^{-1/2}(q-1)^{-1/2}}{\Delta_{\gamma}(\mathbb{R}^+)} \|\Omega\|_{L^\infty(\Sigma)} \times \min\{1,(\Phi(2^{\gamma'}k)s(\xi))^{1/(4\gamma' a_1L)} + (\Phi(2^{\gamma'}k)s(\xi))^{1/(4\gamma' b_1L)}\},
\]
\[
(\Phi(2^{\gamma'}k)s(\xi))^{-1/(4\gamma' a_1L)} + (\Phi(2^{\gamma'}k)s(\xi))^{-1/(4\gamma' b_2L)}\}.
\]

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Let \( S_k \) be as in (2.29). Then, by Minkowski’s inequality and the definition of \( S_k \),
\[
S_{k,\Omega}(f)(x) \leq \sum_{j \in \mathbb{Z}} T_j(f)(x),
\]
where
\[
T_j(f)(x) = \left( \sum_{k \in \mathbb{Z}} \int_{2^j2^k} |\sigma_{h,l} \ast S_{j+k} S_j f(x)|^2 \frac{dt}{t} \right)^{1/2}.
\]

Using (2.30)–(2.31) and invoking Lemma 2.5,
\[
\|T_j(f)\|_{L^p(\mathbb{R}^n)} \leq C(\gamma - 1)^{-1/2} \|\Omega\|_{L^p(\Sigma)} \|f\|_{L^p(\mathbb{R}^n)}, \quad 2 \leq p < \infty,
\]
\[
\|T_j(f)\|_{L^p(\mathbb{R}^n)} \leq C(\gamma - 1)^{1-1} \|\Omega\|_{L^p(\Sigma)} \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < 2.
\]

The constant \( C > 0 \) is independent of \( q \) and \( \gamma \), but depends on \( \Phi \). On the other hand, by Plancherel’s theorem and (3.2),
\[
\|T_j(f)\|_{L^2(\mathbb{R}^n)}^2 \leq \sum_{k \in \mathbb{Z}} \int_{\mathcal{Q}(2^{j}2^{k+1})} \|\Omega\|_{L^2(\Sigma)} \|f\|_{L^2(\mathbb{R}^n)}^2.
\]

where \( B_0 > 1 \) is as in Lemma 2.4. The constants \( C \) and \( \delta \) are independent of \( q \) and \( \gamma \). That is,
\[
\|T_j(f)\|_{L^2(\mathbb{R}^n)} \leq C B_0^{1/2} \|\Omega\|_{L^2(\Sigma)} \|f\|_{L^2(\mathbb{R}^n)}.
\]

Interpolating between (3.4)–(3.5) and (3.6),
\[
\|T_j(f)\|_{L^p(\mathbb{R}^n)} \leq C B_0^{1/2} \|\Omega\|_{L^p(\Sigma)} \|f\|_{L^p(\mathbb{R}^n)}, \quad 2 \leq p < \infty,
\]
\[
\|T_j(f)\|_{L^p(\mathbb{R}^n)} \leq C B_0^{1/2} \|\Omega\|_{L^p(\Sigma)} \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < 2.
\]

The constants \( C, \delta \rho \) and \( \epsilon \rho \) are independent of \( q \) and \( \gamma \). Theorem 1.2 follows from (3.1), (3.3) and (3.7)–(3.8).

PROOF OF THEOREM 1.2. Theorem 1.3 follows directly from Theorem 1.2 and an extrapolation argument as in the proof of [17, Theorem 1.2]) (also see [18, Theorem 1.2]). We omit the details.

4. Additional results

As applications of our main results, we consider the corresponding parametric Marcinkiewicz integral operators \( \mathcal{M}_{h_\lambda,\Phi}\ast \) and \( \mathcal{M}_{h,\Omega,\Phi,S\hat{g}_t}^{\ast} \) related to the Littlewood–Paley \( g_t^\ast\) function and the area integral \( S \), respectively, which are interesting in
themselves. More precisely, let \( \Phi \) be as in (1.2); we define the operators \( \mathcal{M}_{h, \Omega, \Phi, \lambda, \rho} \) and \( \mathcal{M}_{h, \Omega, \Phi, \lambda, \rho}^* \) by

\[
\mathcal{M}_{h, \Omega, \Phi, \lambda, \rho}^*(f)(x) := \left( \int_{\mathbb{R}^{n+1}_+} \left( \frac{t}{1 + r(x - y)} \right)^{\alpha_1} \frac{1}{t^\rho} \int_{r(z) \leq t} \frac{h(r(z)) \Omega(z)}{r(z)^{n-\rho}} f(y - A_\Phi(r(z))z') \, dz \right)^{\frac{1}{2}} dy dt,
\]

where \( \lambda > 0 \) and \( \mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty) \);

\[
\mathcal{M}_{h, \Omega, \Phi, \lambda, \rho}(f)(x) := \left( \int_{\mathbb{R}^{n+1}_+} \left( \frac{t}{1 + r(x - y)} \right)^{\alpha_1} \frac{1}{t^\rho} \int_{r(z) \leq t} \frac{h(r(z)) \Omega(z)}{r(z)^{n-\rho}} f(y - A_\Phi(r(z))z') \, dz \right)^{\frac{1}{2}} dy dt,
\]

where \( \Gamma(x) = \{(y, t) \in \mathbb{R}^{n+1}_+: r(x - y) < t\} \).

**Theorem 4.1.** Let \( \Omega \in L(\log^+ L)^{1/2}(\Sigma) \) satisfying (1.1) and \( h \in N_{1/2}(\mathbb{R}^+) \). Suppose that \( \Phi \in \mathcal{F} \) and \( \lambda > 1 \). Then, for \( 2 \leq p < \infty \),

\[
\begin{align*}
\|\mathcal{M}_{h, \Omega, \Phi, \lambda, \rho}^*(f)\|_{L^p(\mathbb{R}^n)} &\leq C(\lambda, \alpha, \rho, \Phi)(1 + \|\Omega\|_{L(\log^+ L)^{1/2}(\Sigma)})(1 + N_{1/2}(h))\|f\|_{L^p(\mathbb{R}^n)}, \\
\|\mathcal{M}_{h, \Omega, \Phi, \lambda, \rho}(f)\|_{L^p(\mathbb{R}^n)} &\leq C(\alpha, \Phi)(1 + \|\Omega\|_{L(\log^+ L)^{1/2}(\Sigma)})(1 + N_{1/2}(h))\|f\|_{L^p(\mathbb{R}^n)}.
\end{align*}
\]

**Remark 4.2.** Because of (1.3) and (1.4), Theorem 4.1 essentially improves and generalizes [10, Theorem 2], even in the special case \( r(x) = |x| \) and \( \Phi(t) = t \).

The proof of Theorem 4.1 is based on the following lemma.

**Lemma 4.3.** Let \( \lambda > 1 \). Then there exists a constant \( C(\lambda, \alpha) \) such that for any nonnegative locally integrable function \( g \) on \( \mathbb{R}^n \),

\[
\int_{\mathbb{R}^n} (\mathcal{M}_{h, \Omega, \Phi, \lambda, \rho}^*(f)(x))^2 g(x) \, dx \leq C(\lambda, n) \int_{\mathbb{R}^n} (\mathcal{M}_{h, \Omega, \Phi, \lambda, \rho}(f)(x))^2 M(g)(x) \, dx,
\]

where \( M \) is the Hardy–Littlewood maximal operator on \( \mathbb{R}^n \) with respect to the function \( r(\cdot) \).

**Proof.** By the definition of \( \mathcal{M}_{h, \Omega, \Phi, \lambda, \rho}^* \),

\[
\int_{\mathbb{R}^n} (\mathcal{M}_{h, \Omega, \Phi, \lambda, \rho}^*(f)(x))^2 g(x) \, dx
\]

\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n+1}_+} \left( \frac{t}{1 + r(x - y)} \right)^{\alpha_1} \frac{1}{t^\rho} \int_{r(z) \leq t} \frac{h(r(z)) \Omega(z)}{r(z)^{n-\rho}} f(y - A_\Phi(r(z))z') \, dz \right)^{\frac{1}{2}} dy dt \, g(x) \, dx
\]

\[
\leq \int_{\mathbb{R}^n} \int_0^\alpha \frac{1}{t^\rho} \int_{r(z) \leq t} \frac{h(r(z)) \Omega(z)}{r(z)^{n-\rho}} f(y - z) \, dz \right)^{\frac{1}{2}} dy \times \left( \sup_{t > 0} \frac{1}{t^\rho} \int_{r(z) \leq t} \left( \frac{t}{1 + r(x - y)} \right)^{\alpha_1} g(x) \, dx \right) \frac{dt}{t} \, dy
\]

\[
\leq C(\lambda, \alpha) \int_{\mathbb{R}^n} (\mathcal{M}_{h, \Omega, \Phi, \lambda, \rho}(f)(y))^2 M(g)(y) \, dy
\]

for \( \lambda > 1 \). This proves Lemma 4.1. \( \square \)
**Proof of Theorem 4.1.** First we prove (4.1). For $2 \leq p < \infty$, by duality,

$$\| \mathcal{M}_{h,\Omega,\Phi,\lambda,\varrho}^*(f) \|_{L^p(\mathbb{R}^n)^2} = \sup_{\|g\|_{L^q(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} (\mathcal{M}_{h,\Omega,\Phi,\lambda,\varrho}^*(f)(x))^2 g(x) \, dx,$$

where $q = (p/2)'$ and the supremum is taken over all $g$ satisfying $\|g\|_{L^q(\mathbb{R}^n)} \leq 1$. By the $L^p$ bounds of $M$, Hölder’s inequality, Lemma 4.3 and Theorem 1.3, t

$$\| \mathcal{M}_{h,\Omega,\Phi,\lambda,\varrho}^*(f) \|_{L^p(\mathbb{R}^n)^2} \leq C(\lambda, \alpha) \sup_{\|g\|_{L^q(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} (\mathcal{M}_{h,\Omega,\Phi,\lambda,\varrho}^*(f)(x))^2 M(g)(x) \, dx \leq C(\lambda, \alpha) \| \mathcal{M}_{h,\Omega,\Phi,\lambda,\varrho}^*(f) \|_{L^p(\mathbb{R}^n)^2} \leq C(\lambda, \alpha, \varrho, \Phi)(1 + \|\Omega\|_{L(\log^+ L)^{1/2}(\Sigma)})^2 \times (1 + N_{1/2}(h))^2 \|f\|_{L^p(\mathbb{R}^n)}, \quad 2 \leq p < \infty.$$

Thus, (4.1) holds. On the other hand, it is easy to check that

$$\mathcal{M}_{h,\Omega,\Phi,\lambda,\varrho}(f)(x) \leq 2^{\alpha \lambda/2} \mathcal{M}_{h,\Omega,\Phi,\lambda,\varrho}^*(f)(x),$$

which, combined with (4.1), implies (4.2). Theorem 4.1 is proved. □

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Feng Liu and Suzhen Mao


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Feng Liu, College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, Shandong 266590, China
e-mail: liufeng860314@163.com

Suzhen Mao, School of Mathematical Sciences, Xiamen University, Xiamen, Fujian 361005, China
e-mail: suzhen.860606@163.com