This sequence converges to the limit \( \text{hyp}(i) \), where \( \text{hyp}(i) \approx 0.4383 + 0.3606i \).

The behaviour of the sequence \( \{h_n(z)\} \) for different values of \( z \) is illustrated by an interesting fractal image plotted on the Argand Plane at [2].

**References**


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**94.29 On two properties of the inverse Hilbert matrix**

A Hilbert matrix \( H \) of order \( n \) is a square \( n \times n \) matrix with entries \( h_{ij} = 1 / (i + j - 1) \). For instance,

\[
H_4 = \begin{pmatrix}
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\
\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\
\frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\
\frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7}
\end{pmatrix}
\]

and

\[
H_4^{-1} = \begin{pmatrix}
16 & -120 & 240 & -140 \\
-120 & 1200 & -2700 & 1680 \\
240 & -2700 & 6480 & -4200 \\
-140 & 1680 & -4200 & 2800
\end{pmatrix}.
\]

Hilbert matrices are invertible, but they suffer so-called ill-conditioning, which means that they are very sensitive to numerical operations. In general, one should not trust the results of computations on ill-conditioned matrices, unless exact computations are performed. Also, ill-conditioned matrices do not tolerate approximations.

Ill-conditioning is usually measured by the so-called matrix condition number, which is always greater than or equal to 1. Matrices with condition number near 1 are said to be well-conditioned, but if this number is much greater than 1, the matrix is ill-conditioned. For example, the Hilbert matrix of order 5 has a condition number around \( 10^5 \).

Let \( H_n^{-1} = H^{-1} = \{h_{ij}^{-1}\} \) for \( 1 \leq i, j \leq n \), be the inverse Hilbert matrix of order \( n \). Further let \( S_i = \sum_{j=1}^{n} h_{ij}^{-1} \) be the sum of the elements of the
In this note we shall prove two properties of the inverse Hilbert matrix that, to the best of our knowledge, are new.

**Theorem 1**: $S_1 = (-1)^{n-1} n.$

and

**Theorem 2**: $S = \sum_{i=1}^{n} S_i = (S_1)^2.$

These theorems imply a known fact [1], that $S = n^2$.

**Proof of Theorem 1**: Since

$$h_{ij} = (-1)^{i+j}(i+j-1){n+i-1 \choose n-j}{n+j-1 \choose n-i}{i+j-2 \choose i-1}$$

(see [2, 3]), then

$$S_1 = \sum_{j=1}^{n} (-1)^{i+j}(i+j-1){n \choose n-j}{n+j-1 \choose n-1}.$$  

Since $j{n \choose n-j} = n{n-1 \choose j-1}$, then we need to show that

$$\sum_{j=1}^{n} (-1)^{i+j}{n+j-1 \choose j-1}{n+j-1 \choose n-1} = (-1)^n.$$  

(1)

Changing the first $j$ to $j+1$ and then $n-1$ to $n$, we get an equivalent identity

$$\sum_{j=0}^{n} (-1)^{i+j}{n \choose j}{n+j+1 \choose n} = (-1)^n.$$  

(2)

Observe that for $0 < |x| < 1$,

$$\frac{1}{(1+x)^{n+1}} = \sum_{r=0}^{\infty} (-1)^r {n+r \choose n} x^r$$

and

$$\left(1 + \frac{1}{x}\right)^n = \sum_{s=0}^{n} {n \choose s} x^{-s}.$$  

Multiplying these two series we get

$$\frac{(1+x)^n}{x^n} \frac{1}{(1+x)^{n+1}} = \frac{1}{x^n (1+x)} = \sum_{m=0}^{\infty} (-1)^m x^{m-n}.$$  

Comparing coefficients of $x$, that on the right-hand side being $(-1)^{n+1}$ and occurring when $m = n + 1$, we get
\[ \sum_{j=0}^{n} (-1)^{j+1} \binom{n+j+1}{n+j} = (-1)^{n+1}, \]

which is equivalent to (2).

**Proof of Theorem 2:** Let \( p(x) = \sum_{i=0}^{n-1} a_i x^i \) be such that \( \int_{0}^{1} x^k p(x) \, dx = 1 \) for all \( 0 \leq k \leq n-1 \). These \( n \) equations lead to the linear system \( H_n a = 1 \), where \( H_n \) is the Hilbert matrix of order \( n \), \( a = [a_0, a_1, \ldots, a_{n-1}]^t \), \( 1 = [1, 1, \ldots, 1]^t \). Observe that \( p(0) = S_1 \) and \( p(1) = S \). We need to show that \( p(1) = p^2(0) \). We have

\[
\int_{0}^{1} p^2(x) \, dx = \int_{0}^{1} \sum_{i=0}^{n-1} a_i x^i p(x) \, dx = \sum_{i=0}^{n-1} a_i \int_{0}^{1} x^i p(x) \, dx = \sum_{i=0}^{n-1} a_i = p(1). \tag{3}
\]

The conditions imposed on the polynomial \( p(x) \) imply that

\[
\int_{0}^{1} x p(x) p'(x) \, dx = \int_{0}^{1} p(x) p'(x) \, dx \left( = \sum_{i=1}^{n-1} i a_i \right),
\]

so

\[
\int_{0}^{1} p^2(x) \, dx = x p^2(x) \bigg|_{0}^{1} - 2 \int_{0}^{1} x p(x) p'(x) \, dx = x p^2(x) \bigg|_{0}^{1} - 2 \int_{0}^{1} p(x) p'(x) \, dx = p^2(1) - 2 \int_{0}^{1} p(x) p'(x) \, dx, \tag{4}
\]

and

\[
\int_{0}^{1} p(x) p'(x) \, dx = p^2(x) \bigg|_{0}^{1} - \int_{0}^{1} p(x) p'(x) \, dx
\]

so

\[
2 \int_{0}^{1} p(x) p'(x) \, dx = p^2(1) - p^2(0). \tag{5}
\]

Using (3), (4) and (5) we get \( p(1) = p^2(1) - (p^2(1) - p^2(0)), \) i.e. \( p(1) = p^2(0) \).

**Remark:** This note was inspired by problem 11248 proposed by Pál Peter Dályay, Deák Ferenc High School, Szeged, Hungary in the *American
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Mathematical Monthly, Vol. 113, No.8, (October 2006). The authors have discovered the above two new properties of the inverse Hilbert matrix after solving the above problem.

References

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94.30 Ratios of volumes related to the odd extension of a power function

Introduction
Encouraged by correspondence we have received over the past few years, we return here to a situation similar to that of the earlier papers [1, 2, 3, 4]. In this setting, the initial observation was that the line tangent to the curve $y = x^3$ at the point $(a, a^3) \neq (0, 0)$ intersects the curve at exactly one other point. Thus, there is a well-defined region enclosed by that tangent line and the curve. A second region can be formed by drawing another tangent line to the cubic curve at the point where the first tangent line intersects the curve. We will refer to the tangent lines related in this way as successive tangent lines to the curve. That is, each successive tangent line is tangent to the curve at the other point where the previous tangent line crosses the curve. In [1], the authors began by showing that the areas of the regions enclosed by successive tangent lines are proportional to one another. Those authors then generalised that result to the odd extension of $x^3$, defined below; this is an extension of the power function $x^3$ to an odd function of $x$. In [2], we considered the lengths of the chords defined by the successive tangent lines to the odd extension of $x^3$ and found that a similar proportionality result holds, but only in the limit as one chord is compared to the next for an infinite succession of tangent lines. Here, we look at the volumes obtained by revolving each of the enclosed regions about the tangent line that defines it and find several proportionality results that hold, as before, in the limit. We conclude with a suggestion for further work in a similar vein.