COMPACT ACTIONS ON C*-ALGEBRAS

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1. Introduction. In Section 33 of [**2**], Bonsall and Duncan define an element t of a Banach algebra $\mathcal{A}$ to *act compactly* on $\mathcal{A}$ if the map $a \rightarrow tat$ is a compact operator on $\mathcal{A}$. In this paper, the arguments and technique of [**1**] are used to study this question for C*-algebras (see also [**10**]). We determine the elements b of a C*-algebra $\mathcal{A}$ for which the maps $a \rightarrow ba$, $a \rightarrow ab$, $a \rightarrow ab+ba$, $a \rightarrow bab$ are compact (respectively weakly compact), determine the C*-algebras which are compact in the sense of Definition 9, of [**2**, p. 177] and give a characterization of the *-automorphisms of $\mathcal{A}$ which are weakly compact perturbations of the identity.

We introduce the notation which will be used in the sequel. If H is a Hilbert space, $B(H)$ and $K(H)$ denote respectively the W*-algebra of all bounded operators on H and the C*-algebra of all compact operators on H. A C*-algebra $\mathcal{A}$ is said to *act atomically on a Hilbert space H* if there exists an orthogonal family $\{P_\alpha\}$ of projections in $B(H)$, each commuting with $\mathcal{A}$, such that $\bigoplus_\alpha P_\alpha$ is the identity operator on H, $\mathcal{A}P_\alpha$, acts irreducibly on $P_\alpha(H)$, and $\mathcal{A}P_\alpha$ is not unitarily equivalent to $\mathcal{A}P_\beta$ for $\alpha \neq \beta$.

If $\{\mathcal{A}_\lambda : \lambda \in \Lambda\}$ is a family of C*-algebras, the C*-*direct sum* $\bigoplus_\lambda \mathcal{A}_\lambda$ *of the* $\mathcal{A}_\lambda$'s is the C*-algebra of all functions $f(\lambda) \in \mathcal{A}_\lambda$, $\lambda \in \Lambda$, with

$$\|f\| = \sup\{\|f(\lambda)\| : \lambda \in \Lambda\} < \infty,$$

equipped with pointwise operations. The *restricted* C*-*direct sum* $\hat{\bigoplus}_\lambda \mathcal{A}_\lambda$ is the C*-subalgebra of $\bigoplus_\lambda \mathcal{A}_\lambda$ consisting of all functions f with $\{\lambda : \|f(\lambda)\| \geqq \varepsilon\}$ finite for all $\varepsilon > 0$.

A projection p of a C*-algebra $\mathcal{A}$ is said to be *finite-dimensional* if $p\mathcal{A}p$ is finite-dimensional. A C*-algebra is said to be of *elementary type* if it is isomorphic to $K(H)$ for some Hilbert space H.

By an *ideal* of a C*-algebra, we will always mean a uniformly closed, two-sided ideal.

2. The results. We begin with several propositions that determine the operators which act compactly (respectively weak compactly) on $B(H)$. Throughout, H always denotes a (complex) Hilbert space.

2.1. Proposition. *Let* $\Phi : B(H) \rightarrow B(H)$ *be a bounded linear map which is continuous in the ultraweak operator topology, and maps* $K(H)$ *into* $K(H)$. *Then* $\varphi = (\varphi|_{K(H)})^{**}$, *and* φ *is weakly compact if and only if* $\varphi(B(H)) \subseteq K(H)$.

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Proof. Note first that $(\varphi|_{K(H)})^{**}: B(H)\to B(H)$ is ultraweakly continuous and agrees with the ultraweakly continuous map φ on the ultraweakly dense set $K(H)\subseteq B(H)$, whence $\varphi=(\varphi|_{K(H)})^{**}$.

Now assume φ is weakly compact. Then $(\varphi|_{K(H)})^{**}=\varphi$ is weakly compact, whence $\varphi|_{K(H)}$ is weakly compact (Theorem 8, [**4**, p. 485] whence $\varphi(B(H))\subseteq$ norm-closure of $\varphi(K(H))\subseteq K(H)$ (Theorem 2, [**4**, p. 482].

Conversely, assume that $\varphi(B(H))\subseteq K(H)$. Let $K(H)_1$ and $B(H)_1$ denote the closed unit balls of $K(H)$ and $B(H)$, respectively. It follows by ultraweak compactness of $B(H)_1$ and ultraweak continuity of φ that the weak closure of $\varphi(K(H)_1)$ is $\varphi(B(H)_1)$, and this set is $\sigma(K(H)^{**}, K(H)^{*})$-compact. Thus, since $\varphi(B(H)_1)\subseteq K(H)$, and the $\sigma(K(H)^{**}, K(H)^{*})$-topology when restricted to $K(H)$ is the weak topology on $K(H)$, we conclude that the weak closure of $\varphi(K(H)_1)$ is weakly compact. Q.E.D.

2.2 Proposition. *Let b be a nonzero element of $B(H)$ such that any one of the maps*

$$a\to ab,\quad a\to ba,\quad a\to ab+ba,\quad (a\in B(H))$$

is compact (respectively weakly compact). Then $\dim H<\infty$ *(respectively $b\in K(H)$).*

Proof. The "compact" statement is immediate from [**11**], the "weakly compact" statement is immediate from Proposition 2.1.

2.3. Proposition. *If $b, c\in B(H)$ are both nonzero, then the map $a\to bac$ is weakly compact if and only if either b or c is in $K(H)$, and it is compact if and only if both b and c are in $K(H)$.*

Proof. Assume $b, c\notin K(H)$. Then by Corollary 5.10 of [**3**], the ranges of b and c contain closed, infinite-dimensional subspaces. Hence there exists an $a\in B(H)$ which maps a closed, infinite-dimensional subspace of the range of c onto a subspace M of H for which $b(M)$ contains a closed, infinite-dimensional subspace. Thus the range of bac contains a closed, infinite-dimensional subspace, and so by [**3**], Corollary 5.10, $bac\notin K(H)$. Thus by Proposition 2.1 $a\to bac$ is not weakly compact.

Suppose $b\in K(H)$. Then $bac\in K(H)$ for $a\in B(H)$, so that by Proposition 2.1 $a\to bac$ is weakly compact.

The statement about compact $a\to bac$ is a special case of Theorem 3, p. 174 and Corollary 5, p. 175 of [**2**]. The proof is complete.

2.4. Lemma. *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of Banach spaces with compact (respectively weakly compact) maps $\varphi_n: X_n\to X_n$ of uniformly bounded norm. Then $\bigoplus_n \varphi_n:\bigoplus_n X_n\to\bigoplus_n X_n$ is compact (respectively weakly compact) if and only if $\lim_n\|\varphi_n\|=0$. ($\bigoplus_n X_n$ denotes the l_∞-direct sum of $\{X_n\}$.)*

Proof. We need only verify the weakly compact case, the compact case being an immediate corollary. Suppose with no loss of generality that $\sup_n\|\varphi_n\|=1$. Assume the lemma is false. Since the compression of a weakly compact map to a subspace by a

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continuous projection onto that subspace is weakly compact, we may thus find an $x=(x_n)\in\bigoplus_n X_n$ of norm 1 and a $\delta>0$ such that $\|\varphi_n(x_n)\|>\delta$ for all n. Let $M=\hat{\bigoplus}_n X_n$. Since $\varphi(x)\notin M$, there is an $f\in X^*$ such that $f(\varphi(x))=1$ and f vanishes on M.

Define a sequence $\{y_k=(y_n^{(k)})\}\subseteq\bigoplus_n X_n$ by

$$y_n^{(k)}=\begin{cases}0, (n<k),\\ x_n, (n\geq k).\end{cases}$$

Let S denote the l_1-direct sum of $\{X_n^*\}$. With S actong on $X=\bigoplus_n X_n$ in the natural way, we have $S\subseteq X^*$, and since $\{\|\varphi_n\|\}$ is uniformly bounded, $\varphi(y_k)\to 0$ in the $\sigma(X, S)$-topology. Since the $\sigma(X, S)$-topology is Hausdorff and weaker than the weak topology on X, we conclude by weak compactness of φ that $\varphi(y_k)\to 0$ weakly, after perhaps passing to a subsequence and reindexing. But $\varphi(y_k)-\varphi(x)\in M$, for all k, and so by the choice of f, $f(\varphi(y_k))=f(\varphi(x))=1$ for all k, a contradiction. QED

The next result determines the elements of a C*-algebra which act compactly (respectively weak compactly).

2.5. THEOREM. *Let b be a nonzero element of a C*-algebra $\mathcal{A}$. Any one of the maps*

$$a\to ab,\quad a\to ba,\quad a\to ab+ba,\quad (a\in\mathcal{A}) \tag{1}$$

is compact if and only if there exists an orthogonal sequence $\{p_n\}$ of minimal, finite-dimensional, central projections of $\mathcal{A}$ with $b\in\hat{\bigoplus}_n \mathcal{A}p_n$.

Any one of the maps (1) is weakly compact if and only if there exists a sequence $\{I_n\}$ of orthogonal ideals of $\mathcal{A}$ such that each I_n is of elementary type and $b\in\hat{\bigoplus}_n I_n$.

Proof. We may pass to the reduced atomic representation of $\mathcal{A}$([**6**, p. 35] and may hence assume with no loss of generality that $\mathcal{A}$ acts atomically on a Hilbert space $H=\bigoplus_\alpha H_\alpha$. Let $\mathcal{A}^-$ denote the closure of $\mathcal{A}$ in the weak operator topology. We have $\mathcal{A}^-=\bigoplus_\alpha B(H_\alpha)$ by Corollary 4 of [**5**]. Let $q_\alpha=$ the projection of H onto H_α.

Assume, for instance, that $a\to ab+ba$ is weakly compact. Arguing as in the proof of Theorem 3.3 of [**1**], we deduce that $a\to ab+ba$ is weakly compact on $\mathcal{A}^-$ and $\{xb+bx: x\in\mathcal{A}^-\}\subseteq\mathcal{A}$. If $b=\bigoplus_\alpha b_\alpha\in\bigoplus_\alpha B(H_\alpha)$, it follows by the proof of Lemma 3.2 of [**1**] and the fact that the norm of $a\to ab+ba$ is $2\|b\|$ that all but a countable number of the b_α's, say $b_{\alpha_n}=b_n$, are zero, and $\lim_n\|b_n\|=0$ by Lemma 2.4. By Proposition 2.2, $b_n\in K(H_{\alpha_n})=K_n$, and so $\mathcal{A}\cap K_n$ is a nonzero ideal of $\mathcal{A}q_{\alpha_n}\supseteq K_n$, hence a nonzero ideal of K_n. Thus $K_n\subseteq\mathcal{A}$. We conclude that $b=\bigoplus_n b_n\in\hat{\bigoplus}_n K_n$, and the desired result follows.

If $b = \bigoplus_n b_n \in \hat{\bigoplus_n} I_n$ with I_n an ideal of elementary type, by Corollary 4, [**4**, p. 483], it suffices to show that $a \to ab_n + b_n a$, $a \in \mathcal{A}$, is weakly compact for each n. Suppressing the n's, we may assume with no loss of generality that $b \in I$ is nonnegative. By Proposition 2.3, $a \to ab^{1/2}$ and $a \to b^{1/2}a$ are both weakly compact on I, and so $a \to ab + ba = (ab^{1/2})b^{1/2} + b^{1/2}(b^{1/2}a)$ is weakly compact on $\mathcal{A}$ since $b^{1/2}a$ and $ab^{1/2}$ are in I for all $a \in \mathcal{A}$. Similar arguments prove the other statements, and so the proof is complete.

In [**2**] (Definition 9, p. 177), Bonsall and Duncan call a Banach algebra $\mathcal{A}$ *compact* if for each $b \in \mathcal{A}$, the map $a \to bab$ is compact. They show that the Banach algebra of compact operators on a Banach space is compact (Theorem 3(i), [**2**, p. 177]). Proposition 2.3 and the proof of Theorem 2.5 show that an element b of a C*-algebra $\mathcal{A}$ induces a compact map $a \to bab$ if and only if $a \to bab$ is weakly compact, which happens if and only if b is of the form given in the second part of Theorem 2.5. Hence we immediately deduce the following corollary, which determines the C*-algebras compact in the above sense and which improves on some results of [**10**] (see also [**7**]).

2.6. Corollary. *Let $\mathcal{A}$ be a C*-algebra. The following are equivalent.*

(1) *$\mathcal{A}$ is compact in the sense of Bonsall and Duncan.*

(2) *The map $a \to bab$, $a \in \mathcal{A}$ is weakly compact for each $b \in \mathcal{A}$.*

(3) *$\mathcal{A}$ is isomorphic to the restricted direct sum of a family of C*-algebras of elementary type.*

Moreover, at least one of the maps $a \to ab$, $a \to ba$, $a \to ab + ba$, $(a \in \mathcal{A})$ is compact for each $b \in \mathcal{A}$ if and only if $\mathcal{A}$ is isomorphic to the restricted direct sum of a family of finite-dimensional full matrix algebras.

The next results characterize the *-automorphisms of a C*-algebra which are weakly compact perturbations of the identity, but before we state and prove them, the following proposition is needed.

2.7. Proposition. *If $u(\neq 1)$ is a unitary operator in $B(H)$, then $a \to uau^* - a$ is compact (respectively weakly compact) if and only if $\dim H < \infty$ (respectively $(u + \lambda 1) \in K(H)$ for some complex number λ).*

Proof. The map $a \to uau^* - a$ is compact (respectively weakly compact) if and only if the map $a \to ua - au$ is the same, since $b \to bu$ is an isometry of $B(H)$ onto itself. The map $a \to ua - au$ is compact (respectively weakly compact) if and only if $\dim H < \infty$ (respectively $(u + \lambda 1) \in K(H)$ for some complex number λ) by Lemma 2.1 and Theorem 3.1 of [**1**].

If $\mathcal{A}$ is a C*-algebra, Aut$(\mathcal{A})$ will denote the group of *-automorphisms of $\mathcal{A}$, $\pi = \bigoplus_\gamma \pi_\gamma : \mathcal{A} \to B(H_\pi)$ the reduced atomic representation of $\mathcal{A}$. Let $\mathcal{A}_\pi = \pi(\mathcal{A})$, $\mathcal{A}_\pi^- =$ the closure of $\mathcal{A}_\pi$ in the weak operator topology in $B(H_\pi)$. We have $H_\pi = \bigoplus_\gamma H_\gamma$ where H_γ is the representation space of π_γ, and $\mathcal{A}_n^- = \bigoplus_\gamma B(H_\gamma)$. Let $p_\gamma =$ the projection of H_π onto H_γ, $K_\gamma = K(H_\gamma)$. If $\alpha \in$ Aut$(\mathcal{A})$, α_π denotes the *-automorphism of $\mathcal{A}_\pi$ induced by α.

The following two theorems together determine the structure of *-automorphisms of $\mathcal{A}$ which are weakly compact perturbations of the identity automorphism (denoted *id* in the sequel) on $\mathcal{A}$.

2.8. THEOREM. *Let $\mathcal{A}$ be a C*-algebra, $\alpha \in \mathrm{Aut}(\mathcal{A})$. The following are equivalent.*

(1) $\alpha - id$ is weakly compact.

(2) There is a finite-dimensional central projection p of $\mathcal{A}$, an automorphism α_1 of $p\mathcal{A}$, and an automophism α_2 of $(1-p)\mathcal{A}$ such that $\alpha_2 - id$ is weakly compact, α_2 fixes each central element of $(1-p)\mathcal{A}$ and $\alpha = \alpha_1 \oplus \alpha_2$.

Proof. (1)⇒(2). Let $\mathcal{A}^{**}$ denote the enveloping von Neumann algebra of $\mathcal{A}$. If $\sigma = \oplus\{\pi_f : f$ a state on $\mathcal{A}\}$ denotes the universal representation of $\mathcal{A}$, then by Theorem 1.17.2 of [**9**], $\mathcal{A}^{**}$ can be naturally identified with the closure $\sigma(\mathcal{A})^-$ of $\sigma(\mathcal{A})$ in the weak operator topology.

Let $\alpha \in \mathrm{Aut}(\mathcal{A})$. Since α^{**} is a *-automorphism of $\mathcal{A}^{**}$ onto $\mathcal{A}^{**}$, it maps minimal projections onto minimal projections, and, identifying $\mathcal{A}_\pi^-$ in a natural way with the subalgebra of $\mathcal{A}^{**}$ generated by the minimal projections ([**9**, p. 53]), it therefore follows that $\alpha^{**}(\mathcal{A}_\pi^-) \subseteq \mathcal{A}_\pi^-$. Now assume $\alpha - id$ is weakly compact.

Since α^{**} maps minimal central projections onto minimal central projections, it follows that α^{**} permutes the p_γ's in $\mathcal{A}_\pi^- = \bigoplus_\gamma B(H_\gamma)$. Suppose that for $\gamma \neq \lambda$, $\alpha^{**}(p_\gamma) = p_\lambda$. Consider the map $\varphi : p_\gamma \mathcal{A}_\pi^- \to p_\gamma \mathcal{A}_\pi^-$ defined by $\varphi : a \to p_\gamma(\alpha^{**}(a) - a)$. Since $\alpha^{**}(a) \in p_\lambda \mathcal{A}_\pi^-$, $p_\gamma(\mathcal{A}^{**}(a)) = 0$, whence $\varphi = -id$ on $p_\gamma \mathcal{A}_\pi^-$. Since φ is the composition of a bounded map and the weakly compact map $a \to \alpha^{**}(a) - a$, we conclude by Theorem 5, (**4**, p. 484], that φ is weakly compact, whence $p_\gamma \mathcal{A}_\pi^-$ is reflexive, hence finite-dimensional (Proposition 2 of [**8**]). Thus α^{**} can only permute finite-dimensional p_γ's, and it follows by the weak compactness of $\alpha^{**} - id$ and Lemma 2.4 that α^{**} permutes only a finite number of them.

We want to show next that each p_γ permuted by α^{**} is in fact in $\mathcal{A}$. Let p be such a projection, and let $a \in \mathcal{A}$. Then since $p\alpha^{**}(p) = 0$ and p is central,

$$2ap + (\alpha^{**}(ap) - ap) = \alpha^{**}(ap) + ap = (\alpha^{**}(ap) - ap)(\alpha^{**}(p) - p). \tag{1}$$

But by Theorem 2, [**4**, p. 482] $(\alpha^{**} - id)(\mathcal{A}^{**}) \subseteq \mathcal{A}$. Thus by (1), $ap \in \mathcal{A}$, and so $p\mathcal{A} = \mathcal{A}p \subseteq \mathcal{A}$. Now define $\varphi : a \to p(\alpha^{**}(a) - a)$ as before. Then

$$-p = \varphi(p) \in \varphi(\mathcal{A}^{**}) \subseteq p(\alpha^{**} - id)(\mathcal{A}^{**}) \subseteq p\mathcal{A} \subseteq \mathcal{A}.$$

Setting P equal to the sum of all the p_γ's permuted by α^{**}, we conclude that P is a finite-dimensional central projection in $\mathcal{A}$.

Writing $\mathcal{A}^{**} = \mathcal{A}^{**}P \oplus \mathcal{A}^{**}(1-P)$, we have $\alpha^{**} = \alpha^{**}|_{\mathcal{A}^{**}P} \oplus \alpha^{**}|_{a^{**}(1-P)}$ (notice that $\alpha^{**}(P) = P$). Since the center of $\mathcal{A}_\pi^-(1-P)$ is purely atomic and $\alpha^{**}|_{\mathcal{A}_\pi^-(1-P)}$ fixes each atom, it follows that $\alpha^{**}|_{\mathcal{A}_\pi^-(1-P)}$ fixes each central element of $\mathcal{A}_\pi^-(1-P)$. Since $\alpha^{**}|_{\mathcal{A}} = \alpha$, setting $\alpha_1 = \alpha^{**}|_{\mathcal{A}P}$, $\alpha_2 = \alpha^{**}|_{\mathcal{A}(1-P)}$ gives the desired decomposition ofα.

(2)⇒(1). This is clear, and so the proof is complete.

2.9. THEOREM. *Let $\mathcal{A}$ be a C*-algebra, $\alpha \in \mathrm{Aut}(\mathcal{A})$. The following are equivalent.*

(1) $\alpha - id$ is weakly compact and α fixes each central element of $\mathcal{A}$.

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(2) *α_π extends to an inner automorphism $\tilde{\alpha}_\pi$ of $\mathcal{A}_\pi^-$ of the following form: there exists a countable set of indices $\{\gamma_n\}$, unitaries $u_n \in B(H_{\gamma_n})$, and complex numbers $\{z_n\}$ such that (if p_γ is the identity in $B(H_\gamma)$)*

(i) $u_n - z_n p_n \in K_{\gamma_n} \subseteq \mathcal{A}_\pi$ *(where* $p_n = p_{\gamma_n}$*),*

(ii) $\lim_n \|u_n - z_n p_n\| = 0$,

(iii) $\tilde{\alpha}_\pi(a) = uau^*$, $(a \in \mathcal{A}_\pi^-)$, *where* $u = (\bigoplus_{\gamma \neq \gamma_n} p_\gamma) \oplus (\bigoplus_n u_n)$.

Proof. (1)⇒(2). We assert first that $\alpha^{**} \in \mathrm{Aut}(\mathcal{A}^{**})$ fixes each central element of $\mathcal{A}^{**}$. By the spectral theorem and $\sigma(\mathcal{A}^{**}, \mathcal{A}^*)$-continuity of α^{**}, it suffices to show that $\alpha^{**}(z) = z$ for each central projection $z \in \mathcal{A}^{**}$. To see this, note first that by Theorem 2, [**4**, p. 482], and the weak compactness of $\alpha^{**} - id$, $\alpha^{**}(z) - z$ is a central element of $\mathcal{A}$. Since $\alpha^{**}|_{\mathcal{A}} = \alpha$ and α fixes each central element of $\mathcal{A}$, $\alpha^{**}(\alpha^{**}(z) - z) = \alpha^{**}(z) - z$, i.e.,

$$(\alpha^{**})^2(z) + z = 2\alpha^{**}(z). \tag{*}$$

Since $(\alpha^{**})^2(z)$, $\alpha^{**}(z)$, and z are projections in an abelian W*-algebra (the center of $\mathcal{A}^{**}$), they can be viewed as characteristic functions of measurable sets (Proposition 1.18.1 of [**9**]), whence by (*), $\alpha^{**}(z) = z$.

Since α^{**} fixes each central element, we can apply the reasoning of the proof of Theorem 2.8 to extend α_π to an automorphism $\tilde{\alpha}_\pi$ of $\mathcal{A}_\pi^-$ such that $\tilde{\alpha}_\pi - id$ is weakly compact and $\tilde{\alpha}_\pi$ fixes each central element of $\mathcal{A}_\pi^-$. It follows that if $\tilde{\alpha}_{\pi,\gamma} = \tilde{\alpha}_\pi|_{B(H_\gamma)}$, then $\tilde{\alpha}_{\pi,\gamma} \in \mathrm{Aut}(B(H_\gamma))$, $\tilde{\alpha}_{\pi,\gamma} - id|_{B(H_\gamma)}$ is weakly compact, and $\tilde{\alpha}_\pi = \bigoplus_\gamma \tilde{\alpha}_{\pi,\gamma}$. By the proof of Lemma 3.2 of [**1**], all but a countable number of the $\tilde{\alpha}_{\pi,\gamma} - id|_{B(H_\gamma)}$'s are nonzero, and if $\{\gamma_n\}$ is the set of the corresponding indices, $\lim_n \|\tilde{\alpha}_{\pi,\gamma_n} - id|_{B(H_{\gamma_n})}\| = 0$ by Lemma 2.4. It follows by Proposition 2.7 and the preceding that there exist indices $\{\gamma_n\}$, unitaries $u_n \in B(H_{\gamma_n})$, and complex numbers $\{z_n\}$ satisfying (i), (ii), and (iii).

(2)⇒(1). This follows easily from Proposition 2.7 and Lemma 2.4, and so the proof is complete.

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